Korovkin type approximation closures for vector-valued functions

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KOROVKIN TYPE APPROXIMATION CLOSURES FOR VECTOR-VALUED FUNCTIONS

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ABSTRACT. Korovkin type approximation closures are discussed in the spaces of vector-valued functions on compact Hausdorff spaces. For this, it is introduced that the notions of an M_T-envelope, an M_T-affine function as well as an M_T-representing operator and a T-Choquet boundary for a linear subspace M and a positive linear operator T under certain appropriate requirements.

1. Introduction

Let X be a compact Hausdorff space and let E be a Dedekind complete normed vector lattice which contains an element e such that $e > 0, \|e\| = 1$ and $|a| \leq \|a\|e$ for all $a \in E$. We call e the normal order unit of E ([22]). Concerning the general notions and terminology needed from the theory of normed vector lattices, we refer to [25] (cf. [1], [13]). Let $B(X, E)$ denote the normed vector lattice of all E-valued bounded functions on X with the usual pointwise addition, scalar multiplication, ordering and the supremum norm $\|\cdot\|$. We shall use the same symbol $\|\cdot\|$ for underlying norms. $C(X, E)$ denotes the closed linear sublattice of $B(X, E)$ consisting of all E-valued continuous functions on X. In the case when E is equal to the real line $\mathbb{R}$, we simply write $B(X)$ and $C(X)$ instead of $B(X, E)$ and $C(X, E)$, respectively.

For any $v \in B(X)$ and $a \in E$, we define $(v \otimes a)(x) = v(x)a$ for all $x \in X$. Also, for any $v \in B(X)$ and $f \in B(X, E)$, we define

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$(vf)(x) = v(x)f(x)$ for all $x \in X$. Clearly, $v \otimes a$ and vf belong to $B(X,E)$, and $\|v \otimes a\| = \|v\|\|a\|$ and $\|vf\| \leq \|v\|\|f\|$. If $a \in E, v \in C(X)$ and $f \in C(X,E)$, then $v \otimes a$ and vf belong to $C(X,E)$. $C(X) \otimes E$ stands for the linear subspace of $C(X,E)$ consisting of all finite sums of functions of the form $v \otimes a$, where $v \in C(X)$ and $a \in E$. We define $\rho(x) = e$ for all $x \in X$. Notice that ρ is the normal order unit of $B(X,E)$. Let $A(X,E)$ be a linear sublattice of $C(X,E)$ which contains ρ.

The purpose of this paper is to show that our previous approach [22] to approximation theorems of Korovkin type can be adapted to the setting of an arbitrary positive linear operator T in the place of the identity operator I. Consequently, the results of [10] (cf. [3], [4], [5], [6], [7], [14], [15]) can be generalized to the context of functions taking a value in an optional Dedekind complete normed vector lattice with the normal order unit.

Concerning the background of the Korovkin type approximation theory, see the recent book of Altomare and Campiti [2], in which an excellent source and a vast literature of this theory can be found (cf. [9], [11]). Also, for the quantitative treatments on the degree of convergence of approximation processes with respect to positive multiplication operators, we refer to [23] and [24] (cf. [16], [17], [18], [19], [20], [21]).

2. M_T-envelopes and M_T-affine functions

Let M be a linear subspace of $A(X,E)$ which contains ρ and let T be a positive linear operator of $A(X,E)$ into $C(X,E)$. For a function $f \in A(X,E)$ and a point $x \in X$, we set

$$M_T^*(f,x) = \{T(h)(x) : f \leq h, h \in M\}$$

and

$$M_*^T(f,x) = \{T(h)(x) : h \leq f, h \in M\}.$$

Since $|f| \leq \|f\|\rho$, $\|f\|T(\rho)(x) \in M_T^*(f,x)$ and $-\|f\|T(\rho)(x) \in M_*^T(f,x)$ For a given $f \in A(X,E)$, we define

$$f_T^*(x) = \inf M_T^*(f,x) \qquad (x \in X)$$

and

$$f_*^T(x) = \sup M_*^T(f,x) \qquad (x \in X),$$

which are called the upper and lower M_T-envelope of f, respectively. These functions have the following properties, which may follow immediately from the definition:

Lemma 1. *Let $f, g \in A(X, E)$ and $\xi \in \mathbb{R}$. Then we have:*

(a) $\quad -\|f\|T(\rho) \leq f_*^T \leq T(f) \leq f_T^* \leq \|f\|T(\rho)$.

(b) If $f \leq g$, then $f_T^* \leq g_T^*$ and $f_*^T \leq g_*^T$.

