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ESTIMATES FOR THE DEGREE OF BEST APPROXIMATION IN BANACH SPACES

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ABSTRACT. We give estimates for the degree of the best approximation by elements of closed linear subspaces associated with a sequence of projection operators on Banach spaces and obtain inverse theorems of Bernstein type. The results are applied to the best approximation by trigonometric polynomials in homogeneous Banach spaces which include the classical function spaces, as special cases.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\mathbb{Z}$ denote the set of all integers, and let $\{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_jP_n = \delta_{j,n}P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.
(P-2) $\{P_j : j \in \mathbb{Z}\}$ is fundamental, i.e., the linear spann of the set $\cup_{j\in\mathbb{Z}}P_j(X)$ is dense in X.
(P-3) $\{P_j : j \in \mathbb{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

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For any $f \in X$, we associate its (formal) Fourier series expansion (with respect to $\{P_j : j \in \mathbb{Z}\}$)

$$f \quad \sim \quad \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j P_j.$$

Therefore, this implies that $P_j T = \tau_j P_j$ for all $j \in \mathbb{Z}$ (cf. [3], [8], [9], [16]).

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\mathbb{R}$ be the real line and let $\mathfrak{T} = \{T_t : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying

$$A = \sup\{\|T_t\|_{B[X]} : t \in \mathbb{R}\} < \infty$$

and having the expansions

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{\lambda_j t} P_j \qquad (t \in \mathbb{R}),$$

where $\{\lambda_j : j \in \mathbb{Z}\}$ is a sequence of scalars. Then $\mathfrak{T}$ becomes a strongly continuous group of operators in $B[X]$, and there holds

$$G(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \lambda_j P_j(f) \qquad (f \in D(G)),$$

where G is the infinitesimal generator of $\mathfrak{T}$ with domain $D(G)$ ([8; Proposition 2]).

For each $r \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^r = (T_t - I)^r = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} T_{kt} \qquad (r \geq 1),$$

which stands for the r-th iteration of $T_t - I$. Then Δ_t^r belongs to $M[X]$, and

$$\|\Delta_t^r\|_{B[X]} \leq A_r \qquad (t \in \mathbb{R}),$$

where

$$A_r = \min\{(A+1)^r, 2^r A\}$$

and there holds

$$\Delta_t^r \quad \sim \quad \sum_{j=-\infty}^{\infty} (e^{\lambda_j t} - 1)^r P_j \qquad (t \in \mathbb{R}).$$

If $r \in \mathbb{N}_0, f \in X$ and $\delta \geq 0$, then we define

$$\omega_r(X; f, \delta) = \sup\{\|\Delta_t^r(f)\|_X : |t| \leq \delta\},$$

which is called the r-th modulus of continuity of f. For the fundamental properties of $\omega_r(X; f, \delta)$, see [10; Lemma 1].

Let $\mathbb{N}$ be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, let M_n be the linear spann of the set $\{P_j(X) : |j| \leq n\}$, which is a closed linear subspace of X. For a given $f \in X$, we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. Then we have

$$E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0, \quad (1)$$

and Condition (P-2) implies that for every $f \in X$,

$$\lim_{n \to \infty} E_n(X; f) = 0. \qquad (2)$$

In [10], [11] and [12], we studied the relation between the rapidity of convergence (2) and certain smoothness properties of f in terms of its moduli of continuity $\omega_r(X; f, \delta)$. Such results are sometimes called direct theorems and inverse theorems of the best approximation theory (cf. [2], [4], [5], [7], [17]).

In this paper we consider the inverse problem under the hypothesis

$$e_r(X; f) = \sum_{n=1}^{\infty} n^{r-1} E_n(X; f) < \infty, \qquad (3)$$

where $r \in \mathbb{N}$ and $f \in X$. This condition (3) can be standard in the inverse theorems for the best approximation by trigonometric polynomials in the Banach space $C_{2\pi}$ of all 2π-periodic, continuous functions

f defined on $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \le \pi\}$$

and the Banach space $L^p_{2\pi}$ of all 2π-periodic, p-th power Lebesgue integrable functions f defined on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \, dt\right)^{1/p} \qquad (1 \le p < \infty)$$

(e.g, see [7], [15]).

