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THE ORDER OF CONVERGENCE FOR POSITIVE APPROXIMATION PROCESSES *

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Abstract. Quantitative estimates for approximation processes of positive linear operators are derived by using a modulus of continuity and by taking higher order absolute moments with respect to test systems under suitable assumptions. Furthermore, several applications are also provided.

1. Introduction

Let X be a compact Hausdorff space and let $B(X)$ denote the Banach lattice of all real-valued bounded functions on X with the supremum norm $\|\cdot\|$. $C(X)$ denotes the closed sublattice of $B(X)$ consisting of all real-valued continuous functions on X. Let $A(X)$ be a linear subspace of $C(X)$ which contains the unit function defined by $1_X(y) = 1$ for all $y \in X$. Let $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of bounded linear operators of $A(X)$ into $B(X)$, where D is a directed set and Λ is an index set, and let L be a bounded linear operator of $A(X)$ into $B(X)$. Then the family $\{L_{\alpha,\lambda}\}$ is called an approximation process with respect to L on $A(X)$ if for every $f \in A(X)$,

$$(1) \qquad \lim_{\alpha} \|L_{\alpha,\lambda}(f) - L(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

In particular, if $\{L_{\alpha,\lambda}\}$ is an approximation process with respect to the identity operator I on $A(X)$, then we simply say that it is an approximation process on $A(X)$ (cf. [47], [49], [55], [59]).

Let p be a positive real number and let G be a subset of $A(X)$ separating the points of X. Suppose that $A(X)$ contains the set

$$G_p = \{|g - g(y)1_X|^p : g \in G, y \in X\}.$$

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For a function $g \in G$, we define

$$\mu^{(p)}(L; g)(y) = L(|g - g(y)1_X|^p)(y) \qquad (y \in X),$$

whose norm is called the p-th absolute moment for L with respect to g.

Let $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of positive linear operators of $A(X)$ into $B(X)$ and put

$$\mu^{(p)}_{\alpha,\lambda}(g) = \mu^{(p)}(L_{\alpha,\lambda}; g) \qquad (\alpha \in D, \lambda \in \Lambda, g \in G).$$

In [54] we observed that usual convergence of nets of positive linear operators of $A(X)$ into $B(X)$ is valid for the convergence behavior in the sense of (1), where L can be taken to be a positive multiplication operator or a positive projection operator on $A(X)$. That is, we have the following results, which establish a generalized Korovkin-type approximation theorem (cf. [9], [18], [22], [31], [44], [63]):

THEOREM A. *Let U be a multiplication operator given by*

$$U(f) = hf \qquad (f \in A(X)), \tag{2}$$

where h is an arbitrary fixed non-negative function in $B(X)$. If for every $g \in G$,

$$\lim_{\alpha} \|\mu^{(p)}_{\alpha,\lambda}(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

and if there exists a strictly positive function $u \in A(X)$ such that

$$\lim_{\alpha} \|L_{\alpha,\lambda}(u) - U(u)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

then $\{L_{\alpha,\lambda}\}$ is an approximation process with respect to U on $A(X)$.

THEOREM B. *Let T be a positive projection operator on $A(X)$ satisfying $T \neq I$, $T(1_X) = 1_X$ and $L_{\alpha,\lambda}T = T$ for all $\alpha \in D$, $\lambda \in \Lambda$. If for every $g \in G$, $\mu^{(p)}(T; g) \in A(X)$ and*

$$\lim_{\alpha} \|L_{\alpha,\lambda}(\mu^{(p)}(T; g))\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

then $\{L_{\alpha,\lambda}\}$ *is an approximation process with respect to* T *on* $A(X)$.

In [58] we extended Theorem A to the context of functions taking a value in an arbitrary normed linear space under the concept of quasi-positive linear operators including convexity-monotone operators introduced by Campiti [14] (cf. [59]). Moreover, further references concerning the positive approximation processes can be also found in [5] and the Korovkin-type approximation theory is extensively treated in the books of Altomare and Campiti [6], Anastassiou [8], Donner [17] and Keimel and Roth [30].

Now, in [55] we gave a quantitative version of Theorems A and B, in which we estimated the rate of convergence behavior (1) of $L_{\alpha,\lambda}(f)$ by using a suitable modulus of continuity of f under certain requirements (cf. [53], [56]) motivated by the previous works of the author [50, 51, 52] in the setting of compact metric spaces (cf. [60]).

The purpose of this paper is to refine these results for approximation of functions having certain smoothness properties. Actually, the results of the author [45, 49] can be improved by means of the higher order moments. Applications will be made to various approximation processes induced by the method of A-summability due to the author [48] (cf. [49], [57]), which recovers that of Bell [10] (cf. [34], [61]) including the method of almost convergence (F-summability) of Lorentz [32], A_B-summability of Mazhar and Siddiqi [35] and order summability of Jurkat and Peyerimhoff [27, 28].

Consequently, we extend the results of Mohapatra [36] concerning the almost convergence for continuously differentiable functions on the bounded closed interval $[a, b]$ in the real line $\mathbb{R}$ to the case of several variables. Concrete examples of approximating operators can be provided by the Bernstein-Lototsky-Schnabl operators ([49], cf. [18], [22], [23], [62]), the Bernstein-Schnabl operators ([1], cf. [2], [7]), the generalized Stancu-Mühlbach operators ([12], cf. [40]) and the strongly continuous semigroups of Markov operators induced by them (cf. [3], [4], [7], [13], [46], [54], [56]). For the basic theory of semigroups of operators on Banach spaces, we refer to [11], [16], [19], [25] and [43].

2. Auxiliary Results

Let d be a pseudo-metric in X. For $f \in B(X)$ and $\delta \geq 0$, we define

$$\omega(f,\delta) = \omega_d(f,\delta) = \sup\{|f(x) - f(y)| : x, y \in X, d(x,y) \le \delta\},$$

which is called the modulus of continuity of f with respect to d. Obviously, for each $f \in B(X)$, $\omega(f,\cdot)$ is a monotone increasing function on $[0,\infty)$ with

$$0 \le \omega(f,\delta) \le 2\|f\| \qquad (\delta \ge 0)$$

and

$$\omega(f,\delta) = \omega(f,\delta(X)) \qquad (\delta \ge \delta(X)),$$

where $\delta(X)$ denotes the diameter of X. Also,

$$\lim_{\delta \to +0} \omega(f,\delta) = 0$$

if and only if f is uniformly continuous with respect to the topology induced by d.

Here we assume that there exist constants $C, K > 0$ such that

(3) $$\omega(f,\xi\delta) \le (C + \xi K)\omega(f,\delta) \qquad (f \in B(X), \xi, \delta \ge 0).$$

REMARK 1: (cf. [50; Lemma 3]) (a) Suppose that d is convex, i.e., if $d(x,y) = a + b, a, b > 0$, then there exists a point $z \in X$ such that $d(x,z) = a$ and $d(z,y) = b$. Then (3) holds for $C = K = 1$.

(b) Let X be a compact convex subset of a pseudo-metric linear space (Y, d). Assume that d is invariant, i.e., $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in Y$, and that $d(\cdot, 0)$ is starshaped, i.e., $d(\beta x, 0) \le \beta d(x, 0)$ for all $x \in Y$ and all β with $0 \le \beta \le 1$. Then (3) holds for $C = K = 1$.

(c) If (X, d) is a compact metric space having a coefficient of convex deformation $\rho = \rho(X)$, then (3) holds for $C = 1$ and $K = \rho$ ([26; Théorème 2]).

Note that if X is as in Remark 1 (b) with d being invariant and if $d(\beta x, 0) = \beta d(x, 0)$ for all $x \in Y$ and all β with $0 < \beta < 1$, then d is convex. In particular, if d is a pseudo-metric induced by a seminorm, then it is always convex.

Let $p > 1$ and let Φ be a non-negative function in $B(X^2)$, where $X^2 = X \times X$ denotes the product space of X and X, such that $\Phi(\cdot, y) \in A(X)$ for each $y \in X$ and

(4) $$d^p(x,y) \le \Phi(x,y) \qquad \text{for all } (x,y) \in X^2.$$

A function $f \in C(X)$ is said to have the property (mvp) if there exists a finite subset $\{f_1, f_2, \cdots, f_r\}$ of $C(X)$ and a finite subset $\{h_1, h_2, \cdots, h_r\}$ of G such that

$$f(x) - f(y) = \sum_{i=1}^{r} f_i(\xi_i)(h_i(x) - h_i(y)) \tag{5}$$

for all $x, y \in X$, where $\{\xi_1, \xi_2, \cdots, \xi_r\}$ is a set of r points of X with

$$d(\xi_i, y) \le d(x, y) \qquad (i = 1, 2, \cdots, r).$$

In this event, we sometimes say that f has the property (mvp) associated with the system

$$\{f_1, f_2, \cdots, f_r; h_1, h_2, \cdots, h_r\}. \tag{6}$$

REMARK 2: Let X be a compact convex subset of the r-dimensional Euclidean space $\mathbb{R}^r$ equipped with the metric

$$d(x, y) = \max\{|x_i - y_i| : i = 1, 2, \cdots, r\} \tag{7}$$

for $x = (x_1, x_2, \cdots, x_r), y = (y_1, y_2, \cdots, y_r) \in \mathbb{R}^r$ and define

$$\Phi(x, y) = \sum_{i=1}^{r} |x_i - y_i|^p,$$

which clearly satisfies (4). Then (3) holds for $C = K = 1$ and every continuously differentiable function f on X has the property (mvp) associated with the system

$$\{f_1, f_2, \cdots, f_r; e_1, e_2, \cdots, e_r\},$$

where f_i is the i-th partial derivative of f, i.e.,

$$f_i(x) = \frac{\partial f}{\partial x_i}(x) \qquad (x = (x_1, x_2, \cdots, x_r) \in X) \tag{8}$$

and e_i denotes the i-th coordinate function on X, i.e.,

$$e_i(x) = x_i \qquad (x = (x_1, x_2, \cdots, x_r) \in X).$$

From now on, we suppose that $A(X)$ contains the set G_q, where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad i.e., \quad q = \frac{p}{p-1},$$

and that $f \in A(X)$ has the property (mvp) associated with the system (6).