(c) $\quad (f+g)_T^* \leq f_T^* + g_T^*, f_*^T + g_*^T \leq (f+g)_*^T$.

(d) If $\xi \geq 0$, then $(\xi f)_T^* = \xi f_T^*$ and $(\xi f)_*^T = \xi f_*^T$.

(e) If $\xi \leq 0$, then $(\xi f)_T^* = \xi f_*^T$. In particular, $(-f)_T^* = -f_*^T$.

A function $f \in A(X, E)$ is said to be M_T-affine if f_T^* is equal to f_*^T. Obviously, f is M_T-affine if and only if $f_*^T = T(f) = f_T^*$. For a given $x \in X$, we define

$$\hat{M}_T(x) = \{f \in A(X, E) : f_*^T(x) = f_T^*(x)\},$$

which is a linear subspace of $A(X, E)$ containing M. Also, we set

$$\hat{M}_T = \bigcap_{x \in X} \hat{M}_T(x),$$

which is a linear subspace of $A(X, E)$ containing M. Clearly, we have

$$\hat{M}_T = \{f \in A(X, E) : f_*^T = f_T^*\} = \{f \in A(X, E) : f_*^T = T(f) = f_T^*\}.$$

Lemma 2. *Let $f \in \hat{M}_T$. Then for any $\epsilon > 0$, there exist finite subsets $\{g_1, g_2, \cdots, g_m\}$ and $\{h_1, h_2, \cdots, h_m\}$ of M such that*

$$g_T^\vee \leq T(f) \leq h_T^\wedge, \quad h_T^\wedge - g_T^\vee \leq \epsilon\rho, \tag{1}$$

where

$$g_T^\vee = \sup\{T(g_1), T(g_2), \cdots, T(g_m)\}$$

and

$$h_T^\wedge = \inf\{T(h_1), T(h_2), \cdots, T(h_m)\}.$$

Proof. Since $f_*^T(x) = T(f)(x) = f_T^*(x)$ for each $x \in X$, there exist functions $g_x, h_x \in M$ such that

$$g_x \leq f \leq h_x, \quad T(h_x)(x) - T(g_x)(x) < \frac{\epsilon}{2}e.$$

Therefore, there exists an open neighborhood V_x of x such that

$$T(h_x)(t) - T(g_x)(t) < \epsilon e \quad \text{for all } t \in V_x.$$

Since the family $\{V_x : x \in X\}$ is an open covering of X, it has a subcovering $\{V_{x_i} : i = 1, 2, \cdots, m\}$. Then the functions

$$g_i = g_{x_i}, \quad h_i = h_{x_i} \qquad (i = 1, 2, \cdots, m)$$

have the desired properties.

Let A and B be normed vector lattices, and let L be a mapping of A into B. Then L is said to be increasing if $f, g \in A$ and $f \leq g$ imply $L(f) \leq L(g)$. Evidently, if L is linear, then L is increasing if and only if L is positive. Furthermore, if A has a normal order unit a and if L is a positive linear operator of A into B, then L is bounded and $\|L\| = \|L(a)\|$.

From now on let D be a directed set and let Λ be an index set.

Proposition 1. *Let $x \in X$. If $\{\mu_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of increasing mappings of $A(X, E)$ into E satisfying*

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(k) - T(k)(x)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{2}$$

for all $k \in M$, then

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(f) - T(f)(x)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{3}$$

for all $f \in \hat{M}_T(x)$.

Proof. Let g and h be functions in M such that $g \leq f \leq h$. Then for every $\epsilon > 0$, by (2) theres exists an element $\alpha_0 \in D$ such that

$$\|\mu_{\alpha,\lambda}(g) - T(g)(x)\| < \epsilon \quad \text{and} \quad \|\mu_{\alpha,\lambda}(h) - T(h)(x)\| < \epsilon$$

for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$. Since

$$|\mu_{\alpha,\lambda}(u) - T(u)(x)| \leq \|\mu_{\alpha,\lambda}(u) - T(u)(x)\| e \qquad (\alpha \in D, \lambda \in \Lambda)$$

whenever u belongs to $A(X, E)$ and

$$\mu_{\alpha,\lambda}(g) \leq \mu_{\alpha,\lambda}(f) \leq \mu_{\alpha,\lambda}(h) \qquad (\alpha \in D, \lambda \in \Lambda),$$

we conclude that for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$,

$$T(g)(x) - \epsilon e < \mu_{\alpha,\lambda}(f) < T(h)(x) + \epsilon e,$$

which yields

$$f_*^T(x) - \epsilon e \leq \mu_{\alpha,\lambda}(f) \leq f_T^*(x) + \epsilon e.$$

Therefore, we have

$$|\mu_{\alpha,\lambda}(f) - T(f)(x)| \leq \epsilon e \qquad (\alpha \in D, \alpha \geq \alpha_0, \lambda \in \Lambda)$$

because of $f_*^T(x) = T(f)(x) = f_T^*(x)$. Thus we obtain

$$\|\mu_{\alpha,\lambda}(f) - T(f)(x)\| \le \epsilon \qquad (\alpha \in D, \alpha \ge \alpha_0, \lambda \in \Lambda),$$

which implies (3).

