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GENERAL INVERSE PROBLEMS FOR THE BEST APPROXIMATION IN BANACH SPACES

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ABSTRACT. Inverse problems for the degree of the best approximation are discussed in the general settings in Banach spaces and several refined results are obtained. Furthermore, appplications are provided from the viewpoint of Fourier expansions associated with projections on Banach spaces.

1. Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic, continuous functions f on the real line $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \leq \pi\}.$$

Let $\mathbb{N}$ be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, we denote by $\mathfrak{T}_n$ the set of all trigonometric polynomials of degree at most n. For a given function $f \in C_{2\pi}$, we define

$$E_n(C_{2\pi}; f) = \inf\{\|f - g\|_\infty : g \in \mathfrak{T}_n\},$$

which is called the best approximatin of degree n to f with respect to $\mathfrak{T}_n$. Since $\mathfrak{T}_n$ is the $2n+1$-dimensional Chebyshev subspace of $C_{2\pi}$, for each $f \in C_{2\pi}$, there exists a unique trigonometric polynomial $g_n \in \mathfrak{T}_n$ of the best approximation of f with respect to $\mathfrak{T}_n$, i.e., such that

$$E_n(C_{2\pi}; f) = \|f - g_n\|_\infty.$$

(see, e.g., [8; Chap. 2, Theorem 6]).

In this context, the classical Weierstrass approximation theorem simply states that for any $f \in C_{2\pi}$, the sequence $\{E_n(C_{2\pi}; f) : n \in \mathbb{N}_0\}$

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converges to zero as n tends to infinity. Yet it does not say how fast $E_n(C_{2\pi}; f)$ approaches zero. In general, the smoother function, the faster $E_n(C_{2\pi}; f)$ tends to zero. The results that guarantee this event are sometimes called the direct theorems of Jackson-type. Conversely, the inverse theorems of Bernstein-type assert that if $E_n(C_{2\pi}; f)$ tends rapidly enough to zero, then f has certain smoothness properties, which are usually given in terms of its modulus of continuity, Lipschitz classes and differentiability properties.

More precisely, one of Jackson-type theorems states the following: Let $r \in \mathbb{N}_0$ and $0 < \alpha \leq 1$. If the rth derivative $f^{(r)}$ satisfies a Lipschitz condition of order α, i.e.,

$$\|f^{(r)}(\cdot - t) - f^{(r)}(\cdot)\|_\infty = O(|t|^\alpha) \qquad (t \in \mathbb{R}), \tag{1}$$

then

$$E_n(C_{2\pi}; f) = O\left(\frac{1}{n^{\alpha+r}}\right) \qquad (n \to \infty). \tag{2}$$

Conversely, one of Bernstein-type theorems states that if $0 < \alpha < 1$, then (2) implies (1) and that if $\alpha = 1$, then (2) implies

$$\|f^{(r)}(\cdot - t) - f^{(r)}(\cdot)\|_\infty = O(|t \log |t||) \qquad (|t| \to +0).$$

Furthermore, if $\alpha = 1$, then (2) is equivalent to

$$\|f^{(r)}(\cdot + t) + f^{(r)}(\cdot - t) - 2f^{(r)}(\cdot)\|_\infty = O(|t|) \qquad (t \in \mathbb{R}),$$

which is due to Zygmund (cf. [22]). For further related results, we refer to [2], [5], [8] and [9]. The statements analogous to these results also hold for the Banach space $L^p_{2\pi}$ consisting of all 2π-periodic, pth power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \, dt\right)^{1/p} \qquad (1 \leq p < \infty).$$

(cf. [2], [17], [20], [22]). The same results hold if "O" is replaced by "o" (cf. [9], [22]).

These results have been developed further by several authors. There is an elegant generalization by de la Vallée Poussin, which was extended to the higher order moduli of continuity by Butzer and Nessel [3] (cf. [6], [17]). In [13] (cf. [12]), we generalized this result to arbitrary Banach spaces in the setting of the theory of Fourier expansions (cf. [4], [10], [11], [21]). The purpose of this paper is to develop further these techniques in the general situations of [16], and to establish several refined results.