LEMMA 1. *Let φ be a positive linear functional on $A(X)$ and let $y \in X$. Then for all $\delta > 0$,*

$$(9) \qquad |\varphi(f) - f(y)\varphi(1_X)| \leq \sum_{i=1}^{r} |f_i(y)||\varphi(h_i - h_i(y)1_X)|$$

$$+ \quad \left\{C(\varphi(1_X))^{1/p} + \delta^{-1}K(\varphi(\Phi(\cdot, y)))^{1/p}\right\}$$

$$\times \quad \sum_{i=1}^{r} \left(\varphi(|h_i - h_i(y)1_X|^q)\right)^{1/q} \omega(f_i, \delta).$$

In particular, if $\varphi(1_X) = 1$, then (9) reduces to

$$|\varphi(f) - f(y)| \leq \sum_{i=1}^{r} |f_i(y)||\varphi(h_i) - h_i(y)|$$

$$+ \quad \left\{C + \delta^{-1}K(\varphi(\Phi(\cdot, y)))^{1/p}\right\} \sum_{i=1}^{r} \left(\varphi(|h_i - h_i(y)1_X|^q)\right)^{1/q} \omega(f_i, \delta).$$

PROOF: For all $x \in X$, we define

$$F(x) = f(x) - f(y) - \sum_{i=1}^{r} f_i(y)(h_i(x) - h_i(y)).$$

Then we have

$$(10) \quad |\varphi(f) - f(y)\varphi(1_X)| \leq \sum_{i=1}^{r} |f_i(y)||\varphi(h_i - h_i(y)1_X)| + |\varphi(F)|.$$

Now we extend φ to a positive linear functional on the whole space $C(X)$ and denote this functional by the same φ. Since by (3), (4) and (5)

$$|F(x)| \leq \sum_{i=1}^{r} |f_i(\xi_i) - f_i(y)||h_i(x) - h_i(y)|$$

$$\leq \sum_{i=1}^{r} \left(C + \delta^{-1}Kd(\xi_i, y)\right) \omega(f_i, \delta)|h_i(x) - h_i(y)|$$

$$\leq \sum_{i=1}^{r} (C + \delta^{-1} K d(x, y))\, \omega(f_i, \delta) |h_i(x) - h_i(y)|$$

$$\leq \sum_{i=1}^{r} \left(C + \delta^{-1} K (\Phi(x, y))^{1/p}\right) \omega(f_i, \delta) |h_i(x) - h_i(y)|$$

$$= \sum_{i=1}^{r} \left(C|h_i(x) - h_i(y)| + \delta^{-1} K (\Phi(x, y))^{1/p} |h_i(x) - h_i(y)|\right) \omega(f_i, \delta),$$

applying φ to both sides of this inequality with respect to the variable x and using Hölder's inequality, we get

$$|\varphi(F)| \leq \sum_{i=1}^{r} \Big\{ C(\varphi(1_X))^{1/p} \left(\varphi(|h_i - h_i(y)1_X|^q)\right)^{1/q}$$

$$+ \quad \delta^{-1} K(\varphi(\Phi(\cdot, y)))^{1/p} \left(\varphi(|h_i - h_i(y)1_X|^q)\right)^{1/q} \Big\} \omega(f_i, \delta)$$

$$= \sum_{i=1}^{r} \left(C(\varphi(1_X))^{1/p} + \delta^{-1} K(\varphi(\Phi(\cdot, y)))^{1/p} \right)$$

$$\times \quad \left(\varphi(|h_i - h_i(y)1_X|^q)\right)^{1/q} \omega(f_i, \delta),$$

which together with (10) implies the desired inequality (9).

As an immediate consequence of Lemma 1, we have the following:

PROPOSITION 1. *Let L be a positive linear operator of $A(X)$ into $B(X)$. Then for all $y \in X$ and all $\delta > 0$,*

$$(11) \qquad |L(f)(y) - f(y)L(1_X)(y)| \leq \sum_{i=1}^{r} |f_i(y)||L(h_i - h_i(y)1_X)(y)|$$

$$+ \quad \left\{ C(L(1_X)(y))^{1/p} + \delta^{-1} K(m(L; \Phi)(y))^{1/p} \right\}$$

$$\times \quad \sum_{i=1}^{r} \left(\mu^{(q)}(L; h_i)(y)\right)^{1/q} \omega(f_i, \delta),$$

where

$$(12) \qquad m(L; \Phi)(y) = L(\Phi(\cdot, y))(y).$$

In particular, if $L(1_X) = 1_X$, then (11) reduces to

$$(13) \qquad |L(f)(y) - f(y)| \leq \sum_{i=1}^{r} |f_i(y)||L(h_i)(y) - h_i(y)|$$

$$+ \quad \left\{C + \delta^{-1} K(m(L;\Phi)(y))^{1/p}\right\} \sum_{i=1}^{r} \left(\mu^{(q)}(L;h_i)(y)\right)^{1/q} \omega(f_i, \delta).$$

Lemma 2. *Let φ be a positive linear functional on $A(X)$. Let L be a positive linear operator of $A(X)$ into itself such that*

$$(14) \quad m(L;\Phi), |L(h_i) - h_i|, \mu^{(q)}(L;h_i) \in A(X) \qquad (i = 1, 2, \cdots, r),$$

where $m(L;\Phi)$ is the function defined by (12). Then for all $\delta > 0$,

$$(15) \qquad |\varphi(L(f)) - \varphi(f)| \leq \sum_{i=1}^{r} \|f_i\| \varphi(|L(h_i) - h_i|)$$

$$+ \quad \left\{C(\varphi(1_X))^{1/p} + \delta^{-1} K \left(\varphi(m(L;\Phi))\right)^{1/p}\right\}$$

$$\times \quad \sum_{i=1}^{r} \left(\varphi(\mu^{(q)}(L;h_i))\right)^{1/q} \omega(f_i, \delta).$$

In particular, if $\varphi(1_X) = 1$, then (15) reduces to

$$|\varphi(L(f)) - \varphi(f)| \leq \sum_{i=1}^{r} \|f_i\| \varphi(|L(h_i) - h_i|)$$

$$+ \quad \left\{C + \delta^{-1} K(\varphi(m(L;\Phi)))^{1/p}\right\} \sum_{i=1}^{r} \left(\varphi(\mu^{(q)}(L;h_i))\right)^{1/q} \omega(f_i, \delta).$$

Proof: We extend φ to a positive linear functional on the whole space $C(X)$ and denote this functional by the same φ. Then applying φ to both sides of (13) and using Hölder's inequality, we establish the desired estimate (15).

From Lemma 2, we derive the following:

PROPOSITION 2. *Let S and L be positive linear operators of $A(X)$ into itself. Suppose that $L(1_X) = 1_X$ and (14) is satisfied. Then for all $y \in X$ and all $\delta > 0$,*

$$(16)\qquad |S(L(f))(y) - S(f)(y)| \le \sum_{i=1}^{r} \|f_i\| S(|L(h_i) - h_i|)(y)$$

$$+ \quad \left\{ C(S(1_X)(y))^{1/p} + \delta^{-1} K(S(m(L;\Phi))(y))^{1/p} \right\}$$

$$\times \quad \sum_{i=1}^{r} \left(S(\mu^{(q)}(L;h_i))(y) \right)^{1/q} \omega(f_i, \delta).$$

In particular, if $SL = L$, then (16) reduces to

$$|L(f)(y) - S(f)(y)| \le \sum_{i=1}^{r} \|f_i\| S(|L(h_i) - h_i|)(y)$$

$$+ \left\{ C + \delta^{-1} K(S(m(L;\Phi))(y))^{1/p} \right\} \sum_{i=1}^{r} (S(\mu^{(q)}(L;h_i))(y))^{1/q} \omega(f_i, \delta).$$

3. Main Results

Here we assume that $A(X)$ contains G_p for each $p > 1$. If $f \in B(X)$, $\delta \ge 0$ and if $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G, then we define

$$\omega(f; g_1, \cdots, g_m, \delta) = \sup\{|f(x) - f(y)| : x, y \in X, d(x,y) \le \delta\},$$

where

$$(17)\qquad d(x,y) = \max\{|g_i(x) - g_i(y)| : i = 1, 2, \cdots, m\},$$

which is a pseudo-metric in X. This quantity is called the modulus of continuity of f with respect to $g_1, g_2, \cdots, g_m$ ([53], cf. [45], [49]).

In order to achieve our purpose it is always supposed that there exist constants $C, K > 0$ such that

$$(18)\qquad \omega(f; g_1, \cdots, g_m, \xi\delta) \le (C + K\xi)\omega(f; g_1, \cdots, g_m, \delta)$$

for all $f \in B(X), \xi, \delta \geq 0$ and for all finite subsets $\{g_1, g_2, \cdots, g_m\}$ of G.

A function $f \in C(X)$ is said to have the property (MVP) if there exists a finite subset $\{f_1, f_2, \cdots, f_r\}$ of $C(X)$ and a finite subset $\{h_1, h_2, \cdots, h_r\}$ of G satisfying (5), where $\{\xi_1, \xi_2, \cdots, \xi_r\}$ is a set of r points of X with

$$|g(\xi_i) - g(y)| \leq |g(x) - g(y)| \tag{19}$$

for all $g \in G$ and for $i = 1, 2, \cdots, r$. In this event, we sometimes say that f has the property (MVP) associated with the system (6).