Corollary 1. *Let $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of increasing mappings of $A(X,E)$ into $B(X,E)$. Then the following statements hold:*

(a) Let $x \in X$. If

$$\lim_\alpha \|T_{\alpha,\lambda}(g)(x) - T(g)(x)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda$$

for all $g \in M$, then

$$\lim_\alpha \|T_{\alpha,\lambda}(f)(x) - T(f)(x)\| = 0 \qquad \textit{unifomly in } \lambda \in \Lambda$$

for all $f \in \hat{M}_T(x)$.

(b) If

$$\lim_\alpha \|T_{\alpha,\lambda}(g) - T(g)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda$$

for all $g \in M$, then

$$\lim_\alpha \|T_{\alpha,\lambda}(f)(x) - T(f)(x)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda$$

for all $x \in X$ and $f \in \hat{M}_T$.

Proposition 2. *If $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of increasing mappings of $A(X,E)$ into $B(X,E)$ satisfying*

$$\lim_\alpha \|T_{\alpha,\lambda}(g) - T(g)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{4}$$

for all $g \in M$, then

$$\lim_\alpha \|T_{\alpha,\lambda}(f) - T(f)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{5}$$

for all $f \in \hat{M}_T$.

Proof. For any $\epsilon > 0$, by Lemma 2 there exist finite subsets $\{g_1, g_2, \cdots, g_m\}$ and $\{h_1, h_2, \cdots, h_m\}$ of M satisfying (1). By (4), there exists an element $\alpha_0 \in D$ such that

$$\|T_{\alpha,\lambda}(g_i) - T(g_i)\| < \epsilon,\ \|T_{\alpha,\lambda}(h_i) - T(h_i)\| < \epsilon \quad (i = 1, 2, \cdots, m) \tag{6}$$

for all $\alpha \in D, \alpha \ge \alpha_0$ and all $\lambda \in \Lambda$. Since for all $x \in X$

$$|T_{\alpha,\lambda}(k)(x) - T(k)(x)| \le \|T_{\alpha,\lambda}(k)(x) - T(k)(x)\|e \qquad (\alpha \in D, \lambda \in \Lambda)$$

whenever k belongs to $A(X,E)$ and

$$T_{\alpha,\lambda}(g_i) \leq T_{\alpha,\lambda}(f) \leq T_{\alpha,\lambda}(h_i) \qquad (i=1,2,\cdots,m)$$

for all $\alpha \in D$ and all $\lambda \in \Lambda$, by (6) we conclude that

$$T(g_i)(x) - \epsilon e < T_{\alpha,\lambda}(f)(x) < T(h_i)(x) + \epsilon e \qquad (i=1,2,\cdots,m)$$

for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$. Therefore, for all $x \in X$ we obtain

$$g_T^{\vee}(x) - \epsilon e \leq T_{\alpha,\lambda}(f)(x) \leq h_T^{\wedge}(x) + \epsilon e \qquad (\alpha \in D, \alpha \geq \alpha_0, \lambda \in \Lambda),$$

which together with (1) gives

$$|T_{\alpha,\lambda}(f)(x) - T(f)(x)| < 3\epsilon e \qquad (\alpha \in D, \alpha \geq \alpha_0, \lambda \in \Lambda),$$

and so

$$\|T_{\alpha,\lambda}(f)(x) - T(f)(x)\| \leq 3\epsilon \qquad (\alpha \in D, \alpha \geq \alpha_0, \lambda \in \Lambda),$$

which implies

$$\|T_{\alpha,\lambda}(f) - T(f)\| \leq 3\epsilon \qquad (\alpha \in D, \alpha \geq \alpha_0, \lambda \in \Lambda).$$

Hence, (5) remains true.

