# Approximation of the Korovkin type for vector-valued functions

Ryukyu Math. J., 7(1994), 65-81

# APPROXIMATION OF THE KOROVKIN TYPE FOR VECTOR-VALUED FUNCTIONS

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**Abstract.** Korovkin closures for vector-valued functions are characterized by means of the technique of envelopes as well as the representing operators.

## 1. Introduction

Let $X$ be a compact Hausdorff space and $E$ a Dedekind complete normed vector lattice. Concerning the general notions and terminology needed from the theory of normed vector lattices, we refer to [17] (cf. [1], [10]). Let $B(X, E)$ denote the normed vector lattice of all $E$-valued bounded functions on $X$ with the usual pointwise addition, scalar multiplication , ordering and the supremum norm $\|\cdot\|$. We shall use the same symbol $\|\cdot\|$ for underlying norms. $C(X, E)$ denotes the closed sublattice of $B(X, E)$ consisting of all $E$-valued continuous functions on $X$. In the case when $E$ is equal to the real line $\mathbb{R}$, we simply write $B(X)$ and $C(X)$ instead of $B(X, E)$ and $C(X, E)$, respectively.

For any $a \in E$ and $v \in B(X)$, the function $va$ is defined by $(va)(x) = v(x)a$ for all $x \in X$. Also, for any $v \in B(X)$ and $f \in B(X, E)$, we define $(vf)(x) = v(x)f(x)$ for all $x \in X$. Clearly, $va$ and $vf$ belong to $B(X, E)$, and $\|va\| = \|v\|\|a\|$ and $\|vf\| \leq \|v\|\|f\|$. If $a \in E$, $v \in C(X)$ and $f \in C(X, E)$, then $va$ and $vf$ belong to $C(X, E)$. $C(X) \otimes E$ stands for the linear subspace of $C(X, E)$ consisting of all finite sums of functions of the form $va$, where $v \in C(X)$ and $a \in E$.

In this paper we suppose that $E$ contains an element $e \in E$ such that $e > 0, \|e\| = 1$ and $|a| \leq \|a\|e$ for all $a \in E$. We call $e$ the normal order unit of $E$. For instance, $E = \mathbb{R}$ or $E = C(Y, \mathbb{R})$, where $Y$ is a compact Hausdorff space always, has a normal order unit. We define $\rho(x) = e$ for all $x \in X$. Note that $\rho$ is the normal order unit of $B(X, E)$. Let $A(X, E)$ be a sublattice of $C(X, E)$ which contains $\rho$.

Received November 30, 1994.

The purpose of this paper is to study Korovkin closures for positive linear operators of $A(X, E)$ into $B$, where $B$ is equal to $E$ or $B(X, E)$. For this aim, we will make use of the technique of envelopes as well as the representing operators. Actually, we extend the results of Bauer [4](cf. [3], [5]) to the context of functions taking value in an arbitrary Dedekind complete normed vector lattice which contains a normal order unit.

We do not state the classical cases in detail. They have been treated in many places and we globally refer to [7] on the subject and to [2] and [15] for detailed references and summaries on the several other contributions to the area of of Korovkin type approximation theory, which is more recently dealt with in the structures so-called locally convex cones in [8].

## 2. M-Envelopes and M-Affine Functions

Let $M$ be a linear subspace of $A(X, E)$ which contains $\rho$. For a function $f \in A(X, E)$ and a point $x \in X$, we put

$$M^*(f, x) = \{h(x) : f \leq h, h \in M\}$$

and

$$M_*(f, x) = \{h(x) : h \leq f, h \in M\}.$$

Since $|f| \leq \|f\|\rho$, $-\|f\|e \in M_*(f, x)$ and $\|f\|e \in M^*(f, x)$. For a function $f \in A(X, E)$, we define

$$f^*(x) = \inf M^*(f, x) \qquad (x \in X)$$

and

$$f_*(x) = \sup M_*(f, x) \qquad (x \in X),$$

which are called the upper and lower $M$-envelope of $f$, respectively. These functions have the following properties:

LEMMA 1. *Let $f, g \in A(X, E)$ and $\xi \in \mathbb{R}$.*

(i) $-\|f\|\rho \leq f_* \leq f \leq f^* \leq \|f\|\rho$.
(ii) *If $f \leq g$, then $f^* \leq g^*$ and $f_* \leq g_*$.*
(iii) $(f+g)^* \leq f^* + g^*$, $f_* + g_* \leq (f+g)_*$.
(iv) *If $\xi \geq 0$, then $(\xi f)^* = \xi f^*$ and $(\xi f)_* = \xi f_*$.*
(v) *If $\xi \leq 0$, then $(\xi f)^* = \xi f_*$. In particular, $(-f)^* = -f_*$.*

PROOF: This follows immediately from the definition.