Let $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of positive linear operators of $A(X)$ into $B(X)$ with

$$\eta_\alpha = \sup\{\|L_{\alpha,\lambda}(1_X)\| : \lambda \in \Lambda\} < \infty \tag{20}$$

for each $\alpha \in D$. If L is a positive linear operator of $A(X)$ into $B(X)$ and $f \in C(X)$, then we define

$$\|L_\alpha(f) - L(f)\|_\Lambda = \sup\{\|L_{\alpha,\lambda}(f) - L(f)\| : \lambda \in \Lambda\}$$

and

$$\|L_\alpha(f) - fL_\alpha(1_X)\|_\Lambda = \sup\{\|L_{\alpha,\lambda}(f) - fL_{\alpha,\lambda}(1_X)\| : \lambda \in \Lambda\},$$

which are finite by virtue of (20). Obviously, $\{L_{\alpha,\lambda}\}$ is an approximation process with respect to L on $A(X)$ if and only if

$$\lim_\alpha \|L_\alpha(f) - L(f)\|_\Lambda = 0 \qquad \text{for every } f \in A(X).$$

If $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G and $s > 1$, then we define

$$\mu_\alpha^{(s)}(g_1, \cdots, g_m) = \left(\sup\left\{\left\|\sum_{i=1}^m \mu_{\alpha,\lambda}^{(s)}(g_i)\right\| : \lambda \in \Lambda\right\}\right)^{1/s}.$$

Furthermore, for $f \in B(X)$ and $g \in G$, we define

$$\omega_\alpha(f, g) = \inf\{C_\alpha(p, \epsilon)\mu_\alpha^{(p/(p-1))}(g)\omega(f; g_1, \cdots, g_m, \epsilon\mu_\alpha^{(p)}(g_1, \cdots, g_m)) :$$
$$p > 1, \epsilon > 0, g_1, \cdots, g_m \in G, \mu_\alpha^{(p)}(g_1, \cdots, g_m) > 0, m = 1, 2, \cdots\},$$

where

$$C_\alpha(p, \epsilon) = \sup\{\|C(L_{\alpha,\lambda}(1_X))^{1/p} + \epsilon^{-1}K1_X\| : \lambda \in \Lambda\}.$$

We are now in a position to recast Theorem A in a quantitative form with the rate of convergence for functions having the property (MVP). Let f be a function in $A(X)$, which has the property (MVP) associated system (6) and let U be as in (2).

THEOREM 1. *Let u be a strictly positive function in $A(X)$ having the property (MVP) associated the system $\{u_1, u_2, \cdots, u_s; v_1, v_2, \cdots, v_s\}$. Then for all $\alpha \in D$,*

$$\|L_\alpha(f) - U(f)\|_\Lambda \leq \|f/u\| \|L_\alpha(u) - U(u)\|_\Lambda$$

$$+ \quad \|f/u\| \left\{ \sum_{i=1}^{s} \|u_i\| \|L_\alpha(v_i) - v_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{s} \omega_\alpha(u_i, v_i) \right\}$$

$$+ \quad \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{r} \omega_\alpha(f_i, h_i).$$

PROOF: For all $\alpha \in D$, we have

$$\|L_\alpha(f) - U(f)\|_\Lambda \leq \|f/u\| \|L_\alpha(u) - U(u)\|_\Lambda \tag{21}$$

$$+ \|f/u\| \|L_\alpha(u) - uL_\alpha(1_X)\|_\Lambda + \|L_\alpha(f) - fL_\alpha(1_X)\|_\Lambda.$$

Let $p > 1, \delta > 0$ and let $\{g_1, g_2, \cdots, g_m\}$ be a finite subset of G. We define

$$\Phi(x, y) = \sum_{i=1}^{m} |g_i(x) - g_i(y)|^p \tag{22}$$

for all $(x, y) \in X^2$. Then with the pseudo-metric d in X given by (17), inequalities (3) and (4) hold because of (18) and (22). Furthermore f has the property (mvp) associated with the system (6) by virtue of (19). Therefore, taking $L = L_{\alpha,\lambda}$ in Proposition 1, we arrive at

$$|L_{\alpha,\lambda}(f)(y) - f(y)L_{\alpha,\lambda}(1_X)(y)| \leq \sum_{i=1}^{r} \|f_i\| \|L_{\alpha,\lambda}(h_i) - h_i L_{\alpha,\lambda}(1_X)\|$$

$$+ \quad \left\{ C(L_{\alpha,\lambda}(1_X)(y))^{1/p} + \delta^{-1} K \left(\sum_{i=1}^{m} L_{\alpha,\lambda}(|g_i - g_i(y)1_X|^p)(y) \right)^{1/p} \right\}$$

$$\times \quad \sum_{i=1}^{r} \left(\mu_{\alpha,\lambda}^{(p/(p-1))}(h_i)(y) \right)^{1-1/p} \omega(f_i; g_1, \cdots, g_m, \delta)$$

$$\le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda$$

$$+ \quad \left\{ C(L_{\alpha,\lambda}(1_X)(y))^{1/p} + \delta^{-1} K \left\| \sum_{i=1}^{m} \mu_{\alpha,\lambda}^{(p)}(g_i) \right\|^{1/p} \right\}$$

$$\times \quad \sum_{i=1}^{r} \|\mu_{\alpha,\lambda}^{(p/(p-1))}(h_i)\|^{1-1/p} \omega(f_i; g_1, \cdots, g_m, \delta)$$

$$\le \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda$$

$$+ \quad \left\{ C(L_{\alpha,\lambda}(1_X)(y))^{1/p} + \delta^{-1} K \mu_\alpha^{(p)}(g_1, \cdots, g_m) \right\}$$

$$\times \quad \sum_{i=1}^{r} \mu_\alpha^{(p/(p-1))}(h_i) \omega(f_i; g_1, \cdots, g_m, \delta).$$

Now putting $\delta = \epsilon \mu_\alpha^{(p)}(g_1, \cdots, g_m) > 0$ and taking the supremum over all $y \in X$, we get

$$\|L_{\alpha,\lambda}(f) - f L_{\alpha,\lambda}(1_X)\| \le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda$$

$$+ \quad \|C(L_{\alpha,\lambda}(1_X))^{1/p} + \epsilon^{-1} K 1_X\|$$

$$\times \quad \sum_{i=1}^{r} \mu_\alpha^{(p/(p-1))}(h_i) \omega(f_i; g_1, \cdots, g_m, \epsilon \mu_\alpha^{(p)}(g_1, \cdots, g_m)),$$

and so

$$\|L_\alpha(f) - f L_\alpha(1_X)\|_\Lambda \le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda$$

$$+ \quad C_\alpha(p, \epsilon) \sum_{i=1}^{r} \mu_\alpha^{(p/(p-1))}(h_i) \omega(f_i; g_1, \cdots, g_m, \epsilon \mu_\alpha^{(p)}(g_1, \cdots, g_m)),$$

which yields

$$(23) \qquad \|L_\alpha(f) - fL_\alpha(1_X)\|_\Lambda \le \sum_{i=1}^{r} \omega_\alpha(f_i, h_i)$$

$$+ \quad \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda.$$

Similarly, we have

$$\|L_\alpha(u) - uL_\alpha(1_X)\|_\Lambda \le \sum_{i=1}^{s} \|u_i\| \|L_\alpha(v_i) - v_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{s} \omega_\alpha(u_i, v_i),$$

which together with (23) and (21) implies the desired result.

COROLLARY 1. *Let u be as in Theorem 1. Then for all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f/u\| \|L_\alpha(u) - u\|_\Lambda$$

$$+ \quad \|f/u\| \left\{ \sum_{i=1}^{s} \|u_i\| \|L_\alpha(v_i) - v_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{s} \omega_\alpha(u_i, v_i) \right\}$$

$$+ \quad \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{r} \omega_\alpha(f_i, h_i).$$

COROLLARY 2. *For all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\| \|L_\alpha(1_X) - 1_X\|_\Lambda$$

$$+ \quad \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i L_\alpha(1_X)\|_\Lambda + \sum_{i=1}^{r} \omega_\alpha(f_i, h_i).$$

REMARK 3: Let $g \in G$. Then the following estimates hold for all $\alpha \in D$:

$$\|L_\alpha(g) - gL_\alpha(1_X)\|_\Lambda \le \mu_\alpha^{(p)}(g)\eta_\alpha^{1-1/p} \qquad (p > 1);$$

$$\mu_\alpha^{(p/(p-1))}(g) \le \mu_\alpha^{(p)}(g)\eta_\alpha^{(p-2)/p} \qquad (p \ge 2).$$

Suppose that $A(X)$ contains the set

$$F_k(G) = \{g^i : g \in G, i = 0, 1, 2, \cdots, k\}$$

for an even positive integer k. Let $\{g_1, g_2, \cdots, g_m\}$ be a finite subset of G. Then for all $\alpha \in D$,

$$(24)\quad \mu_\alpha^{(k)}(g_1, \cdots, g_m) \le \Big(\sum_{i=1}^{m}\sum_{j=0}^{k}\binom{k}{j}\|g_i\|^{k-j}\|L_\alpha(g_i^j) - U(g_i^j)\|_\Lambda\Big)^{1/k}.$$

In particular, if $h = 1_X$, i.e., $U = I$ and if $L_{\alpha,\lambda}(g^j) = g^j$ for all $\alpha \in D, \lambda \in \Lambda, g \in G$ and for $j = 0, 1, \cdots, k-1$, then (24) reduces to

$$\mu_\alpha^{(k)}(g_1, \cdots, g_m) = \left(\sup\left\{\left\|\sum_{i=1}^{m}(L_{\alpha,\lambda}(g_i^k) - g_i^k)\right\| : \lambda \in \Lambda\right\}\right)^{1/k}.$$

Thus Corollary 2 yields the estimate for $\|L_\alpha(f) - f\|_\Lambda$ in terms of the corresponding quantities for the test system $G^k = \{g^k : g \in G\}$.