2. Results

In this section, we assume that for each $f \in X$, there exists an element $f_n \in M_n$ of the best approximation of f with respect to M_n, i.e., such that

$$E_n(X; f) = \|f - f_n\|_X \qquad (n \in \mathbb{N}_0), \tag{4}$$

and that $\{\lambda_j : j \in \mathbb{Z}\} = \{-ij : j \in \mathbb{Z}\}$. For general problems concerning the existence of elements of the best approximation in normed linear spaces, see e.g., [14], and the literatures cited there.

For $r = 0, 1, 2, \cdots$, the operator G^r is inductively defined by the relations

$$G^0 = I, \quad G^1 = G,$$
$$D(G^r) = \{f : f \in D(G^{r-1}), G^{r-1}(f) \in D(G)\}$$

and

$$G^r(f) = G(G^{r-1}(f)) \qquad (f \in D(G^r), r = 1, 2, 3, \cdots).$$

Then for each $r \in \mathbb{N}$, $D(G^r)$ is a dense linear subspace of X and G^r is a closed linear operator with domain $D(G^r)$ (cf. [1; Propositions 1.1.4 and 1.1.6]). For further extensive list of properties of semigroups of operators from the viewpoint of approximation theory on Banach spaces, we refer to [1].

Note that $D(G^r)$ contains M_n and G^r maps M_n into itself ([11; Lemma 3]). Furthermore, the following Bernstein-type inequality holds ([11; Lemma 5]):

$$\|G^r(f)\|_X \le (2nB)^r \|f\|_X \qquad (r \in \mathbb{N}, n \in \mathbb{N}_0, f \in M_n), \tag{5}$$

where

$$B = \sup\{\|T_t\|_{B[X]} : |t| \le \pi\}.$$

From now on let $a \in \mathbb{N}$ and $a \geq 2$. We are now able to establish the following:

Theorem 1. *If $f \in X$ satisfies (3) for some $r \in \mathbb{N}$, then f belongs to $D(G^r)$ and for all $n \in \mathbb{N}$,*

$$E_n(X; G^r(f)) \leq B_r \sum_{k=[n/a]+1}^{\infty} k^{r-1} E_k(X; f), \tag{6}$$

where

$$B_r = B_r(a, B) = 2(2B)^r \frac{a^{2r}}{a-1}$$

and $[\lambda]$ denotes the largest integer not exceeding $\lambda > 0$.

Proof. Let f_n be an element of the best approximaion of f with respect to M_n. Then we have

$$f - f_n = \sum_{j=1}^{\infty} (f_{a^j n} - f_{a^{j-1} n}) \qquad (n \in \mathbb{N}) \tag{7}$$

because of (2) and (4). Put

$$\alpha_j = a^j n \quad (j \in \mathbb{N}_0), \quad \alpha_{-1} = \frac{n}{a}, \quad \varphi(n) = E_n(X; f) \quad (n \in \mathbb{N}).$$

Then it follows from (1), (4) and (5) that

$$\|G^r(f_{\alpha_j} - f_{\alpha_{j-1}})\|_X \leq (2a^j n B)^r \|f_{\alpha_j} - f_{\alpha_{j-1}}\|_X \leq 2(2B)^r (\alpha_j)^r \varphi(\alpha_{j-1}).$$

Therefore, by [12; Lemma 5] (note that this remains true for $\alpha_{-1} > 0$ and $\alpha_0 > [\alpha_{-1}]$), we have

$$\sum_{j=1}^{m+1} \|G^r(f_{\alpha_j} - f_{\alpha_{j-1}})\|_X \leq 2(2B)^r \frac{a^{2r}}{a-1} \sum_{k=[\alpha_{-1}]+1}^{\alpha_m} k^{r-1} \varphi(k),$$

and so letting m tend to ∞ implies

$$\sum_{j=1}^{\infty} \|G^r(f_{\alpha_j} - f_{\alpha_{j-1}})\|_X \leq 2(2B)^r \frac{a^{2r}}{a-1} \sum_{k=[\alpha_{-1}]+1}^{\infty} k^{r-1} \varphi(k) < \infty \tag{8}$$

by virtue of (3). Thus there exists an element $g \in X$ such that

$$g = \sum_{j=1}^{\infty} G^r(f_{\alpha_j} - f_{\alpha_{j-1}}). \tag{9}$$

Since G^r is a closed linear operator, it follows from (7) and (9) that $f - f_n \in D(G^r)$ and

$$G^r(f - f_n) = \sum_{j=1}^{\infty} G^r(f_{\alpha_j} - f_{\alpha_{j-1}}). \tag{10}$$

Hence, since $f_n \in M_n \subseteq D(G^r)$ and $G^r(f_n) \in M_n$, f belongs to $D(G^r)$ and (8) and (10) yields the desired estimate (6).