2. General settings and results

Let X be a Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\{M_n : n \in \mathbb{N}_0\}$ be a sequence of closed linear subspaces of X satisfying the following conditions:

(M-1) $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$.

(M-2) $\cup_{n=0}^{\infty} M_n$ is dense in X.

Let $\{G_k : k \in \mathbb{N}_0\}$ be a sequence of closed linear operators with domain $D(G_k)$ in X and range in X such that for all $n, k \in \mathbb{N}_0$,

(G-1) $M_n \subseteq D(G_k), \quad G_k(M_n) \subseteq M_n$,
and for all $k \in \mathbb{N}, n \in \mathbb{N}_0$ and all $f \in M_n$,

(G-2) $\|G_k(f)\|_X \le A_k n^k \|f\|_X$,
where A_k is a positive constant independent of n and f.

For a given $f \in X$, we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. By Condition (M-1), we have

$$E_n(X; f) \ge E_{n+1}(X; f) \ge 0 \qquad (n = 0, 1, 2, \ldots)$$

and Condition (M-2) implies that for every $f \in X$,

$$\lim_{n \to \infty} E_n(X; f) = 0.$$

In this paper, we derive certain smoothness properties of an element $f \in X$ from the hypothesis that $E_n(X; f)$ approaches zero with a given rapidity. For this aim, we consider the following class of functions:

Let $\{\omega_k : k \in \mathbb{N}_0\}$ be a sequence of non-negative functions on $X \times [0, \infty)$, which satisfy the following conditions:

$(\omega - 1)$ For each $k \in \mathbb{N}_0$, there exists a positive constant B_k such that

$$\omega_k(f, \delta) \le B_k \|f\|_X$$

for all $f \in X$ and all $\delta \ge 0$.

$(\omega - 2)$ $\lim_{\delta \to +0} \omega_k(f, \delta) = 0$ for all $f \in X$ and all $k \in \mathbb{N}$.

$(\omega - 3)$ For each $k \in \mathbb{N}_0$, there exists a positive constant C_k such that

$$\omega_k(f, \delta) \le C_k \delta^k \|G_k(f)\|_X$$

for all $f \in D(G_k)$ and all $\delta \ge 0$.

$(\omega - 4)$ $\omega_k(\cdot, \delta)$ is a seminorm on X for each $k \in \mathbb{N}_0$, $\delta \geq 0$.

From now on, we suppose that for each $f \in X$, there is an element $g_n \in M_n$ of the best approximation of f with respect to M_n, i.e., such that

$$E_n(X; f) = \|f - g_n\|_X \quad (n \in \mathbb{N}_0).$$

For the general theory of the existence of the best approximation in normed linear spaces, we refer to [19].

Let $a \in \mathbb{N}$ and $a \geq 2$. Let $\varphi : [a, \infty) \to [0, \infty)$ be a bounded function and we define

$$\varphi^*(x) = \sup\{\varphi(t) : x \leq t\} \qquad (x \geq a).$$

Then $\varphi^* : [a, \infty) \to [0, \infty)$ is a bounded, monotone decreasing function and

$$\varphi(x) \leq \varphi^*(x)$$

for all $x \geq a$. In particular, if φ is monotone decreasing, then $\varphi = \varphi^*$. Also,

$$\lim_{x \to \infty} \varphi(x) = 0 \quad \Longleftrightarrow \quad \lim_{x \to \infty} \varphi^*(x) = 0.$$

Let $\Omega : [a, \infty) \to [0, \infty)$ be a non-negative, monotone decreasing function satisfying

$$\lim_{x \to \infty} \Omega(x) = 0 \quad \text{and} \quad \int_a^\infty \frac{\Omega(x)}{x} \, dx < \infty.$$

Then we have the following result:

Theorem 1. *Let $f \in X, r \in \mathbb{N}_0, a < \min\{b, c\}, b^p \leq c, p \geq 2$ and $\delta > 0$. Suppose that for all $n \geq a$,*

$$E_n(X; f) \leq \frac{\varphi(n)\Omega(n)}{n^r}.$$

Then f belongs to $D(G_r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(G_r(f), \delta) \leq C\Big\{\delta^k \int_a^{ab} x^{k-1}\Omega(x)\, dx$$

$$+ \varphi^*(ab)\Big(\delta^k \int_a^{ac} x^{k-1}\Omega(x)\, dx + \int_c^\infty \frac{\Omega(x)}{x}\, dx\Big)\Big\},$$

where C is a positive constant independent of b, c, p and δ.