Let T as in Theorem B and suppose that

$$|T(h_i) - h_i| \in A(X) \qquad (i = 1, 2, \cdots, r)$$

and

$$\mu^{(s)}(T; g) \in A(X) \qquad (s > 1, g \in G).$$

For $\alpha \in D$ and for $i = 1, 2, \cdots, r$, we define

$$\|L_\alpha(|T(h_i) - h_i|)\|_\Lambda = \sup\{\|L_{\alpha,\lambda}(|T(h_i) - h_i|)\| : \lambda \in \Lambda\}.$$

If $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G and $s > 1$, then we define

$$\mu_\alpha^{(s)}(T; g_1, \cdots, g_m) = \left(\sup\left\{\left\|\sum_{i=1}^{m} L_{\alpha,\lambda}(\mu^{(s)}(T; g_i))\right\| : \lambda \in \Lambda\right\}\right)^{1/s}.$$

Furthermore, for $f \in B(X)$ and $g \in G$, we define

$$\begin{aligned} \omega_\alpha(T; f, g) = \inf\{&(C + \epsilon^{-1}K)\mu_\alpha^{(p/(p-1))}(T; g) \\ &\times \ \omega(f; g_1, \cdots, g_m, \epsilon\mu_\alpha^{(p)}(T; g_1, \cdots, g_m)) : \\ p > 1, \epsilon > 0, g_1, &\cdots, g_m \in G, \mu_\alpha^{(p)}(T; g_1, \cdots, g_m) > 0, m = 1, 2, \cdots\}. \end{aligned}$$

Now concerning the degree of convergence in Theorem B we have the following:

THEOREM 2. *For all $\alpha \in D$,*

$$\|L_\alpha(f) - T(f)\|_\Lambda \leq \sum_{i=1}^{r} \|f_i\| \|L_\alpha(|T(h_i) - h_i|)\|_\Lambda + \sum_{i=1}^{r} \omega_\alpha(T; f_i, h_i).$$

PROOF: Let $p > 1, \delta > 0$ and let $\{g_1, g_2, \cdots, g_m\}$ be a finite subset of G. Let Φ be the function given by (22). Then taking $S = L_{\alpha,\lambda}$ and $L = T$ in Proposition 2, we have

$$|L_{\alpha,\lambda}(f)(y) - T(f)(y)| \leq \sum_{i=1}^{r} \|f_i\| L_{\alpha,\lambda}(|L(h_i) - h_i|)(y)$$

$$+ \quad \left\{ C + \delta^{-1} K \left(\sum_{i=1}^{m} L_{\alpha,\lambda}(\mu^{(p)}(T; g_i))(y) \right)^{1/p} \right\}$$

$$\times \quad \sum_{i=1}^{r} \left(L_{\alpha,\lambda}(\mu^{(p/(p-1))}(T; h_i))(y) \right)^{1-1/p} \omega(f_i; g_1, \cdots, g_m, \delta),$$

which gives

$$\|L_\alpha(f) - T(f)\|_\Lambda \leq \sum_{i=1}^{r} \|f_i\| \|L_\alpha(|T(h_i) - h_i|)\|_\Lambda$$

$$+ \quad \{ C + \delta^{-1} K \mu_\alpha^{(p)}(T; g_1, g_2, \cdots, g_m) \}$$

$$\times \quad \sum_{i=1}^{r} \mu_\alpha^{(p/(p-1))}(T; h_i) \omega(f_i; g_1, g_2, \cdots, g_m, \delta).$$

Therefore, Putting $\delta = \epsilon \mu_\alpha^{(p)}(T; g_1, \cdots, g_m) > 0$ and taking the infimum over all $p > 1, \epsilon > 0, g_1, \cdots, g_m \in G, \mu_\alpha^{(p)}(T; g_1, \cdots, g_m) > 0$ and $m = 1, 2, \cdots$, we obtain the desired result.

COROLLARY 3. *If $T(g) = g$ for all $g \in G$, then*

$$\|L_\alpha(f) - T(f)\|_\Lambda \leq \sum_{i=1}^{r} \omega_\alpha(T; f_i, h_i)$$

for all $\alpha \in D$.

REMARK 4: For all $\alpha \in D$, we have:

$$\|L_\alpha(|T(h_i) - h_i|)\|_\Lambda \le \mu_\alpha^{(p)}(T; h_i) \qquad (p > 1, i = 1, 2, \cdots, r);$$

$$\mu_\alpha^{(p/(p-1))}(T; g) \le \mu_\alpha^{(p)}(T; g) \qquad (p \ge 2, g \in G).$$

If $A(X)$ contains $F_k(G)$ for an even positive integer k and

$$T(g^i) = g^i \qquad (g \in G, i = 0, 1, 2, \cdots, k-1), \tag{25}$$

then we have

$$\mu_\alpha^{(k)}(T; g_1, \cdots, g_m) \le \left(\sum_{i=1}^{m} \|L_\alpha(g_i^k) - T(g_i^k)\|_\Lambda\right)^{1/k},$$

and so Corollary 3 gives an estimate for $\|L_\alpha(f) - T(f)\|_\Lambda$ in terms of the corresponding quantities for the test system G^k.

In the rest of this section it is assumed that $A(X)$ contains $F_k(G)$ for an even positive integer k. Let T be a positive projection operator on $A(X)$ with $T \ne I$, which satisfies (25) and $L_{\alpha,\lambda}T = T$ for all $\alpha \in D, \lambda \in \Lambda$. In addition, we suppose that each $L_{\alpha,\lambda}$ maps $A(X)$ into itself and

$$L_{\alpha,\lambda}(g^k) = g^k + \xi_{\alpha,\lambda}(T(g^k) - g^k)$$

for all $\alpha \in D, \lambda \in \Lambda$ and all $g \in G$, where $\{\xi_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of real numbers with $0 < \xi_{\alpha,\lambda} < 1$.

For $f \in B(X)$ and $\delta > 0$, we define

$$\Gamma(f, \delta) = \inf\left\{ (C + K/\epsilon)\omega\left(f; g_1, \cdots, g_m, \delta\epsilon \left\|\sum_{i=1}^{m}(T(g_i^k) - g_i^k)\right\|^{1/k}\right) : \right.$$

$$\left. \epsilon > 0, g_1, \cdots, g_m \in G, \left\|\sum_{i=1}^{m}(T(g_i^k) - g_i^k)\right\| > 0, m = 1, 2, \cdots \right\}.$$

Using this quantity, we have the following result which is more convenient for later applications.

COROLLARY 4. *Let $\{n_\alpha : \alpha \in D\}$ be a net of positive integers and let $U_{\alpha,\lambda} = L^{n_\alpha}_{\alpha,\lambda}$ be the n_α-iteration of $L_{\alpha,\lambda}$ for each $\alpha \in D, \lambda \in \Lambda$. Then for all $\alpha \in D$, we have:*

$$(26) \qquad \|U_\alpha(f) - f\|_\Lambda \le \sum_{i=1}^{r} \mu_\alpha^{(k/(k-1))}(h_i)\Gamma(f_i, \zeta_\alpha)$$

$$\le \sum_{i=1}^{r} \mu_\alpha^{(k/(k-1))}(h_i)\Gamma(f_i, (n_\alpha \xi_\alpha)^{1/k}),$$

where

$$\mu_\alpha^{(k/(k-1))}(h_i) = \left(\sup\{\|\mu^{(k/(k-1))}(U_{\alpha,\lambda}; h_i)\| : \lambda \in \Lambda\}\right)^{1-1/k},$$

$$\zeta_\alpha = \left(\sup\{1 - (1 - \xi_{\alpha,\lambda})^{n_\alpha} : \lambda \in \Lambda\}\right)^{1/k}$$

and

$$\xi_\alpha = \sup\{\xi_{\alpha,\lambda} : \lambda \in \Lambda\};$$

$$(27) \qquad \|U_\alpha(f) - T(f)\|_\Lambda \le \sum_{i=1}^{r} \mu_\alpha^{(k/(k-1))}(T; h_i)\Gamma(f_i, \gamma_\alpha),$$

where

$$\mu_\alpha^{(k/(k-1))}(T; h_i) = \left(\sup\{\|U_{\alpha,\lambda}(\mu^{(k/(k-1))}(T; h_i))\| : \lambda \in \Lambda\}\right)^{1-1/k}$$

and

$$\gamma_\alpha = \left(\sup\{1 - \xi_{\alpha,\lambda} : \lambda \in \Lambda\}\right)^{n_\alpha/k}.$$

Indeed, by induction on the degree of iteration, it can be verified that

$$U_{\alpha,\lambda} T = T \qquad (\alpha \in D, \lambda \in \Lambda)$$

and

$$U_{\alpha,\lambda}(g^k) = T(g^k) + (1 - \xi_{\alpha,\lambda})^{n_\alpha}(g^k - T(g^k)) \qquad (\alpha \in D, \lambda \in \Lambda, g \in G).$$

Thus (26) and (27) follow from Corollaries 2 and 3, respectively.

4. Applications

Let $A(X)$ be a closed linear subspace of $C(X)$ which contains 1_X. A mapping L of $A(X)$ into itself is called a Markov operator on $A(X)$ if it is a positive linear operator with $L(1_X) = 1_X$. Let $\mathbb{N}$ denote the set of all non-negative integers. Let $\{a_{\alpha,n}^{(\lambda)} : \alpha \in D, \lambda \in \Lambda, n \in \mathbb{N}\}$ be a family of non-negative real numbers with

$$\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} = 1 \qquad \text{for each} \quad \alpha \in D, \lambda \in \Lambda. \tag{28}$$

For examples of such families, see, for instance, [48], [49], [50], [52] and [57]. Let $\{j_n : n \in \mathbb{N}\}$ be a sequence of positive integers and $\{k_n : n \in \mathbb{N}\}$ a sequence of non-negative integers. Let $\{L_n : n \geq 1\}$ be a sequence of Markov operators on $A(X)$. For any $f \in A(X)$, we define

$$T_{\alpha,\lambda}(f) = \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} L_{j_n}^{k_n}(f) \qquad (\alpha \in D, \lambda \in \Lambda), \tag{29}$$

which converges in $A(X)$ because of (28). Let $\{W(t) : t \geq 0\}$ be a family of Markov operators on $A(X)$ such that for each $f \in A(X)$, the map $t \mapsto W(t)(f)$ is strongly continuous on $[0, \infty)$, $\{\Phi_\lambda : \lambda \in \Lambda\}$ a family of non-negative continuous functions on $[0, \infty)$ and $\{v_\alpha : \alpha \in D\}$ a net of positive real numbers with $\lim_\alpha v_\alpha = 0$ or $\lim_\alpha v_\alpha = +\infty$. For any $f \in A(X)$, we define

$$C_{\alpha,\lambda}(f) = \frac{1}{v_\alpha} \int_0^{v_\alpha} W(\Phi_\lambda(t))(f)dt \qquad (\alpha \in D, \lambda \in \Lambda) \tag{30}$$

and

$$R_{\alpha,\lambda}(f) = v_\alpha \int_0^{\infty} \exp(-v_\alpha t) W(\Phi_\lambda(t))(f)dt \quad (\alpha \in D, \lambda \in \Lambda), \tag{31}$$

which exist in $A(X)$.