Remark. Let $f \in X$ and $r \in \mathbb{N}$. If there exists a sequence $\{a_n : n \in \mathbb{N}\}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} a_n < \infty, \quad n^{r-1} E_n(X; f) = O(a_n) \qquad (n \to \infty),$$

then Condition (3) is satisfied. If

$$\sum_{n=1}^{\infty} n^{r-1} \omega_s(X; f, 1/n) < \infty$$

for some $s \in \mathbb{N}$, then Condition (3) is also satisfied by virtue of [10; Theorem 2], which is a direct result of the Jackson type.

Theorem 2. *Let $\Omega \neq 0$ be a non-negative, monotone decreasing function on $[1, \infty)$ satisfying*

$$\int_1^{\infty} \frac{\Omega(x)}{x}\, dx < \infty, \tag{11}$$

$f \in X$ and $r \in \mathbb{N}$. If

$$E_n(X; f) \le C \frac{\Omega(n)}{n^r} \qquad \text{for all } n \ge 1, \tag{12}$$

where C is a positive constant independent of n, then f belongs to $D(G^r)$ and

$$E_n(X; G^r(f)) \le B_r C \int_{[n/a]}^{\infty} \frac{\Omega(x)}{x}\, dx \qquad (n > a). \tag{13}$$

Proof. By (11) and (12), we have

$$\sum_{n=m+1}^{\infty} n^{r-1} E_n(X; f) \le C \int_m^{\infty} \frac{\Omega(x)}{x}\, dx < \infty \qquad (m \in \mathbb{N}),$$

and so the claim of the theorem follows from Theorem 1.

Theorem 3. *Let $b \in \mathbb{N}, a > b$ and let Ω be as in Theorem 2, and suppose that $f \in X$ satisfies (12). Then f belongs to $D(G^r)$ and for every $s \in \mathbb{N}$,*

$$\omega_s(X; G^r(f), \delta) \le C_s \delta^s \Big\{ E_0(X; G^r(f)) + B_r C \Big(e_r(X; f) \sum_{n=1}^{a} n^{s-1}$$

$$+ \sum_{n=a+1}^{[b/\delta]} n^{s-1} \int_{[n/a]}^{\infty} \frac{\Omega(x)}{x}\, dx \Big) \Big\} \qquad (0 < \delta \le b/a), \tag{14}$$

where

$$C_s = C_s(a, b, A, B) = \max\left\{ \frac{A_s}{b^s}, 2^{s+1} A B^s \right\} \max\left\{ b^s, \frac{a^{2s}}{a-1} \right\}.$$

Proof. By [12; Theorem 1] and Theorem 1, we have

$$\omega_s(X; G^r(f), \delta) \le C_s \delta^s \Big\{ E_0(X; G^r(f)) + \sum_{n=1}^{[b/\delta]} n^{s-1} E_n(X; G^r(f)) \Big\}$$

$$\le C_s \delta^s \Big\{ E_0(X; G^r(f)) + B_r \sum_{n=1}^{[b/\delta]} n^{s-1} \Big(\sum_{k=[n/a]+1}^{\infty} k^{r-1} E_k(X; f) \Big) \Big\}. \tag{15}$$

Now, (3) and (12) give

$$\sum_{n=1}^{[b/\delta]} n^{s-1} \Big(\sum_{k=[n/a]+1}^{\infty} k^{r-1} E_k(X; f) \Big) \le C \Big\{ e_r(X; f) \sum_{n=1}^{a} n^{s-1}$$

$$+ \sum_{n=a+1}^{[b/\delta]} n^{s-1} \Big(\sum_{k=[n/a]+1}^{\infty} \frac{\Omega(k)}{k} \Big) \Big\}$$

$$\le C \Big(e_r(X; f) \sum_{n=1}^{a} n^{s-1} + \sum_{n=a+1}^{[b/\delta]} n^{s-1} \int_{[n/a]}^{\infty} \frac{\Omega(x)}{x}\, dx \Big),$$

which together with (15) yields the desired estimate (14).