The proof of this theorem is the same argument as that of [16; Theorem 1] for the case of $p = 2$ and is therefore omitted. Now, in the rest of this section we always suppose that $r \in \mathbb{N}_0$ and $f \in X$ satisfies

$$E_n(X; f) \le \frac{\varphi(n)\Omega(n)}{n^r}$$

for all $n \ge a$, and let $k \in \mathbb{N}$. Therefore, by Theorem 1, f always belongs to $D(G_r)$.

Theorem 2. *Let $f \in X, r \in \mathbb{N}_0, a < b, p \ge 2$ and $\delta > 0$. Then we have*

$$\omega_k(G_r(f), \delta) \le C\Big\{\delta^k \int_a^{ab} x^{k-1}\Omega(x)dx$$

$$+ \varphi^*(ab)\Big(\delta^k \int_a^{ab^p} x^{k-1}\Omega(x)dx + \int_{b^p}^{\infty} \frac{\Omega(x)}{x}dx\Big)\Big\}$$

$$\le C\Big\{\big(1+\varphi^*(ab)\big)\delta^k \int_a^{ab^p} x^{k-1}\Omega(x)dx + \varphi^*(ab)\int_{b^p}^{\infty} \frac{\Omega(x)}{x}dx\Big\},$$

where C is a positive constatnt independent of b and δ.

Proof. Take $c = b^p$ in Theorem 1.

Corollary 1. *Let ξ be a positive function on $(0, \infty), p \ge 2$ and $\mu > 0$. If $\xi(\delta) < a^{-\mu}$, $\delta > 0$, then*

$$\omega_k(G_r(f), \delta) \le C\Big\{\delta^k \int_a^{a/\xi(\delta)^{1/\mu}} x^{k-1}\Omega(x)dx$$

$$+ \varphi^*(a/\xi(\delta)^{1/\mu})\Big(\delta^k \int_a^{a/\xi(\delta)^{p/\mu}} x^{k-1}\Omega(x)\,dx + \int_{1/\xi(\delta)^{p/\mu}}^{\infty} \frac{\Omega(x)}{x}\,dx\Big)\Big\}$$

$$\le C\Big\{\big(1+\varphi^*(a/\xi(\delta)^{1/\mu})\big)\delta^k \int_a^{a/\xi(\delta)^{p/\mu}} x^{k-1}\Omega(x)\,dx$$

$$+ \varphi^*(a/\xi(\delta)^{1/\mu})\int_{1/\xi(\delta)^{p/\mu}}^{\infty} \frac{\Omega(x)}{x}\,dx\Big\},$$

where C is a positive constant independent of ξ, μ, δ and p. In particular, if $\lambda > 0, \mu > 0$ and $0 < \delta^\lambda < a^{-\mu}$, then

$$\omega_k(G_r(f), \delta) \le C\Big\{\delta^k \int_a^{a/\delta^{\lambda/\mu}} x^{k-1}\Omega(x)dx$$

$$+ \varphi^*(a/\delta^{\lambda/\mu})\Big(\delta^k \int_a^{a/\delta^{p\lambda/\mu}} x^{k-1}\Omega(x)dx + \int_{\delta^{-p\lambda/\mu}}^{\infty} \frac{\Omega(x)}{x}dx\Big)\Big\}$$

$$\leq C\Big\{\Big(1+\varphi^*(a/\delta^{\lambda/\mu})\Big)\delta^k\int_a^{a/\delta^{p\lambda/\mu}} x^{k-1}\Omega(x)\,dx$$
$$+\ \varphi^*(a/\delta^{\lambda/\mu})\int_{\delta^{-p\lambda/\mu}}^{\infty}\frac{\Omega(x)}{x}\,dx\Big\},$$

where C is a positive constant independent of λ,μ,δ and p.