A function $f \in A(X, E)$ is called $M$-affine, if $f^*$ is equal to $f_*$. Obviously, $f$ is $M$-affine if and only if $f^* = f = f_*$. For a point $x \in X$, we define

$$\hat{M}_x = \{f \in A(X, E) : f_*(x) = f^*(x)\},$$

which is a linear subspace of $A(X, E)$ containing $M$. Also, we define

$$\hat{M} = \bigcap_{x \in X} \hat{M}_x,$$

which is a linear subspace of $A(X, E)$ containing $M$. Plainly, we have

$$\hat{M} = \{f \in A(X, E) : f_* = f^*\} = \{f \in A(X, E) : f_* = f = f^*\},$$

which can be characterized in the sense of the following:

LEMMA 2. *Let $f \in \hat{M}$. Then for any $\epsilon > 0$, there exist finite subsets $\{g_1, g_2, \cdots, g_m\}$ and $\{h_1, h_2, \cdots, h_m\}$ of $M$ such that*

$$\underline{g} \le f \le \overline{h} \quad \text{and} \quad \overline{h} - \underline{g} < \epsilon\rho, \tag{1}$$

*where*

$$\underline{g} = \sup\{g_1, g_2, \cdots, g_m\} \quad \text{and} \quad \overline{h} = \inf\{h_1, h_2, \cdots, h_m\}.$$

PROOF: For each $x \in X$, since $f_*(x) = f(x) = f^*(x)$, there exist functions $g_x$ and $h_x \in M$ such that

$$g_x \le f \le h_x, \quad h_x(x) - g_x(x) < (\epsilon/2)e.$$

Therefore, since $h_x - g_x$ belongs to $C(X, E)$, there exists an open neighborhood $V_x$ of $x$ such that

$$h_x(y) - g_x(y) < \epsilon e \quad \text{for all } y \in V_x.$$

The family $\{V_x : x \in X\}$ is an open covering of $X$, and so it has a finite subcovering, say $\{V_{x_i} : i = 1, 2, \cdots, m\}$. Then the associated functions

$$g_i = g_{x_i}, \quad h_i = h_{x_i} \qquad (i = 1, 2, \cdots, m)$$

have the desired properties.

Let $A$ and $B$ be normed vector lattices, and let $L$ be a mapping of $A$ into $B$. $L$ is said to be increasing if $f, g \in A$ and $f \geq g$, then $L(f) \geq L(g)$. Also, $L$ is said to be positive if $f \in A$ and $f \geq 0$, then $L(f) \geq 0$. Clearly, if $L$ is linear, then $L$ is increasing if and only if $L$ is positive. Furthermore, if $A$ has a normal order unit $a$ and if $L$ is a positive linear operator of $A$ into $B$, then $L$ is bounded and $\|L\| = \|L(a)\|$.

From now on let $D$ be a direct set and let $\Lambda$ be an index set.

PROPOSITION 1. *Let $x \in X$. If $\{\mu_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of increasing mappings of $A(X, E)$ into $E$ satisfying*

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(k) - k(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda \tag{2}$$

*for all $k \in M$, then*

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(f) - f(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda \tag{3}$$

*for all $f \in \hat{M}_x$.*

PROOF: Let $g$ and $h$ be functions in $M$ such that $g \leq f \leq h$. Then for every $\epsilon > 0$, by (2) there exists an element $\alpha_0 \in D$ such that

$$\|\mu_{\alpha,\lambda}(g) - g(x)\| < \epsilon \quad \text{and} \quad \|\mu_{\alpha,\lambda}(h) - h(x)\| < \epsilon$$

for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$. Since

$$|\mu_{\alpha,\lambda}(u) - u(x)| \leq \|\mu_{\alpha,\lambda}(u) - u(x)\| e \qquad (\alpha \in D, \ \lambda \in \Lambda)$$

whenever $u$ belongs to $A(X, E)$ and

$$\mu_{\alpha,\lambda}(g) \leq \mu_{\alpha,\lambda}(f) \leq \mu_{\alpha,\lambda}(h) \qquad (\alpha \in D, \ \lambda \in \Lambda),$$

we conclude that for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$,

$$g(x) - \epsilon e < \mu_{\alpha,\lambda}(f) < h(x) + \epsilon e,$$

which yields

$$f_*(x) - \epsilon e \leq \mu_{\alpha,\lambda}(f) \leq f^*(x) + \epsilon e.$$

Thus we have

$$|\mu_{\alpha,\lambda}(f) - f(x)| \leq \epsilon e \qquad (\alpha \in D, \ \alpha \geq \alpha_0, \ \lambda \in \Lambda)$$

because of $f_*(x) = f(x) = f^*(x)$. Consequently, we obtain

$$\|\mu_{\alpha,\lambda}(f) - f(x)\| \leq \epsilon \qquad (\alpha \in D, \ \alpha \geq \alpha_0, \ \lambda \in \Lambda),$$

which implies (3).