Applying Theorems 2 and 3 to the special case where

$$\Omega(x) = \frac{1}{x^{\alpha}}, \quad \alpha > 0,$$

we have the following:

Corollary 1. *Let $\alpha > 0, f \in X$ and $r \in \mathbb{N}$. Assume that*

$$E_n(X; f) \le C\frac{1}{n^{\alpha+r}} \qquad \textit{for all } n \ge 1,$$

where C is a positive constant independent of n. Then f belongs to $D(G^r)$ and the following assertions hold:

(a) For all $n > a$,

$$E_n(X; G^r(f)) \le \frac{B_r C}{\alpha}\left(\frac{a}{n-a}\right)^{\alpha}.$$

(b) Let $b, s \in \mathbb{N}, a > b$ and $0 < \delta \le b/a$. If $s > \alpha$, then

$$\omega_s(X; G^r(f), \delta) \le C_s\delta^s\Big\{E_0(X; f) + B_r C\Big(e_r(X; f)\sum_{n=1}^{a} n^{s-1}$$

$$+ K_\alpha\Big(\big([b/\delta]^{s-\alpha} + 1\big) - a^{s-\alpha}\Big)\Big)\Big\},$$

where

$$K_\alpha = K_\alpha(a, s) = \frac{a^\alpha}{a(s-\alpha)}\sup\left\{\left(\frac{n}{n-a}\right)^{\alpha} : n \ge a+1\right\}.$$

If $s = \alpha$, then

$$\omega_s(X; G^r(f), \delta) \le C_s\delta^s\Big\{E_0(X; f) + B_r C\Big(e_r(X; f)\sum_{n=1}^{a} n^{s-1}$$

$$+ L_\alpha \log\left([b/\delta]/a\right)\Big)\Big\},$$

where

$$L_\alpha = L_\alpha(a) = \frac{a^\alpha}{a}\sup\left\{\left(\frac{n}{n-a}\right)^{\alpha} : n \ge a+1\right\}.$$

If $s < \alpha$, then

$$\omega_s(X; G^r(f), \delta) \le C_s\delta^s\Big\{E_0(X; f) + B_r C\Big(e_r(X; f)\sum_{n=1}^{a} n^{s-1}$$

$$+ K_\alpha\Big(a^{s-\alpha} - [b/\delta]^{s-\alpha}\Big)\Big)\Big\}.$$

Corollary 2. *Let $\alpha > 0, f \in X$ and $r \in \mathbb{N}$. Suppose that*

$$E_n(X; f) = O\left(\frac{1}{n^{\alpha+r}}\right) \qquad (n \to \infty).$$

Then f belongs to $D(G^r)$ and the following assertions hold:

(a)

$$E_n(X; G^r(f)) = O\left(\frac{1}{n^{\alpha}}\right) \qquad (n \to \infty).$$

(b) ([11; Corollary 1]) For every $s \in \mathbb{N}$,

$$\omega_s(X; G^r(f), \delta) = \begin{cases} O(\delta^{\alpha}) & (\alpha < s, \ \delta \to +0) \\ O(\delta^{s}|\log \delta|) & (\alpha = s, \ \delta \to +0) \\ O(\delta^{s}) & (\alpha > s, \ \delta \to +0). \end{cases}$$

Corollary 3. *Let $f \in X, r \in \mathbb{N}$ and $r < \alpha$. If*

$$E_n(X; f) = O\left(\frac{1}{n^{\alpha}}\right) \qquad (n \to \infty),$$

then f belongs to $D(G^r)$ and the following assertions hold:

(a)

$$E_n(X; G^r(f)) = O\left(\frac{1}{n^{\alpha-r}}\right) \qquad (n \to \infty).$$

(b) ([11; Corollary 2]) For every $s \in \mathbb{N}$,

$$\omega_s(X; G^r(f), \delta) = \begin{cases} O(\delta^{\alpha-r}) & (\alpha - r < s, \ \delta \to +0) \\ O(\delta^{s}|\log \delta|) & (\alpha - r = s, \ \delta \to +0) \\ O(\delta^{s}) & (\alpha - r > s, \ \delta \to +0). \end{cases}$$

3. Applications

In this section we apply the results obtained in the preceding section to homogeneous Banach spaces, which can be defined as follows (cf. [6], [8], [13]):

A linear subspace X of $L^1_{2\pi}$ is called a homogeneous Banach space if it satisfies the following conditions (H-1) - (H-4):

(H-1) X is a Banach space with a norm $\|\cdot\|_X$.

(H-2) X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $C > 0$ such that

$$\|f\|_1 \leq C\|f\|_X \qquad \text{for all } f \in X.$$

(H-3) The translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X)$$

is isometric on X for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

We now give several examples of homogeneous Banach spaces.

(1°) $C_{2\pi}$.

(2°) $L^p_{2\pi}, 1 \leq p < \infty$.

(3°) $C^{(n)}_{2\pi}$ = the linear subspace of $C_{2\pi}$ of all n-times continuously differentiable functions f with the norm

$$\|f\|_{C^{(n)}_{2\pi}} = \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!} \qquad (n \in \mathbb{N}).$$

(4°) $AC_{2\pi}$ = the linear subspace of $L^1_{2\pi}$ of all 2π-periodic, absolutely continuous functions f with the norm

$$\|f\|_{AC_{2\pi}} = \|f\|_1 + \|f'\|_1.$$

(5°) $0 < \alpha < 1, lip^\alpha_{2\pi}$ = the linear subspace of $C_{2\pi}$ of all functions f for which

$$q(f) = \sup\left\{\frac{|f(t+h) - f(t)|}{|h|^\alpha} : h \neq 0, t \in \mathbb{R}\right\} < \infty$$

and

$$\lim_{h\to 0}\left(\sup\left\{\frac{|f(t+h) - f(t)|}{|h|^\alpha} : t \in \mathbb{R}\right\}\right) = 0,$$

with the norm

$$\|f\|_{lip^\alpha_{2\pi}} = \|f\|_\infty + q(f).$$

(6°) $D(L)$ = the domain in $L^1_{2\pi}$ of a closed linear operator L with range in $L^1_{2\pi}$ such that for each $t \in \mathbb{R}$, T_t commutes with L, with the norm

$$\|f\|_{D(L)} = \|f\|_1 + \|L(f)\|_1.$$

Now, let X be a homogeneous Banach space with norm $\|\cdot\|_X$ and recall that $\mathfrak{T} = \{T_t : t \in \mathbb{R}\}$ be the family of translation operators on X. We define the sequence $\{P_j : j \in \mathbb{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f}(j)e^{ij\cdot} \qquad (f \in X),$$

where

$$\hat{f}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt \qquad (j \in \mathbb{Z})$$

denotes the j-th Fourier coefficient of f. Then $\{P_j : j \in \mathbb{Z}\}$ satisfies Conditions (P-1), (P-2)and (P-3) just as Section 1 (cf. [6], [8]), and

$$\Delta_t^r(f)(\cdot) \quad \sim \quad \sum_{j=-\infty}^{\infty} (e^{-ijt}-1)^r \hat{f}(j)e^{ij\cdot} \qquad (t \in \mathbb{R}, r \in \mathbb{N}_0, f \in X).$$

Also, there holds

$$\Delta_t^r(f)(\cdot) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(\cdot - jt) \qquad (t \in \mathbb{R}, r \in \mathbb{N}_0, f \in X).$$

Furthermore, M_n coincides with the set $\mathfrak{T}_n$ of all trigonometric polynomials of degree at most n, and so every $f \in X$ has always the trigonometric polynomial of best approximation of f with respect to $\mathfrak{T}_n, n \in \mathbb{N}_0$, since $\mathfrak{T}_n$ is the $2n+1$-dimensional linear subspace of X.

Consequently, all the results obtained in Section 2 hold under the above setting and in particular, Theorem 1 extends [7; Chapter 4, § 5,Theorem 8] to arbitrary homogeneous Banach spaces.

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