Remark 1. *If $g \otimes e \in \hat{M}_T$ and $T(g \otimes e) = g \otimes e$ for every $g \in C(X)$, then $T(M)$ separates the points of X.*

3. M_T-representing operators and T-Choquet boundaries

For a given $x \in X$, a positive linear operator μ of $A(X,E)$ into E is called an M_T-representing operator for x if $\mu(g) = T(g)(x)$ for all $g \in M$. For each $x \in X$, we define $\delta_x(f) = f(x)$ for every $f \in B(X,E)$. The operator δ_x is called the evaluation operator at x. Clearly, we have $\delta_x(\rho) = e$ and $\|\delta_x\| = 1$. Also, $\delta_x \circ T$ is always the M_T-representing operator for x. Let $\mathcal{R}_x^T(M)$ denote the set of all M_T-representing operators for x.

For $x \in X$ and $f \in A(X,E)$, we denote by $[f_*^T(x), f_T^*(x)]$ the order interval in E, i.e.,

$$[f_*^T(x), f_T^*(x)] = \{a \in E : f_*^T(x) \leq a \leq f_T^*(x)\}.$$

Then we have the following result, which gives the close connection between the M_T-envelpoes and M_T-representing operators.

Lemma 3. *Let $x \in X$ and let $f \in A(X, E)$. Then we have*

$$[f_*^T(x), f_T^*(x)] = \{\mu(f) : \mu \in \mathcal{R}_x^T(M)\}.$$

Proof. Let $\mu \in \mathcal{R}_x^T(M)$, and let g and h be functions in M such that $g \leq f \leq h$. Then we have

$$T(g)(x) = \mu(g) \leq \mu(f) \leq \mu(h) = T(h)(x),$$

which establishes

$$f_*^T(x) \leq \mu(f) \leq f_T^*(x).$$

Conversely, let a be an arbitrary element in $[f_*^T(x), f_T^*(x)]$, and let V be a linear subspace of $A(X, E)$ spanned by f. We define

$$p(g) = g_T^*(x) \qquad \text{for every } g \in A(X, E)$$

and

$$\mu_0(\xi f) = \xi a \qquad \text{for every } \xi \in \mathbb{R}.$$

Then, by Lemma 1, the mapping $p : A(X, E) \to E$ is sublinear and μ_0 is a linear operator of V into E satisfying $\mu_0(g) \leq p(g)$ for all $g \in V$. Therefore, by the vector-valued Hahn-Banach theorem ([1; Theorem 2.1], cf. [13; Theorem 1.5.4]), there exists a linear operator μ of $A(X, E)$ into E such that $\mu(g) = \mu_0(g)$ for all $g \in V$ and $\mu(h) \leq p(h)$ for all $h \in A(X, E)$. If $h \in A(X, E)$ and $h \leq 0$, then Lemma 1 (b) gives

$$\mu(h) \leq p(h) = h_T^*(x) \leq 0_T^*(x) = 0,$$

which implies that μ is positive. Furthermore, for every $g \in M$ we have

$$\mu(g) \leq p(g) = g_T^*(x) = g_*^T(x) = T(g)(x)$$

and

$$-\mu(g) = \mu(-g) \leq p(-g) = (-g)_T^*(x) = -T(g)(x),$$

and so $\mu(g) = T(g)(x)$. Thus μ belongs to $\mathcal{R}_x^T(M)$ and $\mu(f) = \mu_0(f) = a$. The proof of the lemma is now complete.

As an immediate consequence of Lemma 3, we have the following.

Lemma 4. *Let $f \in A(X, E)$. Then the following assertions hold:*

(a) Let $x \in X$. Then f belongs to $\hat{M}_T(x)$ if and only if $\mu(f) = (\delta_x \circ T)(f)$ for all $\mu \in \mathcal{R}_x^T(M)$.

(b) f belongs to $\hat{M}_T$ if and only if $\mu(f) = (\delta_x \circ T)(f)$ for all $x \in X$ and all $\mu \in \mathcal{R}_x^T(M)$.

We define

$$\partial_M^T(X) = \{x \in X : \mathcal{R}_x^T(M) = \{\delta_x \circ T\}\},$$

which is called the T-Choquet boundary of X with respect to M. This can be characterized by Lemma 4 (a) as follows:

Lemma 5. *A point $x \in X$ belongs to $\partial_M^T(X)$ if and only if $f_*^T(x) = f_T^*(x)$ for all $f \in A(X,E)$, i.e., $\hat{M}_T(x) = A(X,E)$.*

According to Lemma 4 (b) and Lemma 5, we have the following.