Theorem 3. *Let ξ and γ be positive functions on $(0,\infty)$ such that*

$$\lim_{\delta\to+0}\xi(\delta)=+\infty,\qquad \lim_{\delta\to+0}\gamma(\delta)=0,\qquad \lim_{\delta\to+0}\gamma(\delta)\xi^p(\delta)=0 \tag{3}$$

for some $p\geq 2$. Then we have

$$\omega_k(G_r(f),\delta)\leq O\Big\{\delta^k\int_a^{a\xi(\delta)} x^{k-1}\Omega(x)\,dx$$
$$+\ \varphi^*(a\xi(\delta))\Big(\delta^k\int_a^{a/\gamma(\delta)} x^{k-1}\Omega(x)dx+\int_{1/\gamma(\delta)}^{\infty}\frac{\Omega(x)}{x}dx\Big)\Big\}$$
$$(\delta\to+0).$$

In particular, if $\lambda,\mu>0$, then

$$\omega_k(G_r(f),\delta)\leq O\Big\{\delta^k\int_a^{a\mu|\log\delta|} x^{k-1}\Omega(x)\,dx$$
$$+\ \varphi^*(a\mu|\log\delta|)\Big(\delta^k\int_a^{a/\delta^\lambda} x^{k-1}\Omega(x)dx+\int_{1/\delta^\lambda}^{\infty}\frac{\Omega(x)}{x}dx\Big)\Big\}$$
$$(\delta\to+0).$$

Proof. We take $b=\xi(\delta)$ and $c=1/\gamma(\delta)$ for sufficiently small $\delta>0$, and use Theorem 1.

3. Results of Bernstein and Zygmund type

In this section, we restrict ourselves to the special case where

$$\Omega(x)=\frac{1}{x^\alpha},\qquad \alpha>0. \tag{4}$$

Here, let $f\in X, r\in\mathbb{N}_0, k\in\mathbb{N}$ and suppose that for all $n\geq a$,

$$E_n(X;f)\leq\frac{\varphi(n)}{n^{\alpha+r}}.$$

Therefore, by Theorem 1, f always belongs to $D(G_r)$.

Theorem 4. *Let $a < \min\{b, c\}, b^p \leq c, p \geq 2$ and $\delta > 0$ Then we have:*

$$\omega_k(G_r(f), \delta) \leq C\Big\{\frac{a^{k-\alpha}}{k-\alpha}\Big(b^{k-\alpha} - 1 + \varphi^*(ab)(c^{k-\alpha} - 1)\Big)\delta^k + \frac{\varphi^*(ab)}{\alpha c^\alpha}\Big\}$$

$$(k \neq \alpha),$$

and

$$\omega_k(G_r(f), \delta) \leq C\Big\{\Big(\log b + \varphi^*(ab)\log c\Big)\delta^k + \frac{\varphi^*(ab)}{kc^k}\Big\}$$

$$(k = \alpha),$$

where C is a positive constant independent of b, c, p and δ.

Proof. Let $\Omega(x)$ be as in (4). If $a < d$, then

$$\int_a^{ad} x^{k-1}\Omega(x)\,dx = \begin{cases} \frac{a^{k-\alpha}}{k-\alpha}(d^{k-\alpha} - 1) & (k \neq \alpha) \\ \log d & (k = \alpha), \end{cases}$$

and

$$\int_c^\infty \frac{\Omega(x)}{x}\,dx = \frac{1}{\alpha c^\alpha}$$

Therefore, the desired result follows from Theorem 1.