All the operators given above are Markov operators on $A(X)$ and our general results obtained in the preceding section are applicable to them. As illustrations of these general results we restrict ourselves to the following setting:

Let X be a compact convex subset of a real locally convex Hausdorff vector space E and let $G = G(X)$ be the space of all real-valued continuous affine functions on X. Note that (18) holds for $C = K = 1$ (see, [45; Lemma 1]). Also, it is assumed that each point ξ_i in (5) is an internal point of the segment joining x and y (cf. [45; Definition 2]). Therefore, (19) is automatically fulfilled. Let T be a positive projection operator of $C(X)$ onto a closed linear subspace containing 1_X and G (which is the case where $A(X) = C(X)$ and $k = 2$).

For applications of Corollary 4 it is convenient to make the following definition: Let $\{P_\gamma : \gamma \in \Gamma\}$ be a family of Markov operators on $C(X)$ and $\{x_\gamma : \gamma \in \Gamma\}$ a family of non-negative real numbers. We say that $\{P_\gamma\}$ is of type $[T; x_\gamma]$ if

$$P_\gamma T = T \quad \text{and} \quad P_\gamma(g^2) = g^2 + x_\gamma(T(g^2) - g^2)$$

for all $\gamma \in \Gamma$ and all $g \in G$.

Now we first consider the case where $E = \mathbb{R}^r$, in which the metric $d(x, y)$ is given by (7). Then we have

$$\omega(f; e_1, \cdots, e_r, \delta) = \omega(f, \delta) \qquad (f \in B(X), \delta \geq 0).$$

Let $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of positive linear operators of $C(X)$ into $B(X)$. If $\mu_\alpha^{(2)}(e_1, \cdots, e_r) = 0$ for all $\alpha \in D$, then $L_{\alpha,\lambda}(f) = fL_{\alpha,\lambda}(1_X)$ for all $f \in C(X), \alpha \in D$ and all $\lambda \in \Lambda$ ([cf. [45; Lemma 2], [50; Lemma 1]). Thus we always consider the case where

$$\mu_\alpha^{(2)}(e_1, \cdots, e_r) > 0 \qquad (\alpha \in D).$$

Then for all $f \in B(X), \alpha \in D$ and for $i = 1, 2, \cdots, r$, we have

$$\omega_\alpha(f, e_i) \leq \inf\Big\{C_\alpha(\epsilon)\mu_\alpha^{(2)}(e_i)\omega(f, \epsilon\mu_\alpha^{(2)}(e_1, \cdots, e_r)) : \epsilon > 0\Big\},$$

where

$$C_\alpha(\epsilon) = \sup\Big\{\|(L_{\alpha,\lambda}(1_X))^{1/2} + \epsilon^{-1}1_X\| : \lambda \in \Lambda\Big\}.$$

Therefore, in view of Remark 2, we extend the results of Mohapatra [36] (cf. [15], [38], [39], [65]) and give a quantitative version of the

Korovkin type convergence theorem due to Karlin and Ziegler [29; Theorem 1 and Remark 2] for all functions in $C^{(1)}(X)$, which denotes the space of all continuously differentiable functions on X.

Take $X = \mathbb{I}_r$, the unit r-cube, i.e.,

$$\mathbb{I}_r = \{x = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r : 0 \le x_i \le 1, i = 1, 2, \cdots, r\}$$

and let F be the closed linear subspace of $C(\mathbb{I}_r)$ spanned by the set

$$\{e_1^{m_1} e_2^{m_2} \cdots e_r^{m_r} : m_i \in \{0, 1\}, i = 1, 2, \cdots, r\}.$$

Let $\{B_n : n \ge 1\}$ be the sequence of the Bernstein operators on $C(\mathbb{I}_r)$ given by

$$B_n(f)(x) = \sum_{m_1=0}^{n} \cdots \sum_{m_r=0}^{n} f(m_1/n, \cdots, m_r/n) \prod_{i=1}^{r} \binom{n}{m_i} x_i^{m_i}(1-x_i)^{n-m_i}$$

for $f \in C(\mathbb{I}_r)$ and $x = (x_1, x_2, \cdots, x_r) \in \mathbb{I}_r$ (see, e.g., [33]). It can be verified that B_1 is a positive projection operator of $C(\mathbb{I}_r)$ onto F and that $\{B_n\}$ is of type $[B_1; 1/n]$. Consequently, if $L_n = B_n, n \ge 1$, then $\{T_{\alpha,\lambda}\}$ is of type $[B_1; 1 - x_{\alpha,\lambda}]$, where

$$x_{\alpha,\lambda} = \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \left(1 - \frac{1}{j_n}\right)^{k_n} \qquad (\alpha \in D, \lambda \in \Lambda),$$

and so Corollary 4 can be applied to these operators. In particular, concerning the degree of approximation by iterations of the Bernstein operators, we have the following estimates: For all $f \in C^{(1)}(\mathbb{I}_r)$ and all $n \in \mathbb{N}$,

$$\text{(32)} \qquad \|B_{j_n}^{k_n}(f) - f\| \le \frac{r}{2}\left(1 - (1 - 1/j_n)^{k_n}\right)^{1/2} \sum_{i=1}^{r} \inf\Big\{(1 + \epsilon^{-1})$$

$$\times \quad \omega\Big(f_i, \left(1 - (1 - 1/j_n)^{k_n}\right)^{1/2} \epsilon \frac{\sqrt{r}}{2}\Big) : \epsilon > 0\Big\}$$

$$\le \frac{r}{2}\sqrt{\frac{k_n}{j_n}} \sum_{i=1}^{r} \inf\left\{(1 + \epsilon^{-1})\omega\left(f_i, \epsilon\sqrt{\frac{k_n}{j_n}}\frac{\sqrt{r}}{2}\right) : \epsilon > 0\right\}$$

and

$$(33)\qquad \|B_{j_n}^{k_n}(f)-B_1(f)\| \le \frac{r}{2}\Big(1-\frac{1}{j_n}\Big)^{k_n/2}\sum_{i=1}^{r}\inf\Big\{(1+\epsilon^{-1}) \times \ \omega\Big(f_i,(1-1/j_n)^{k_n/2}\epsilon\frac{\sqrt{r}}{2}\Big):\epsilon>0\Big\},$$

where f_i stands for the i-th partial derivative of f given by (8).

Taking $\epsilon = 2/\sqrt{r}$, (32) and (33) yield

$$(34)\qquad \|B_{j_n}^{k_n}(f)-f\| \le \frac{r}{2}\Big(1+\frac{\sqrt{r}}{2}\Big)\Big(1-(1-1/j_n)^{k_n}\Big)^{1/2} \times \ \sum_{i=1}^{r}\omega\Big(f_i,\Big(1-(1-1/j_n)^{k_n}\Big)^{1/2}\Big) \le \frac{r}{2}\Big(1+\frac{\sqrt{r}}{2}\Big)\sqrt{\frac{k_n}{j_n}}\sum_{i=1}^{r}\omega\left(f_i,\sqrt{\frac{k_n}{j_n}}\right)$$

and

$$(35)\qquad \|B_{j_n}^{k_n}(f)-B_1(f)\| \le \frac{r}{2}\Big(1+\frac{\sqrt{r}}{2}\Big)\Big(1-\frac{1}{j_n}\Big)^{k_n/2} \times \ \sum_{i=1}^{r}\omega\Big(f_i,(1-1/j_n)^{k_n/2}\Big),$$

respectively. In particular, if $r=1$, then (34) and (35) reduce to

$$\|B_{j_n}^{k_n}(f)-f\| \le \frac{3}{4}\Big(1-\Big(1-\frac{1}{j_n}\Big)^{k_n}\Big)^{1/2}\omega\Big(f',\Big(1-(1-1/j_n)^{k_n}\Big)^{1/2}\Big) \le \frac{3}{4}\sqrt{\frac{k_n}{j_n}}\,\omega\left(f',\sqrt{\frac{k_n}{j_n}}\right),$$

which is given in [33; Theorem 1.6.2] for $\{k_n\}=\{1\}$ and $\{j_n\}=\{n\}$ and

$$\|B_{j_n}^{k_n}(f)-B_1(f)\| \le \frac{3}{4}\Big(1-\frac{1}{j_n}\Big)^{k_n/2}\omega\Big(f',(1-1/j_n)^{k_n/2}\Big),$$

respectively (cf. [39], [41], [42]).

Statements analogous to the above-mentioned results may be derived for the case where $B_n, n \geq 1$, are the Bernstein operators on $C(\Delta_r)$ with the standard r-simplex

$$\Delta_r = \{x = (x_1, \cdots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, \cdots, r, x_1 + \cdots + x_r \leq 1\},$$

given by

$$B_n(f)(x) = \sum_{m_i \geq 0, m_1 + \cdots + m_r \leq n} f(m_1/n, \cdots, m_r/n)$$
$$\times \frac{n!}{m_1! m_2! \cdots m_r! (n - m_1 - m_2 - \cdots - m_r)!}$$
$$\times \quad x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} (1 - x_1 - x_2 \cdots - x_r)^{n - m_1 - m_2 - \cdots - m_r}$$

for $f \in C(\Delta_r)$ and $x = (x_1, x_2, \cdots, x_r) \in \Delta_r$ (see, e.g., [33]). These can be obtained in the following very general setting.