Proposition 3. *$\hat{M}_T = A(X,E)$ if and only if $\partial_M^T(X) = X$.*

4. T-Korovkin closures and T-Korovkin spaces

Let A and B be normed linear spaces, and let $\mathfrak{L}$ be a class of mappings of A into B. Let S be a subset of A and let $L \in \mathfrak{L}$. Then we define $Kor(\mathfrak{L}; S, L)$ to be the set of all $f \in A$ with the property that if $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an arbitrary family of mappings in $\mathfrak{L}$ satisfying

$$\lim_{\alpha} \|L_{\alpha,\lambda}(g) - L(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $g \in S$, then

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f) - L(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

We call $Kor(\mathfrak{L}; S, L)$ an L-Korovkin closure of S with respect to $\mathfrak{L}$. Also, S is said to be an L-Korovkin set with respect to $\mathfrak{L}$ if $Kor(\mathfrak{L}; S, L)$ is identical with A. In this event, if S is a linear subspace of A, then we shall say that S is an L-Korovkin space with respect to $\mathfrak{L}$.

If A and B are normed vector lattice, then $\mathfrak{I}[A,B]$ and $\mathfrak{P}[A,B]$ denote the classes of all increasing mappings of A into B and positive linear operators of A into B, respectively.

Theorem 1. *Let $x \in X$. Then the following statements are equivalent:*

(a) x belongs to $\partial_M^T(X)$.
(b) M is a $\delta_x \circ T$- Korovkin space with respect to $\mathfrak{I}[A(X,E),E]$.
(c) M is a $\delta_x \circ T$-Korovkin space with respect to $\mathfrak{P}[A(X,E),E]$.

Proof. If $x \in \partial_M^T(X)$, then we have

$$Kor(\mathfrak{I}[A(X,E),E]; M, \delta_x \circ T) = A(X,E)$$

because of Lemma 5 and Proposition 1. Therefore, (a) implies (b). It is obvious that (b) implies (c). Now, suppose that

$$Kor(\mathfrak{P}[A(X,E),E]; M, \delta_x \circ T) = A(X,E)$$

and let μ be an arbitrary element in $\mathcal{R}_x^T(M)$. Then for all $\alpha \in D$ and all $\lambda \in \Lambda$, we define $\mu_{\alpha,\lambda} = \mu$, which is an operator in $\mathfrak{P}[A(X,E),E]$ satisfying

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(g) - (\delta_x \circ T)(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $g \in M$. Therefore for every $f \in A(X,E)$, we have

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(f) - (\delta_x \circ T)(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

which yields $\mu = \delta_x \circ T$, and so $x \in \partial_M^T(X)$. Thus (c) implies (a).

Theorem 2. *In the following assertions, the implications* (*a*) $\Rightarrow$ (*b*) $\Rightarrow$ (*c*) *hold:*

> (*a*) $\partial_M^T(X)$ *is identical with* X.
> (*b*) M *is a* T*-Korovkin space with respect to* $\mathfrak{I}[A(X,E), B(X,E)]$.
> (*c*) M *is a* T*-Korovkin space with respect to* $\mathfrak{P}[A(X,E), B(X,E)]$.

Proof. If $\partial_M^T(X) = X$, then it follows from Propositions 2 and 3 that

$$A(X,E) = \hat{M}_T \subseteq Kor(\mathfrak{I}[A(X,E), B(X,E)]; M, T) \subseteq A(X,E),$$

and so we have

$$Kor(\mathfrak{I}[A(X,E), B(X,E)]; M, T) = A(X,E).$$

Thus (a) implies (b). It is clear that (b) implies (c).

In order to show that the implication (c) $\Rightarrow$ (a) in the statements of Theorem 2, we assume that X is a first countable, compact Hausdorff space and that D is the set $\mathbb{N}$ of all natural numbers in the remaining part of this section.

Proposition 4. *We have*

$$\hat{M}_T = Kor(\mathfrak{I}[A(X,E), B(X,E)]; M, T) = Kor(\mathfrak{P}[A(X,E), B(X,E)]; M, T).$$

Proof. It follows from Proposition 2 and Theorem 2 that

$$\hat{M}_T \subseteq Kor(\mathfrak{I}[A(X,E),B(X,E)];M,T) \subseteq Kor(\mathfrak{P}[A(X,E),B(X,E)];M,T).$$

Now, let f be an arbitrary function in $Kor(\mathfrak{P}[A(X,E),B(X,E)];M,T)$. Let $x \in X$ and $\mu \in \mathcal{R}_x^T(M)$. Since X satisfies the first countability axiom, there is a fundamental system $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of x such that

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq V_{n+1} \supseteq \cdots .$$

For each $n \in \mathbb{N}$, by Urysohn's lemma there exists a function $u_n \in C(X)$ such that

$$0 \le u_n(t) \le 1 \quad (t \in X), \quad u_n(x) = 1, \quad u_n(t) = 0 \quad (t \in X \setminus V_n).$$