Corollary 2. *Let $p \geq 2$, and let γ be a positive function on $(0, \infty)$ with $\lim_{\delta\to+0}\gamma(\delta) = 0$. Then*

$$\omega_k(G_r(f), \delta) \leq C\Big\{\frac{a^{k-\alpha}}{k-\alpha}\Big(\frac{\delta}{\gamma^p(\delta)}\Big)^k\Big(1+\varphi^*(a/\gamma(\delta))\Big) + \frac{\varphi^*(a/\gamma(\delta))}{\alpha}\Big\}\gamma^{p\alpha}(\delta)$$

$$(\alpha < k, \gamma(\delta) < a^{-1});$$

$$\omega_k(G_r(f), \delta) \leq C\Big\{\Big(1 + p\varphi^*(a/\gamma(\delta))\Big)\Big(\frac{\delta}{\gamma(\delta)}\Big)^k + \frac{\varphi^*(a/\gamma(\delta))}{k}\gamma^{k(p-1)}(\delta)\Big\}\gamma^k(\delta)|\log\gamma(\delta)|$$

$$(\alpha = k, \gamma(\delta) < \min\{a^{-1}, e^{-1}\});$$

$$\omega_k(G_r(f), \delta) \leq C\Big\{\frac{a^{k-\alpha}}{\alpha-k}\Big(\frac{\delta}{\gamma(\delta)}\Big)^k\Big(1 + \varphi^*(a/\gamma(\delta))\Big) + \frac{\varphi^*(a/\gamma(\delta))}{\alpha}\gamma^{p\alpha-k}(\delta)\Big\}\gamma^k(\delta)$$

$$(\alpha > k, \gamma(\delta) < a^{-1}).$$

Here, C is a positive constant independent of δ, γ and p. In particular, if $p \geq 2$ and $\lambda > 0$, then

$$\omega_k(G_r(f), \delta) \leq C\Big\{\frac{a^{k-\alpha}}{k-\alpha}\big(1+\varphi^*(a/\delta^\lambda)\big)\delta^{(1-p\lambda)k} + \frac{\varphi^*(a/\delta^\lambda)}{\alpha}\Big\}\delta^{p\alpha\lambda}$$

$$(\alpha < k, \delta < a^{-1/\lambda}),$$

$$\omega_k(G_r(f), \delta) \leq C\Big\{\big(\big(1+p\varphi^*(a/\delta^\lambda)\big) + \frac{\varphi^*(a/\delta^\lambda)}{k}\delta^{(p\lambda-1)k}\Big\}\delta^k\lambda|\log\delta|$$

$$(\alpha = k, \delta < \min\{a^{-1/\lambda}, e^{-1/\lambda}\})$$

and

$$\omega_k(G_r(f), \delta) \leq C\Big\{\frac{a^{k-\alpha}}{\alpha-k}\big(1+\varphi^*(a/\delta^\lambda)\big) + \frac{\varphi^*(a/\delta^\lambda)}{\alpha}\delta^{p\alpha\lambda-k}\Big\}\delta^k$$

$$(\alpha > k, \delta < a^{-1/\lambda}),$$

where C is a positive constant independent of δ, λ and p.

Corollary 3 ([16; Corollary 2]). *Let $\alpha > 0, f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = O\Big(\frac{1}{n^{\alpha+r}}\Big) \qquad (n \to \infty),$$

then f belongs to $D(G_r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(G_r(f), \delta) = \begin{cases} O(\delta^\alpha) & (\alpha < k) \\ O(\delta^k|\log\delta|) & (\alpha = k) \\ O(\delta^k) & (\alpha > k) \end{cases}$$

$$(\delta \to +0).$$

Corollary 4. *Let $r \in \mathbb{N}_0, r < \alpha$ and suppose that*

$$E_n(X; f) = O\left(\frac{1}{n^\alpha}\right) \quad (n \to \infty).$$

Then f belongs to $D(G_r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(G_r(f), \delta) = \begin{cases} O(\delta^{\alpha-r}) & (\alpha - r < k) \\ O(\delta^k|\log\delta|) & (\alpha - r = k) \\ O(\delta^k) & (\alpha - r > k) \end{cases}$$