Recall that X is a compact convex subset of a real locally convex Hausdorff vector space E and let $G = G(X)$. If L is a Markov operator on $C(X)$, then for point $x \in X$, a Radon probability measure ν_x on X is called an $L(G)$-representing measure for x if

$$L(f)(x) = \int_X f \, d\nu_x \qquad \text{for every} f \in G$$

(cf. [24], [37]). Let $\mathbb{M} = \{M_n : n \geq 1\}$ be a sequence of Markov operators on $C(X)$, $\nu^{(\mathbb{M})} = \{\nu_{x,n} : x \in X, n \geq 1\}$ a family of Radon probability measures on X such that $\nu_{x,n}$ is an $M_n(G)$-representing measure for x, $\mathbb{P} = (p_{nj})_{n,j \geq 1}$ an infinite lower triangular stochastic matrix, $\mathbb{Y} = \{y_x : x \in X\}$ a family of points of X, and $\rho = \{\rho_n : n \geq 1\}$ a sequence of functions of X into $[0, 1]$. Then we define

$$\nu_{x,n,\rho}^{(\mathbb{M},\mathbb{Y})} = \rho_n(x)\nu_{x,n} + (1 - \rho_n(x))\epsilon_{y_x} \circ M_n,$$

where ϵ_t denotes the Dirac measure at t, and also define the mapping

$$\pi_{n,\mathbb{P}} : X^n \to X \quad \text{by} \quad (x_1, x_2, \cdots, x_r) \mapsto \sum_{j=1}^{n} p_{nj} x_j.$$

For a given $f \in C(X)$, we define

$$B_n(f)(x) = B_{n,\mathbb{P},\rho}^{(\nu^{(\mathsf{M})},\Upsilon)}(f)(x) = \int_{X^n} f \circ \pi_{n,\mathbb{P}}\, d \bigotimes_{1 \le j \le n} \nu_{x,j,\rho}^{(\mathsf{M},\Upsilon)} \quad (x \in X),$$

which is called the n-th Bernstein Lototsky-Schnabl function of f on X with respect to $\nu^{\mathsf{M}}, \mathbb{P}, \Upsilon$ and ρ ([49], cf. [18], [22], [23], [62]).

For any $f \in C(X)$, we define

$$H_n(f)(x) = \nu_{x,n}(f) \qquad (x \in X, n \ge 1).$$

Obviously, H_n is a positive linear operator of $C(X)$ into $B(X)$ with $H_n(1_X) = 1_X$. If $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G and $\alpha \in D$, then we define

$$\mu_\alpha(g_1, \cdots, g_m) = \left(\sup_{\lambda \in \Lambda} \left\| \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \sum_{k \ge 1} p_{j_n k}^2 \rho_k \sum_{i=1}^{m} (H_k(g_i^2) - g_i^2) \right\| \right)^{1/2}.$$

Hereafter, let f be a function in $C(X)$ having the property (MVP) associated with the system (6). Now take

$$k_n = 1 \qquad (n = 0, 1, 2, \cdots), \quad L_n = B_n \qquad (n = 1, 2, \cdots),$$

and define the operators $T_{\alpha,\lambda}$ by (29). Then we have the following.

THEOREM 3. *Suppose that $M_n(g) = g$ for all $n \ge 1$ and all $g \in G$. Then the following statements hold:*

(a) *If $y_x = x$ for every $x \in X$, then for all $\alpha \in D$,*

$$\|T_\alpha(f) - f\|_\Lambda \le \sum_{i=1}^{r} \Psi_\alpha(f_i, h_i), \tag{36}$$

where

$$\Psi_\alpha(f_i, h_i) = \inf\{(1 + \epsilon^{-1})\mu_\alpha(h_i)\omega(f_i; g_1, \cdots, g_m, \epsilon\mu_\alpha(g_1, \cdots, g_m)) :$$
$$\epsilon > 0, g_1, \cdots, g_m \in G, \mu_\alpha(g_1, \cdots, g_m) > 0, m = 1, 2, \cdots\}.$$

(b) *If $\rho_n = 1_X$ for all $n \geq 1$, then (36) holds with*

$$\mu_\alpha(h_i) = \left(\sup_{\lambda\in\Lambda}\left\|\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \sum_{k\geq 1} p_{j_n k}^2 (H_k(h_i^2) - h_i^2)\right\|\right)^{1/2}$$

and

$$\mu_\alpha(g_1, \cdots, g_m) = \left(\sup_{\lambda\in\Lambda}\left\|\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \sum_{k\geq 1} p_{j_n k}^2 \sum_{i=1}^{m}(H_k(g_i^2) - g_i^2)\right\|\right)^{1/2}.$$

PROOF: Assume that $y_x = x$ for all $x \in X$. Then, by [49; Lemma 4], it can be seen that $T_{\alpha,\lambda}(g) = g$ and

$$\mu^{(2)}(T_{\alpha,\lambda}, g) = T_{\alpha,\lambda}(g^2) - g^2 = \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \sum_{k\geq 1} p_{j_n k}^2 \rho_k (H_k(g^2) - g^2)$$

for all $\alpha \in D, \lambda \in \Lambda$ and all $g \in G$. Therefore, the desired estimate (36) follows from Corollary 2. The proof of Part (b) is similar.

If $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G, then we define

$$\theta_n(g_1, \cdots, g_m) = \left\|\sum_{j\geq 1} p_{nj}^2 \rho_j \sum_{i=1}^{m}(H_j(g_i^2) - g_i^2)\right\|^{1/2}.$$

COROLLARY 5. *Let M be as in Theorem 3. Then the following assertions hold:*

(a) *If $y_x = x$ for every $x \in X$, then for all $n \geq 1$,*

$$\|B_n(f) - f\| \leq \sum_{i=1}^{r} \psi_n(f_i, h_i), \tag{37}$$

where

$$\psi_n(f_i, h_i) = \inf\{(1+\epsilon^{-1})\theta_n(h_i)\omega(f_i; g_1, \cdots, g_m, \epsilon\theta_n(g_1, \cdots, g_m)) :$$

$$\epsilon > 0, g_1, \cdots, g_m \in G, \theta_n(g_1, \cdots, g_m) > 0, m = 1, 2, \cdots\}.$$

(b) *If* $\rho_n = 1_X$ *for all* $n \geq 1$, *then (37) holds with*

$$\theta_n(h_i) = \left\| \sum_{j \geq 1} p_{nj}^2 (H_j(h_i^2) - h_i^2) \right\|^{1/2}$$

and

$$\theta_n(g_1, \cdots, g_m) = \left\| \sum_{j \geq 1} p_{nj}^2 \sum_{i=1}^{m} (H_j(g_i^2) - g_i^2) \right\|^{1/2}.$$

From now on we suppose that

$$M_n = I \quad (n \geq 1), \qquad y_x = x \quad (x \in X),$$
$$\rho_n = 1_X \quad (n \geq 1), \qquad \nu_{x,n} = \nu_x \quad (x \in X, n \geq 1),$$

where ν_x is a representing measure for x (i.e., an $I(G)$-representing measure for x) such that the mapping

$$x \mapsto H(f)(x) = \nu_x(f) = \int_X f \, d\nu_x$$

belongs to G for every $f \in C(X)$. Consequently, each B_n maps $C(X)$ into itself and $B_1 = H$ is a positive projection operator of $C(X)$ onto G (cf. [23; Proposition], [49; Remark 7]).

For any $f \in B(X)$ and $\delta > 0$, we define

$$\Omega(f, \delta) = \inf\left\{ (1 + \epsilon^{-1})\omega \left(f; g_1, \cdots, g_m, \delta\epsilon \left\| \sum_{i=1}^{m} (H(g_i^2) - g_i^2) \right\|^{1/2} \right) : \right.$$
$$\left. \epsilon > 0, g_1, \cdots, g_m \in G, \left\| \sum_{i=1}^{r} (H(g_i^2) - g_i^2) \right\| > 0, m = 1, 2, \cdots \right\}.$$

Let $\{m_\alpha : \alpha \in D\}$ be a net of positive integers. If $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of Markov operators on $C(X)$ and if L is a Markov operator on $C(X)$, then we define

$$\|L_\alpha^{m_\alpha}(f) - L(f)\|_\Lambda = \sup\{\|L_{\alpha,\lambda}^{m_\alpha}(f) - L(f)\| : \lambda \in \Lambda\}.$$

Now take

$$L_n = B_n \qquad (n = 1, 2, \cdots),$$

and defines the operators $T_{\alpha,\lambda}$ by (29). Then we have the following.

THEOREM 4. *Let $\{m_\alpha : \alpha \in D\}$ be a net of positive integers. Then for all $\alpha \in D$,*

$$(38) \qquad \|T_\alpha^{m_\alpha}(f) - f\|_\Lambda \le \tau_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \tau_\alpha),$$

where

$$\tau_\alpha = \left(\sup \left\{ 1 - \left(\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \left(1 - \sum_{i \ge 1} p_{j_n i}^2 \right)^{k_n} \right)^{m_\alpha} : \lambda \in \Lambda \right\} \right)^{1/2},$$

and

$$(39) \qquad \|T_\alpha^{m_\alpha}(f) - H(f)\|_\Lambda \le v_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \theta_\alpha),$$

where

$$\theta_\alpha = \left(\sup \left\{ \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \left(1 - \sum_{i \ge 1} p_{j_n i}^2 \right)^{k_n} : \lambda \in \Lambda \right\} \right)^{m_\alpha/2}.$$

PROOF: By the induction on the degree k of iteration of B_n, it can be verified that $\{B_n^k\}$ is of type

$$\left[H; 1 - \left(1 - \sum_{j \ge 1} p_{nj}^2 \right)^k \right] \qquad (n, k = 1, 2, \cdots).$$

Therefore, $\{T_{\alpha,\lambda}\}$ is of type

$$\left[H; 1 - \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \left(1 - \sum_{i \ge 1} p_{j_n i}^2 \right)^{k_n} \right] \qquad (\alpha \in D, \lambda \in \Lambda),$$

and so the desired result follows from Corollary 4.