Now, for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$ we define

$$T_{n,\lambda}(g) = u_n \otimes \mu(g) + (1_X - u_n)T(g) \qquad (g \in A(X,E)),$$

where

$$1_X(t) = 1 \qquad \text{for every } t \in X. \tag{7}$$

Then $\{T_{n,\lambda} : n \in \mathbb{N}, \lambda \in \Lambda\}$ is a family of operators in $\mathfrak{P}[A(X,E),C(X,E)]$ satisfying

$$\lim_{n\to\infty} \|T_{n,\lambda}(h) - T(h)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $h \in M$. Therefore we have

$$\lim_{n\to\infty} \|T_{n,\lambda}(f) - T(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

and so

$$\lim_{n\to\infty} \|T_{n,\lambda}(f)(x) - T(f)(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

This yields $\mu(f) = (\delta_x \circ T)(f)$, since $T_{n,\lambda}(f)(x) = \mu(f)$ for all $n \in \mathbb{N}$ and all $\lambda \in \Lambda$. Hence, by Lemma 4 (b), f belongs to $\hat{M}_T$, and so we have

$$\hat{M}_T \supseteq Kor(\mathfrak{P}[A(X,E),B(X,E)];M,T).$$

The proof of the proposition is now complete.

Theorem 3. *The following assertions are equivalent:*

(a) $\partial_M^T(X) = X$.
(b) $Kor(\mathfrak{I}[A(X,E),B(X,E)];M,T) = A(X,E)$.
(c) $Kor(\mathfrak{P}[A(X,E),B(X,E)];M,T) = A(X,E)$.
(d) $\hat{M}_T = A(X,E)$.

Proof. This immediately follows from Theorem 2, Propositions 3 and 4.

Now, we are interested in that T is a finitely defined operator of order n, which is defined as follows (cf. [8], [14], [15]):

Let $\{v_1, v_2, \cdots, v_n\}$ be a finite set of non-negative functions in $C(X)$ and let $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ be a finite set of continuous mappings of X into itself. Then we define

$$T(f) = \sum_{i=1}^{n} v_i \cdot (f \circ \varphi_i) \qquad \text{for every } f \in A(X, E).$$

Consequently, all the results can be applicable to this setting.

From now on we restrict ourselves to the case where $n = 1, v_1 = 1_X$ and $\varphi_1 = \varphi$, where $\varphi : X \to X$ is an arbitrary continuous mapping. We set $T_\varphi = T$. That is, this operator is given by

$$T_\varphi(f) = f \circ \varphi \qquad \text{for every } f \in A(X, E). \tag{8}$$

Remark 2. *We set*

$$f^* = f^*_I \quad \textit{and} \quad f_* = f^I_* \qquad (f \in A(X, E)),$$

which are called the upper and lower M-envelope of f, respetively and

$$\partial_M(X) = \partial^I_M(X),$$

which is called the Choquet boundary (cf. [22]). Let $x \in X$ and $f \in A(X, E)$. Then it follows immediately from the definition that

$$f^*_{T_\varphi}(x) = f^*(\varphi(x)), \quad f^{T_\varphi}_*(x) = f_*(\varphi(x)).$$

Thus, by Lemma 5, we have

$$x \in \partial^{T_\varphi}_M(X) \iff \varphi(x) \in \partial_M(X). \tag{9}$$

As a consequence of Theorem 3 and the above equivalence (9), we have the following corollary which can be more convenient for later applications.

Corollary 2. *Let T_φ be as in (8). If $\partial_M(X)$ coincides with X, then M is both a T_φ-Korovkin space with respect to $\mathfrak{I}[A(X, E), B(X, E)]$ and a T_φ-Korovkin space with respect to $\mathfrak{P}[A(X, E), B(X, E)]$.*

Remark 3. *The following assertions hold:*

(a) If M is an I-Korovkin space with respect to $\mathfrak{I}[A(X, E), B(X, E)]$, then it is a T_φ-Korovkin space with respect to $\mathfrak{I}[A(X, E), B(X, E)]$.

(b) If M is an I-Korovkin space with respect to $\mathfrak{P}[A(X,E),B(X,E)]$, then it is a T_φ-Korovkin space with respect to $\mathfrak{P}[A(X,E),B(X,E)]$.