$$(\delta \to +0).$$

In particular,

$$\omega_k(G_0(f),\delta) = \begin{cases} O(\delta^\alpha) & (\alpha < k) \\ O(\delta^k |\log\delta|) & (\alpha = k) \\ O(\delta^k) & (\alpha > k) \end{cases}$$

$$(\delta \to +0).$$

Theorem 5. *Let $k \in \mathbb{N}$ and $p \geq 2$. Let ξ and γ be positive functions on $(0,\infty)$. If $\alpha < k$, then*

$$\omega_k(G_r(f),\delta) \leq C\Big\{\frac{a^{k-\alpha}}{k-\alpha}\frac{\delta^k}{\gamma(\delta)}\xi^{k-\alpha}(\delta)$$

$$+ \varphi^*(a\xi(\delta))\Big(\frac{a^{k-\alpha}}{k-\alpha}\frac{\delta^k}{\gamma(\delta)}\xi^{p(k-\alpha)}(\delta) + \frac{1}{\alpha}\frac{1}{\xi^{p\alpha}(\delta)\gamma(\delta)}\Big)\Big\}\gamma(\delta) \qquad (5)$$

$$(a < \xi(\delta)),$$

where C is a positive constant independent of δ, ξ, γ and p. If (3) holds and $\alpha = k$, then

$$\omega_k(G_r(f),\delta) = O\Big[\Big\{\Big(\frac{\delta}{\gamma(\delta)}\Big)^k \frac{\log\xi(\delta)}{|\log\gamma(\delta)|}$$

$$+ \varphi^*(a\xi(\delta))\Big(\Big(\frac{\delta}{\gamma(\delta)}\Big)^k + \frac{1}{k}\Big)\Big\}\gamma^k(\delta)|\log\gamma(\delta)|\Big] \qquad (\delta \to +0). \qquad (6)$$

In particular, if $\beta, \lambda > 0$ and $\alpha < k$, then

$$\omega_k(G_r(f),\delta) \leq C\Big\{\frac{a^{k-\alpha}}{k-\alpha}\delta^\nu + \Big(\frac{a^{k-\alpha}}{k-\alpha}\delta^\rho + \frac{1}{\alpha}\delta^{2\alpha\lambda-\beta}\Big)\varphi^*(a/\delta^\lambda)\Big\}\delta^\beta$$

$$(0 < \delta < a^{-1/\lambda}),$$

where

$$\nu = (\alpha - k)\lambda + k - \beta, \quad \rho = 2(\alpha - k)\lambda + k - \beta$$

and C is a positive constant independent of β, δ and λ. If $\mu > 0$ and $\alpha = k$, then

$$\omega_k(G_r(f),\delta) = O\Big\{\Big(\frac{\log|\mu\log\delta|}{\lambda|\log\delta|} + \varphi^*(a\mu|\log\delta|)\Big(1 + \frac{\delta^{(\lambda-1)k}}{k}\Big)\Big)\lambda\delta^k|\log\delta|\Big\}$$

$$(\delta \to +0).$$

Proof. If $\alpha < k$, then by Theorem 4, we have

$$\omega_k(G_r(f), \delta) \leq C\left(\frac{a^{k-\alpha}}{k-\alpha}\left(b^{k-\alpha} + \varphi^*(ab)b^{p(k-\alpha)}\right)\delta^k + \frac{\varphi^*(ab)}{\alpha b^{p\alpha}}\right)$$

whenever $a < b$ and $\delta > 0$. Thus taking $b = \xi(\delta)$, we get the estimate (5). If $\alpha = k$, then by Theorem 4, we have

$$\omega_k(G_r(f), \delta) = O\left\{\delta^k \log \xi(\delta) + \varphi^*(a\xi(\delta))\left(\delta^k |\log \gamma(\delta)| + \frac{\gamma^k(\delta)}{k}\right)\right\}$$
$$(\delta \to +0),$$

which implies (6).