COROLLARY 1. *Let* $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *be a family of increasing mappings of* $A(X,E)$ *into* $B(X,E)$. (a) Let $x \in X$. If

$$\lim_{\alpha} \|L_{\alpha,\lambda}(g)(x) - g(x)\| = 0 \quad \text{uniformly in } \lambda \in \Lambda$$

for all $g \in M$, then

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f)(x) - f(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $f \in \hat{M}_x$. (b) If

$$\lim_{\alpha} \|L_{\alpha,\lambda}(g) - g\| = 0 \quad \text{uniformly in } \lambda \in \Lambda$$

for all $g \in M$, then

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f)(x) - f(x)\| = 0 \quad \text{uniformly in } \lambda \in \Lambda$$

for all $x \in X$ and all $f \in \hat{M}$.

PROPOSITION 2. *If* $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *be a family of increasing mappings of* $A(X,E)$ *into* $B(X,E)$ *satisfying*

$$\lim_{\alpha} \|L_{\alpha,\lambda}(g) - g\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{4}$$

*for all* $g \in M$, *then*

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f) - f\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{5}$$

*for all* $f \in \hat{M}$.

PROOF: For any $\epsilon > 0$, by Lemma 2 there exist finite subsets $\{g_1, g_2, \cdots, g_m\}$ and $\{h_1, h_2, \cdots, h_m\}$ of $M$ satisfying (1). By (4), there exists an element $\alpha_0 \in D$ such that

$$\|L_{\alpha,\lambda}(g_i) - g_i\| \le \epsilon \text{ and } \|L_{\alpha,\lambda}(h_i) - h_i\| \le \epsilon \quad (i = 1, 2, \cdots, m) \tag{6}$$

for all $\alpha \in D, \alpha \ge \alpha_0$ and all $\lambda \in \Lambda$. Since for all $x \in X$

$$|L_{\alpha,\lambda}(k)(x) - k(x)| \le \|L_{\alpha,\lambda}(k)(x) - k(x)\| e \qquad (\alpha \in D, \ \lambda \in \Lambda)$$

whenever $k$ belongs to $A(X, E)$ and

$$L_{\alpha,\lambda}(g_i) \leq L_{\alpha,\lambda}(f) \leq L_{\alpha,\lambda}(h_i) \qquad (i = 1, 2, \cdots, m)$$

for all $\alpha \in D$ and all $\lambda \in \Lambda$, by (6) we conclude that

$$g_i(x) - \epsilon e < L_{\alpha,\lambda}(f)(x) < h_i(x) + \epsilon e \qquad (i = 1, 2, \cdots, m)$$

for all $\alpha \in D, \alpha \geq \alpha_0$ and all $\lambda \in \Lambda$. Hence, for all $x \in X$ we obtain

$$\underline{g}(x) - \epsilon e \leq L_{\alpha,\lambda}(f)(x) \leq \overline{h}(x) + \epsilon e \qquad (\alpha \in D, \ \alpha \geq \alpha_0, \ \lambda \in \Lambda),$$

which together with (1) gives

$$|L_{\alpha,\lambda}(f)(x) - f(x)| < 3\epsilon e \qquad (\alpha \in D, \ \alpha \geq \alpha_0, \ \lambda \in \Lambda),$$

and so

$$\|L_{\alpha,\lambda}(f)(x) - f(x)\| \leq 3\epsilon \qquad (\alpha \in D, \ \alpha \geq \alpha_0, \ \lambda \in \Lambda),$$

which implies

$$\|L_{\alpha,\lambda}(f) - f\| \leq 3\epsilon \qquad (\alpha \in D, \ \alpha \geq \alpha_0 \ \lambda \in \Lambda).$$

Thus (5) is proved.

REMARK 1: Suppose that $A(X, E)$ contains $C(X) \otimes E$. If $\hat{M} = A(X, E)$, then $M$ separates the points of $X$, *i.e.*, for any $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists a function $h \in M$ such that $h(x_1) \neq h(x_2)$.

In fact, by Urysohn's lemma there exists a function $g \in C(X)$ such that $g(x_1) = 0$ and $g(x_2) = 1$. Let $f = ge$. Then $f \in C(X) \otimes E \subseteq A(X, E) = \hat{M}$, and so there is a function $h \in M$ satisfying $h \leq f$ and $f(x_2) - e < h(x_2)$. Therefore we have $f(x_1) = 0 < h(x_2)$ and $h(x_1) \leq f(x_1)$. Hence $h(x_1) < h(x_2)$.