5. T_φ-Korovkin sets and T_φ- Korovkin spaces

Recall that 1_X is the normal order unit of $C(X)$ defined by (7), and let T_φ be as in (8). Here we consider the case of $A(X,E) = C(X,E)$, and let M be a linear subspace of $C(X,E)$ which contains $1_X \otimes a$ for all $a \in E$. We set

$$\mathfrak{I} = \mathfrak{I}[C(X,E),B(X,E)]$$

and

$$\mathfrak{P} = \mathfrak{P}[C(X,E),B(X,E)].$$

Theorem 4. *If for each point $x \in X$, there exists a function $h_x \in C(X)$ such that*

$$h_x \geq 0,\ h_x(x) = 0 \text{ and } h_x(t) > 0 \text{ for all } t \in X \text{ with } t \neq x \tag{10}$$

and

$$h_x \otimes a \in M \qquad \text{for every } a \in E, \tag{11}$$

then M is both a T_φ-Korovkin space with respect to $\mathfrak{I}$ and a T_φ-Korovkin space with respect to $\mathfrak{P}$.

Proof. This follows immediately from [22;Lemma 7] and Corollary 2.

For a given subset S of $C(X)$, we define

$$S \otimes E = \{v \otimes a : v \in S, a \in E\}$$

and let span$(S \otimes E)$ denote the linear subspace of $C(X,E)$ spanned by $S \otimes E$. Notice that $S \otimes E$ is a T_φ-Korovkin set with respect to $\mathfrak{P}$ if and only if span$(S \otimes E)$ is a T_φ-Korovkin space with respect to $\mathfrak{P}$.

Corollary 3. *Let $\{u_1, u_2, \cdots, u_m\}$ be a finite subset of $C(X)$ and let*

$$U = \{1_X, u_1, u_2, \cdots, u_m\} \otimes E.$$

Suppose that for each point $x \in X$, there exists a finite subset $\{a_1(x), a_2(x), \cdots, a_m(x)\}$ of $\mathbb{R}$ such that the function

$$h_x = \sum_{i=1}^{m} a_i(x) u_i$$

satisfies (10) and (11). Then span(U) is a T_φ-Korovkin space with respect to $\mathfrak{I}$ and U is a T_φ-Korovkin set with respect to $\mathfrak{P}$.

From now on let p be an arbitrary fixed even positive integer.

Corollary 4. *Let $\{v_1, v_2, \cdots, v_n\}$ of $C(X)$ which separates the points of X and let*

$$V = \left\{1_X, v_1, \cdots, v_n, v_1^2, \cdots, v_n^2, \cdots, v_1^{p-1}, \cdots, v_n^{p-1}, \sum_{i=1}^{n} v_i^p\right\} \bigotimes E.$$

Then span(V) is a T_φ-Korovkin space with respect to $\mathfrak{J}$ and V is a T_φ-Korovkin set with respect to $\mathfrak{P}$.

Indeed, with the help of the function

$$h_x = \sum_{i=1}^{n} (v_i - v_i(x))^p \qquad (x \in X),$$

this follows from Corollary 3.

Theorem 5. *Let G be a subset of $C(X)$ separating the points of X and let*

$$W = \{g^i : g \in G, i = 0, 1, 2, \cdots, p\} \otimes E,$$

where $g^0 = 1_X$. Then span(W) is a T_φ-Korovkin space with respect to $\mathfrak{J}$ and W is a T_φ-Korovkin set with respect to $\mathfrak{P}$.

Proof. It was shown in the proof of [22; Theorem 5] that

$$\partial_{span(W)}(X) = X.$$

Therefore, the desired result follows from Corollary 2.

Corollary 5. *Let X be a compact subset of a real locally convex Hausdorff vector space F with its dual space F^*, and let*

$$H = \{(h|_X)^i : h \in F^*, i = 0, 1, 2, \cdots, p\} \otimes E,$$

where $h|_X$ denotes the restriction of h to X. Then span(H) is a T_φ-Korovkin space with respect to $\mathfrak{J}$ and H is a T_φ-Korovkin set with respect to $\mathfrak{P}$.

Applying Corollary 4 and Theorem 5 to the case of $p = 2$, we shall now mention some examples of T_φ-Korovkin sets with respect to $\mathfrak{P}$ and T_φ-Korovkin spaces with respect to $\mathfrak{J}$, which include the classical cases for $T_\varphi = I$, thus, $\varphi : X \to X$ is the identity mapping (cf. [12], [22]).