COROLLARY 6. *For all $n \in \mathbb{N}$,*

$$\|B_{j_n}^{k_n}(f) - f\| \le \epsilon_n \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \epsilon_n)$$

$$\le \left(k_n \sum_{i\ge 1} p_{j_n i}^2\right)^{1/2} \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega\left(f_i, \left(k_n \sum_{i\ge 1} p_{j_n i}^2\right)^{1/2}\right),$$

where

$$\epsilon_n = \left(1 - \left(1 - \sum_{i\ge 1} p_{j_n i}^2\right)^{k_n}\right)^{1/2},$$

and

$$\|B_{j_n}^{k_n}(f) - H(f)\| \le \delta_n \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \delta_n),$$

where

$$\delta_n = \left(1 - \sum_{i\ge 1} p_{j_n i}^2\right)^{k_n/2}.$$

COROLLARY 7. *Let $D = \mathbb{N} \setminus \{0\}$, $\Lambda = \mathbb{N}$ and we define*

$$T_{\alpha,\lambda} = \frac{1}{\alpha} \sum_{n=\lambda}^{\alpha+\lambda-1} B_{j_n}^{k_n} \qquad (\alpha \in D, \lambda \in \Lambda).$$

Then for all $\alpha \in D$, (38) and (39) hold with

$$(40) \quad \tau_\alpha = \left(\sup\left\{1 - \left(\frac{1}{\alpha} \sum_{n=\lambda}^{\alpha+\lambda-1} \left(1 - \sum_{i\ge 1} p_{j_n i}^2\right)^{k_n}\right)^{m_\alpha} : \lambda \in \Lambda\right\}\right)^{1/2}$$

and

$$(41) \quad \theta_\alpha = \left(\sup\left\{\frac{1}{\alpha} \sum_{n=\lambda}^{\alpha+\lambda-1} \left(1 - \sum_{i\ge 1} p_{j_n i}^2\right)^{k_n} : \lambda \in \Lambda\right\}\right)^{m_\alpha/2},$$

respectively.

REMARK 5: In particular, if $\{m_\alpha\} = \{1\}$, then (40) and (41) reduce to

$$\tau_\alpha = \left(\sup\left\{\frac{1}{\alpha}\sum_{n=\lambda}^{\alpha+\lambda-1}\left(1-\left(1-\sum_{i\geq 1} p_{j_n i}^2\right)^{k_n}\right) : \lambda \in \Lambda\right\}\right)^{1/2}$$

and

$$\theta_\alpha = \left(\sup\left\{\frac{1}{\alpha}\sum_{n=\lambda}^{\alpha+\lambda-1}\left(1-\sum_{i\geq 1} p_{j_n i}^2\right)^{k_n} : \lambda \in \Lambda\right\}\right)^{1/2},$$

respectively and Corollary 7 gives the estimate on the degree of almost convergence (F-summability) (in the sense of Lorentz [32]) of $\{B_{j_n}^{k_n} : n \in \mathbb{N}\}$.

Let $\{n_\alpha : \alpha \in D\}$ be a net of non-negative integers and $\{t_\alpha : \alpha \in D\}$ a net of numbers in the unit open interval $(0,1)$. If L is a Markov operator on $C(X)$, then for any $f \in C(X)$ we define

$$\sigma_{\alpha,i}(L;f) = \frac{1}{n_\alpha + 1}\sum_{j=0}^{n_\alpha} L^{i+j}(f) \qquad (\alpha \in D, i \in \mathbb{N})$$

and

$$A_{\alpha,i}(L;f) = (1-t_\alpha)\sum_{j=0}^{\infty} t_\alpha^j L^{i+j}(f) \qquad (\alpha \in D, i \in \mathbb{N}).$$

Note that if $\{L\}$ is of type $[T;x]$, then $\{\sigma_{\alpha,i}(L;\cdot)\}$ and $\{A_{\alpha,i}(L;\cdot)\}$ are of types

$$\left[H; 1 - \frac{(1-x)^i(1-(1-x)^{n_\alpha+1})}{x(n_\alpha+1)}\right]$$

and

$$\left[H; 1 - \frac{(1-t_\alpha)(1-x)^i}{1-t_\alpha(1-x)}\right],$$

respectively. Therefore, in view of this fact, making use of Corollary 4 we have the following quantitative ergodic type theorem for iterations of the discrete Cesàro and Abel means of the Bernstein-Lototsky-Schnabl operators.

THEOREM 5. *Let $m, j \geq 1$ be fixed, and set*

$$\beta = \beta(m,j) = \Big(1 - \sum_{i\geq 1} p_{mi}^2\Big)^j .$$

Let $\{k_\alpha : \alpha \in D\}$ be a net of positive integers and $\Lambda = \mathbb{N}$. Then the following statements hold:

(a) *For all $\alpha \in D$,*

$$\|\sigma_\alpha^{k_\alpha}(B_m^j; f) - H(f)\|_\Lambda \leq x_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2}\Omega(f_i, x_\alpha),$$

where

$$x_\alpha = \left(\frac{1-\beta^{n_\alpha+1}}{(1-\beta)(n_\alpha+1)}\right)^{k_\alpha/2}. \tag{42}$$

(b) *For all $\alpha \in D$,*

$$\|A_\alpha^{k_\alpha}(B_m^j; f) - H(f)\|_\Lambda \leq y_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2}\Omega(f_i, y_\alpha),$$

where

$$y_\alpha = \left(\frac{1-t_\alpha}{1-\beta t_\alpha}\right)^{k_\alpha/2}. \tag{43}$$

In particular, for the Bernstein operators on $C(\Delta_r)$ we have:

COROLLARY 8. *Let $m, j \geq 1$ be fixed. Let x_α and y_α be given by (42) and (43) with $\beta = \beta(m,j) = (1-1/m)^j$, respectively. Let $\{k_\alpha : \alpha \in D\}$ be a net of positive integers and $\Lambda = \mathbb{N}$. Then for all $f \in C^{(1)}(\Delta_r)$ and all $\alpha \in D$,*

$$\|\sigma_\alpha^{k_\alpha}(B_m^j; f) - B_1(f)\|_\Lambda \leq \frac{r}{2}\left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) x_\alpha \sum_{i=1}^{r} \omega(f_i, x_\alpha)$$

$$\leq \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right) x_\alpha \sum_{i=1}^{r} \omega(f_i, x_\alpha)$$

and

$$\|A_\alpha^{k_\alpha}(B_m^j; f) - B_1(f)\|_\Lambda \leq \frac{r}{2}\left(1+\left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) y_\alpha \sum_{i=1}^{r} \omega(f_i, y_\alpha)$$

$$\leq \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right) y_\alpha \sum_{i=1}^{r} \omega(f_i, y_\alpha),$$

where f_i denotes the i-th partial derivative of f given by (8).

We also note that the corresponding result of Corollary 8 holds for the Bernstein operators on $C(\mathbb{I}_r)$.

Finally, we restrict ourselves to the case where $\mathbb{P} = (p_{nj})_{n,j\geq 1}$ is the arithmetic Toeplitz matrix, i.e.,

$$p_{nj} = \frac{1}{n} \quad (n \geq 1, j = 1, 2, \cdots, n), \qquad p_{nj} = 0 \quad (j > n).$$

In [46] we showed that there exists a unique strongly continuous semigroup $\{S(t) : t \geq 0\}$ of Markov operators on $C(X)$ such that for every $f \in C(X)$ and for every sequence $\{m_n\}$ of positive integers with $\lim_{n\to\infty} m_n/n = t$, $t \geq 0$,

$$\lim_{n\to\infty} \|B_n^{m_n}(f) - S(t)(f)\| = 0$$

and

$$\lim_{n\to\infty} \left\| \frac{1}{m_n+1} \sum_{i=0}^{m_n} B_n^i(f) - \int_0^1 S(tu)(f)\,du \right\| = 0.$$

Let $\{k_n : n \in \mathbb{N}\}$ be a sequence of non-negative integers and $\{t_n : n \in \mathbb{N}\}$ a sequence of non-negative real numbers. For any $f \in C(X)$, we define

$$S_{\alpha,\lambda}(f) = \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} S(t_n)^{k_n}(f) = \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} S(k_n t_n)(f),$$

which converges in $C(X)$. Then we have the following.

THEOREM 6. *Let $\{m_\alpha : \alpha \in D\}$ be a net of positive integers. Then for all $\alpha \in D$,*

$$\|S_\alpha^{m_\alpha}(f) - f\|_\Lambda \le \tau_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \tau_\alpha),$$

where

$$\tau_\alpha = \left(\sup\left\{1 - \left(\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \exp(-k_n t_n)\right)^{m_\alpha} : \lambda \in \Lambda\right\}\right)^{1/2}$$

and

$$\|S_\alpha^{m_\alpha}(f) - H(f)\|_\Lambda \le \theta_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, \theta_\alpha),$$

where

$$\theta_\alpha = \left(\sup\left\{\sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \exp(-k_n t_n) : \lambda \in \Lambda\right\}\right)^{m_\alpha/2}.$$

PROOF: From the proof of [46; Theorem 4], $\{S(t)\}$ is of type $[H; 1 - \exp(-t)]$. Therefore, $\{S_{\alpha,\lambda}\}$ is of type

$$\left[H; 1 - \sum_{n=0}^{\infty} a_{\alpha,n}^{(\lambda)} \exp(-k_n t_n)\right],$$

and so the desired result follows from Corollary 4.