Corollary 5. *Let $\alpha > 0, f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = o\left(\frac{1}{n^{\alpha+r}}\right) \quad (n \to \infty),$$

then f belongs to $D(G_r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(G_r(f), \delta) = \begin{cases} o(\delta^\alpha) & (\alpha < k) \\ o(\delta^k |\log \delta|) & (\alpha = k) \end{cases}$$
$$(\delta \to +0).$$

Corollary 6. *Let $r \in \mathbb{N}_0, r < \alpha, f \in X$ and suppose that*

$$E_n(X; f) = o\left(\frac{1}{n^\alpha}\right) \quad (n \to \infty).$$

Then f belongs to $D(G_r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(G_k(f), \delta) = \begin{cases} o(\delta^{\alpha-r}) & (\alpha - r < k) \\ o(\delta^k |\log \delta|) & (\alpha - r = k) \end{cases}$$
$$(\delta \to +0).$$

In particular,

$$\omega_k(G_0(f), \delta) = \begin{cases} o(\delta^\alpha) & (\alpha < k) \\ o(\delta^k |\log \delta|) & (\alpha = k) \end{cases}$$
$$(\delta \to +0).$$

Remark. Results similar to those in [14] and [15] hold in the present general settings with the following additional conditions:

$(\omega - 5)$ $G_k(g) = 0$ for all $k \in \mathbb{N}$ and all $g \in M_0$.

$(\omega - 6)$ $\omega_k(f, \delta) = \omega_k(f + g, \delta)$ for all $k \in \mathbb{N}, \delta \geq 0, f \in X$ and all $g \in M_0$.

We omit the details.

4. Applications

Let $\mathbb{Z}$ denote the set of all integers, and let $\{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\{P_j : j \in \mathbb{Z}\}$ is fundamental, i.e., the linear spann of $\cup_{j \in \mathbb{Z}} P_j(X)$ is dense in X.

(P-3) $\{P_j : j \in \mathbb{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

For any $f \in X$, we associate its (formal) Fourier series expansion (with respect to $\{P_j\}$)

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j.$$

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\{T_t : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying

$$A = \sup\{\|T_t\|_{B[X]} : t \in \mathbb{R}\} < \infty$$

and having the expansions

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{-ijt} P_j \qquad (t \in \mathbb{R}).$$

Then we have the following ([10; Proposition 2]):

Proposition 1. *$\{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and there holds*

$$G(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} (-ij) P_j(f) \qquad (f \in D(G)),$$

where G is the infinitesimal generator of $\{T_t\}$ with domain $D(G)$.

For $k = 0, 1, 2, \cdots$, the operator G^k is inductively defined by

$$G^0 = I, \quad G^1 = G,$$

$$D(G^k) = \{f : f \in D(G^{k-1}), G^{k-1}(f) \in D(G)\}$$

and

$$G^k(f) = G(G^{k-1}(f)) \qquad (f \in D(G^k), k = 1, 2, 3, \dots).$$

Then for each $k \in \mathbb{N}$, $D(G^k)$ is a dense linear subspace of X and G^k is a closed linear operator with domain $D(G^k)$ (cf. [1; Propositions 1.1.4 and 1.1.6]). Furthermore, it follows from induction on k that

$$G^k(P_j(g)) = (-ij)^k P_j(g) \qquad (g \in X, j \in \mathbb{Z}, k \in \mathbb{N}) \tag{7}$$

and

$$G^k(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} (-ij)^k P_j(f) \qquad (f \in D(G^k), k \in \mathbb{N}).$$

For each $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^k = (T_t - I)^k = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} T_{mt} \qquad (k \geq 1).$$

Then Δ_t^k belongs to $M[X]$, and

$$\Delta_t^k \quad \sim \quad \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)^k P_j \tag{8}$$

and

$$\|\Delta_t^k\|_{B[X]} \leq B_k,$$

where

$$B_k = \min\{(A+1)^k, 2^k A\}.$$

If $k \in \mathbb{N}_0, f \in X$ and $\delta \geq 0$, then we define

$$\omega_k(X; f, \delta) = \sup\{\|\Delta_t^k(f)\|_X : |t| \leq \delta\},$$

which is called the k-th modulus of continuity of f with respect to the familiy $\{T_t\}$. This quantity has the following properties ([12; Lemma 1]):

Proposition 2. *Let $k \in \mathbb{N}$ and $f \in X$. Then the following statements hold:*

(a) $\quad \omega_k(X; f, \delta) \leq B_k \|f\|_X \qquad (\delta \geq 0).$

(b) $\omega_k(X; f, \cdot)$ *is a monotone increasing function on* $[0, \infty)$ *and* $\omega_k(X; f, 0) = 0$.