(1°) Let X be a compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$. For each $k = 1, 2, \cdots, n$, p_k denotes the k-th coordinate function defined by

$$p_k(x) = x_k \qquad \text{for every } x = (x_1, x_2, \cdots, x_n) \in X.$$

Then

$$K_1 = \left\{1_X, p_1, p_2, \cdots, p_n, \sum_{k=1}^{n} p_k^2\right\} \bigotimes E$$

and

$$K_2 = \{1_X, p_1, p_2, \cdots, p_n, p_1^2, p_2^2, \cdots, p_n^2\} \otimes E$$

are $T\varphi$-Korovkin sets with respect to $\mathfrak{P}$. Also, span(K_1) and span(K_2) are T_φ-Korovkin spaces with respect to $\mathfrak{I}$.

(2°) Let X be a compact subset of the n-dimensional unitary space $\mathbb{C}^n$, where $\mathbb{C}$ denotes the complex plane. For each $k = 1, 2, \cdots, n$, we define

$$q_k(z) = \mathrm{Re}(z_k) \quad \text{and} \quad r_k(z) = \mathrm{Im}(z_k)$$

for every $z = (z_1, z_2, \cdots, z_n) \in X$, where $\mathrm{Re}(z_k)$ and $\mathrm{Im}(z_k)$ stand for the real part and the imaginary part of z_k, respectively. Then

$$K_3 = \left\{1_X, q_1, q_2, \cdots, q_n, r_1, r_2, \cdots, r_n, \sum_{k=1}^{n} (q_k^2 + r_k^2)\right\} \bigotimes E$$

and

$$K_4 = \{1_X, q_1, \cdots, q_n, r_1, \cdots, r_n, q_1^2, \cdots, q_n^2, r_1^2, \cdots, r_n^2\} \otimes E$$

are T_φ-Korovkin sets with respect to $\mathfrak{P}$. Also, span(K_3) and span(K_4) are T_φ-Korovkin spaces with respect to $\mathfrak{I}$.

(3°) Let X be the n-dimensional torus $\mathbb{T}^n$, i.e,

$$\mathbb{T}^n = \{z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : |z_k| = 1, k = 1, 2, \cdots, n\},$$

and q_k and r_k $(k = 1, 2, \cdots, n)$ be as in the example (2°). Then

$$K_5 = \{1_X, q_1, q_2, \cdots, q_n, r_1, r_2, \cdots, r_n\} \otimes E$$

is a T_φ-Korovkin set with respect to $\mathfrak{P}$ and span(K_5) is a T_φ-Korovkin space with respect to $\mathfrak{I}$.

(4°) Let $C_{2\pi}(\mathbb{R}^n, E)$ denote the normed vector lattice of all E-valued continuous functions f on $\mathbb{R}^n$ which are periodic with period 2π in each variable with the norm

$$\|f\| = \sup\{\|f(x)\| : x \in \mathbb{R}^n\}.$$

Then $C(\mathbb{T}^n, E)$ is isometrically isomorphic to $C_{2\pi}(\mathbb{R}^n, E)$. For each $k = 1, 2, \cdots, n$, we define

$$c_k(x) = \cos x_k \quad \text{and} \quad s_k(x) = \sin x_k$$

for every $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then

$$K_6 = \{1_{\mathbb{R}^n}, c_1, c_2, \cdots, c_n, s_1, s_2, \cdots, s_n\} \otimes E$$

is a T_φ-Korovkin set with respect to $\mathfrak{P}$ and $\mathrm{span}(K_6)$ is a T_φ-Korovkin space with respec to $\mathfrak{I}$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping.

Remark 4. *Let A and B be normed linear spaces, and let $\mathfrak{L}$ be a set of mappings of A into B. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathfrak{A} = \{a_{\alpha,n}^{(\lambda)} : \alpha \in D, \lambda \in \Lambda, n \in \mathbb{N}_0\}$ be a family of scalars. Let $\{L_n : n \in \mathbb{N}_0\}$ be a sequence of mappings in $\mathfrak{L}$, $L \in \mathfrak{L}$ and $f \in A$. Then the sequence $\{L_n(f) : n \in \mathbb{N}_0\}$ is said to be $\mathfrak{A}$-summable to $L(f)$ with respect to $\mathfrak{L}$ if*

$$\lim_{\alpha} \| \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} L_n(f) - L(f) \| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, \tag{12}$$

where it is supposed that the series in (12) converges for each $\alpha \in D$ and $\lambda \in \Lambda$.

As the examples mentioned in [19] (cf. [17], [18]) show, there are a wide variety of families $\mathfrak{A} = \{a_{\alpha,n}^{(\lambda)} : \alpha \in D, \lambda \in \Lambda, n \in \mathbb{N}_0\}$ of particular interest which cover several summability methods scatered in the literatures. Consequently, all the results obtained in this paper can be applied to the $\mathfrak{A}$-summability in the sense of (12).

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