In particular, for the semigroup induced by the Bernstein operators on $C(\Delta_r)$ we have:

COROLLARY 9. *Let $\tau_\alpha, \theta_\alpha$ and $\{m_\alpha\}$ be as in Theorem 6. Then for all $f \in C^{(1)}(\Delta_r)$ and all $\alpha \in D$,*

$$\|S_\alpha^{m_\alpha}(f) - f\|_\Lambda \le \frac{r}{2}\left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) \tau_\alpha \sum_{i=1}^{r} \omega(f_i, \tau_\alpha)$$

$$\leq \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right)\tau_\alpha \sum_{i=1}^{r} \omega(f_i,\tau_\alpha)$$

and

$$\|S_\alpha^{m_\alpha}(f)-H(f)\|_\Lambda \leq \frac{r}{2}\left(1+\left\|\sum_{i=1}^{r}(e_i-e_i^2)\right\|^{1/2}\right)\theta_\alpha \sum_{i=1}^{r}\omega(f_i,\theta_\alpha)$$

$$\leq \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right)\theta_\alpha \sum_{i=1}^{r}\omega(f_i,\theta_\alpha),$$

where f_i denotes the i-th partial derivative of f given by (8).

THEOREM 7. *Let $t \geq 0$ be fixed. Let $\{k_\alpha : \alpha \in D\}$ be a net of positive integers and $\Lambda = \mathbb{N}$. Then for all $\alpha \in D$,*

$$\|\sigma_\alpha^{k_\alpha}(S(t);f)-H(f)\|_\Lambda \leq x_\alpha \sum_{i=1}^{r}\|H(h_i^2)-h_i^2\|^{1/2}\Omega(f_i,x_\alpha),$$

where

$$x_\alpha = \left\{\frac{1-\exp(-t(n_\alpha+1))}{(1-\exp(-t))(n_\alpha+1)}\right\}^{k_\alpha/2}$$

and

$$\|A_\alpha^{k_\alpha}(S(t);f)-H(f)\|_\Lambda \leq y_\alpha \sum_{i=1}^{r}\|H(h_i^2)-h_i^2\|^{1/2}\Omega(f_i,y_\alpha),$$

where

$$y_\alpha = \left(\frac{1-t_\alpha}{1-t_\alpha\exp(-t)}\right)^{k_\alpha/2}.$$

PROOF: Since $\{S(t)\}$ is of type $[H;1-\exp(-t)]$, $\{\sigma_{\alpha,i}(S(t);\cdot)\}$ and $\{A_{\alpha,i}(S(t);\cdot)\}$ are of types

$$\left[H;1-\frac{\exp(-it)(1-\exp(-t(n_\alpha+1)))}{(1-\exp(-t))(n_\alpha+1)}\right]$$

and

$$\left[H;1-\frac{\exp(-it)(1-t_\alpha)}{1-t_\alpha\exp(-t)}\right],$$

respectively. Thus the desired result follows from Corollary 4.

In particular, for the semigroup induced by the Bernstein operators on $C(\Delta_r)$ we have:

COROLLARY 10. *Let* $x_\alpha, y_\alpha, \{k_\alpha\}$ *and* Λ *be as in Theorem 7. Then for all* $f \in C^{(1)}(\Delta_r)$ *and all* $\alpha \in D$,

$$\|\sigma_\alpha^{k_\alpha}(S(t);f)-B_1(f)\|_\Lambda \le \frac{r}{2}\left(1+\left\|\sum_{i=1}^{r}(e_i-e_i^2)\right\|^{1/2}\right) x_\alpha \sum_{i=1}^{r}\omega(f_i,x_\alpha)$$

$$\le \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right) x_\alpha \sum_{i=1}^{r}\omega(f_i,x_\alpha)$$

and

$$\|A_\alpha^{k_\alpha}(S(t);f)-B_1(f)\|_\Lambda \le \frac{r}{2}\left(1+\left\|\sum_{i=1}^{r}(e_i-e_i^2)\right\|^{1/2}\right) y_\alpha \sum_{i=1}^{r}\omega(f_i,y_\alpha)$$

$$\le \frac{r}{2}\left(1+\frac{\sqrt{r}}{2}\right) y_\alpha \sum_{i=1}^{r}\omega(f_i,y_\alpha),$$

where f_i *denotes the i-th partial derivative of* f *given by (8).*

We take

$$W(t)=S(t) \quad (t\ge 0), \qquad \Phi_\lambda(t)=t+c_\lambda \quad (t\ge 0, \lambda\in\Lambda),$$

where $\{c_\lambda : \lambda \in \Lambda\}$ is a family of non-negative real numbers. Let $\{C_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ and $\{R_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be the families of operators defined by (30) and (31), respectively. Then we have the following quantitative ergodic type theorem for iterations of continuous Cesàro and Abel means of the semigroup $\{S(t) : t \ge 0\}$.

THEOREM 8. *Let* $\{k_\alpha : \alpha \in D\}$ *be a net of positive integers and put* $c=\sup\{\exp(-c_\lambda) : \lambda \in \Lambda\}$. *Then for all* $\alpha \in D$,

$$\|C_\alpha^{k_\alpha}(f)-H(f)\|_\Lambda \le a_\alpha \sum_{i=1}^{r}\|H(h_i^2)-h_i^2\|^{1/2}\Omega(f_i,a_\alpha),$$

where

$$a_\alpha=\left\{\frac{c(1-\exp(-v_\alpha))}{v_\alpha}\right\}^{k_\alpha/2}$$

and

$$\|R_\alpha^{k_\alpha}(f) - H(f)\|_\Lambda \le b_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, b_\alpha),$$

where

$$b_\alpha = \left(\frac{cv_\alpha}{v_\alpha + 1}\right)^{k_\alpha/2}.$$

PROOF: Since $\{S(t)\}$ is of type $[H; 1 - \exp(-t)]$, $\{C_{\alpha,\lambda}\}$ and $\{R_{\alpha,\lambda}\}$ are of types

$$\left[H; 1 - \frac{\exp(-c_\lambda)(1 - \exp(-v_\alpha))}{v_\alpha}\right]$$

and

$$\left[H; 1 - \frac{v_\alpha \exp(-c_\lambda)}{v_\alpha + 1}\right],$$

respectively. Hence the desired result follows from Corollary 4.

In particular, for the semigroup induced by the Bernstein operators on $C(\Delta_r)$ we have:

COROLLARY 11. *Let $\{a_\alpha\}, \{b_\alpha\}$ and $\{k_\alpha\}$ be as in Theorem 8. Then for all $f \in C^{(1)}(\Delta_r)$ and all $\alpha \in D$,*

$$\|C_\alpha^{k_\alpha}(f) - B_1(f)\|_\Lambda \le \frac{r}{2}\left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) a_\alpha \sum_{i=1}^{r} \omega(f_i, a_\alpha)$$

$$\le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right) a_\alpha \sum_{i=1}^{r} \omega(f_i, a_\alpha)$$

and

$$\|R_\alpha^{k_\alpha}(f) - B_1(f)\|_\Lambda \le \frac{r}{2}\left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) b_\alpha \sum_{i=1}^{r} \omega(f_i, b_\alpha)$$

$$\le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right) b_\alpha \sum_{i=1}^{r} \omega(f_i, b_\alpha),$$

where f_i denotes the i-th partial derivative of f given by (8).

REMARK 6: Let $\Lambda = \{0\}$ and $c_0 = 0$. Thus for all $f \in C(X)$ and all $\alpha \in D$, we have

$$C_{\alpha,0} = \frac{1}{v_\alpha} \int_0^{v_\alpha} S(t)(f)\, dt$$

and

$$R_{\alpha,0}(f) = v_\alpha \int_0^{\infty} \exp(-v_\alpha t) S(t)(f)\, dt.$$

For these operators, we have the following: For all $\alpha \in D$,

$$\|C_{\alpha,0}^{k_\alpha}(f) - f\| \leq a_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, a_\alpha), \tag{44}$$

where

$$a_\alpha = \left(1 - \left(\frac{1 - \exp(-v_\alpha)}{v_\alpha}\right)^{k_\alpha}\right)^{1/2},$$

and

$$\|R_{\alpha,0}^{k_\alpha}(f) - f\| \leq b_\alpha \sum_{i=1}^{r} \|H(h_i^2) - h_i^2\|^{1/2} \Omega(f_i, b_\alpha), \tag{45}$$

where

$$b_\alpha = \left(1 - \left(1 - \frac{1}{v_\alpha + 1}\right)^{k_\alpha}\right)^{1/2}.$$

Specially, in case of the semigroup induced by the Bernstein operators on $C(\Delta_r)$, (44) and (45) reduce to

$$\|C_{\alpha,0}^{k_\alpha}(f) - f\| \leq \frac{r}{2} \left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) a_\alpha \sum_{i=1}^{r} \omega(f_i, a_\alpha)$$

$$\leq \frac{r}{2} \left(1 + \frac{\sqrt{r}}{2}\right) a_\alpha \sum_{i=1}^{r} \omega(f_i, a_\alpha)$$

and

$$\|R^{k_\alpha}_{\alpha,0}(f) - f\| \le \frac{r}{2}\left(1 + \left\|\sum_{i=1}^{r}(e_i - e_i^2)\right\|^{1/2}\right) b_\alpha \sum_{i=1}^{r} \omega(f_i, b_\alpha)$$

$$\le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right)\sum_{i=1}^{r}\omega(f_i, b_\alpha),$$

respectively,where f_i denotes the i-th partial derivative of f given by (8).

REMARK 7: Applying Corollary 4, all the corresponding results of this section are also obtained for the Bernstein-Schnabl operators due to Altomare [1](cf. [2], [7]), the generalized Stancu-Mühlbach operators of Campiti [12] (cf. [40]) and the strongly continuous semigroups of Markov operators induced by them (cf. [3], [4], [7], [13]). We omit the details.

We refer to [20], [21] and [64] for detailed references on the other contributions to approximation of functions by Bernstein-type operators (cf. [5], [6]).

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