(c) $\quad \omega_{k+r}(X; f, \delta) \leq B_k \omega_r(X; f, \delta) \qquad (r \in \mathbb{N}_0,\ \delta \geq 0).$

In particular, we have

$$\lim_{\delta \to +0} \omega_k(X; f, \delta) = 0.$$

(d) $\quad \omega_k(X; f, \xi\delta) \leq A(1+\xi)^k \omega_k(X; f, \delta) \qquad (\xi,\ \delta \geq 0).$

(e) If $0 < \delta \leq \xi$, *then*

$$\omega_k(X; f, \xi)/\xi^k \leq 2^k A \omega_k(X; f, \delta)/\delta^k.$$

(f) If $f \in D(G^k)$, *then*

$$\omega_{k+r}(X; f, \delta) \leq A\delta^k \omega_r(X; f, \delta) \qquad (r \in \mathbb{N}_0,\ \delta \geq 0).$$

In particular, we have

$$\omega_k(X; f, \delta) \leq A\delta^k \|G^k(f)\|_X \qquad (\delta \geq 0).$$

(g) $\omega_k(X; \cdot, \delta)$ *is a seminorm on* X.

For each $n \in \mathbb{N}_0$, let M_n be the linear spann of $\{P_j(X) : |j| \leq n\}$, which is a closed linear subspace of X. Then the following result is the Bernstein-type inequality [13; Lemma 5]):

Proposition 3. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then*

$$\|G^k(f)\|_X \leq (2nB)^k \|f\|_X$$

holds for all $f \in M_n$, where

$$B = \sup\{\|T_t\|_{B[X]} : |t| \leq \pi\}.$$

Now, the sequence $\{M_n : n \in \mathbb{N}_0\}$ clearly satisfies (M-1) and (P-2) implies (M-2). Take

$$G_k = G^k \quad (k \in \mathbb{N}_0), \quad \omega_k(f, \delta) = \omega_k(X; f, \delta) \quad (f \in X, k \in \mathbb{N}_0, \delta \geq 0).$$

Then (7) and Proposition 3 imply (G-1) and (G-2) with $A_k = (2B)^k$, respectively. Furthermore, all the conditions $(\omega - 1) - (\omega - 4)$ are satisfied by Proposition 2. Also, $(\omega - 5)$ and $(\omega - 6)$ hold by (7) and (8), respectively.

Consequently, all the results obtained in the preceding sections hold in the above settings (cf. [12], [13]). In particular, if we restrict ourselves to the case where X is a homogeneous Banach space which includes $C_{2\pi}$ and $L^p_{2\pi}$, $1 \leq p < \infty$, as special cases (cf. [7], [10], [18]), then the sequence $\{P_j : j \in \mathbb{Z}\}$ is defined by

$$P_j(f)(\cdot) = \hat{f}(j)e^{ij\cdot} \qquad (f \in X),$$

where

$$\hat{f}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt \qquad (j \in \mathbb{Z})$$

is the j-th Fourier coefficient of f and $\{T_t : t \in \mathbb{R}\}$ is the group of right translations in $B[X]$ defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X).$$

Therefore, $M_n = \mathfrak{T}_n$ for all $n \in \mathbb{N}_0$ and

$$\Delta_t^k(f)(\cdot) = \sum_{m=0}^{k} (-1)^{k-m}\binom{k}{m} f(\cdot - mt)$$

$$\sim \quad \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)^k \hat{f}(j)e^{ij\cdot}$$

for all $f \in X, k \in \mathbb{N}_0$ and all $t \in \mathbb{R}$.

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