## 3. Choquet Boundaries

For a given point $x \in X$, a positive linear operator $\mu$ of $A(X, E)$ into $E$ is called an $M$-representing operator for $x$ if $\mu(g) = g(x)$ for every $g \in M$. For each $x \in X$, we define $\delta_x(f) = f(x)$ for every

$f \in A(X, E)$. This operator $\delta_x$ is called the evaluation operator at $x$. Evidently, $\delta_x(\rho) = e, \|\delta_x\| = 1$ and $\delta_x$ is always the $M$-representing operator for $x$. Let $\mathcal{R}_x(M)$ denote the set of all $M$-representing operators for $x$. For example, let $X = [0, 1]$ be the closed unit interval in $\mathbb{R}$, $0 \leq t < 1/2$ and let $M$ be a linear subspace of $A(X, E)$ spanned by $\{\rho, \rho_1\}$, where $\rho_1(x) = xe$ for every $x \in X$. Then $\mu = 1/2(\delta_t + \delta_{(1-t)})$ belongs to $\mathcal{R}_{1/2}(M)$ and $\mu \neq \delta_{1/2}$.

For $x \in X$ and $f \in A(X, E)$, we denote by $[f_*(x), f^*(x)]$ the order interval in $E$, i.e.,

$$[f_*(x), f^*(x)] = \{a \in E : f_*(x) \leq a \leq f^*(x)\}.$$

Then the following lemma give the close connection between the $M$-envelopes and the $M$-representing operators.

LEMMA 3. *Let $x \in X$ and let $f \in A(X, E)$. Then*

$$[f_*(x), f^*(x)] = \{\mu(f) : \mu \in \mathcal{R}_x(M)\}.$$

PROOF: If $f = 0$, then we have

$$[f_*(x), f^*(x)] = \{0\} = \{\mu(f) : \mu \in \mathcal{R}_x(m)\}.$$

Now let $f \neq 0$. Let $\mu \in \mathcal{R}_x(M)$, and let $g$ and $h$ be functions in $M$ such that $g \leq f \leq h$. Then

$$g(x) = \mu(g) \leq \mu(f) \leq \mu(h) = h(x),$$

which yields

$$f_*(x) \leq \mu(f) \leq f^*(x).$$

Conversely, let $a$ be an arbitrary element in $[f_*(x), f^*(x)]$, and let $V$ be the linear subspace of $A(X, E)$ spanned by $f$. We define

$$p(g) = g^*(x) \qquad \text{for every } g \in A(X, E)$$

and

$$\mu_0(\xi f) = \xi a \qquad \text{for every } \xi \in \mathbb{R}.$$

Then by Lemma 1, the mapping $p : A(X, E) \to E$ is sublinear and $\mu_0$ is a linear operator of $V$ into $E$ satisfying $\mu_0(g) \leq p(g)$ for all $g \in V$.

Therefore, by the vector-valued Hahn-Banach theorem ([1; Theorem 2.1]. cf [10; Theorem 1.5.4]) there exists a linear operator $\mu$ of $A(X,E)$ into $E$ such that $\mu(g) \le p(g)$ for all $g \in A(X,E)$ and $\mu(h) = \mu_0(h)$ for all $h \in V$. If $g \in A(X,E)$ and $g \le 0$, then Lemma 1 (ii) gives

$$\mu(g) \le p(g) = g^*(x) \le 0^*(x) = 0,$$

which implies that $\mu$ is positive. Furthermore, for every $h \in M$ we have

$$\mu(h) \le p(h) = h^*(x) = h_*(x) = h(x)$$

and

$$-\mu(h) = \mu(-h) \le p(-h) = (-h)^*(x) = -h(x),$$

and so $\mu(h) = h(x)$. Thus $\mu$ belongs to $\mathcal{R}_x(M)$ and $\mu(f) = \mu_0(f) = a$. The proof of the lemma is complete.

LEMMA 4. *Let $f \in A(X,E)$.*

(i) *Let $x \in X$. Then $f$ belongs to $\hat{M}_x$ if and only if $\mu(f) = \delta_x(f)$ for all $\mu \in \mathcal{R}_x(M)$.*

(ii) *$f$ belongs to $\hat{M}$ if and only if $\mu(f) = \delta_x(f)$ for all $x \in X$ and all $\mu \in \mathcal{R}_x(M)$.*

PROOF: This follows immediately from Lemma 3.

We define

$$\partial_M(X) = \{x \in X : \mathcal{R}_x(M) = \{\delta_x\}\},$$

which is called the Choquet boundary of $X$ with respect to $M$. This can be characterized as follows:

LEMMA 5. *A point $x \in X$ belongs to $\partial_M(X)$ if and only if $f_*(x) = f^*(x)$ for all $f \in A(X,E)$, i.e., $\hat{M}_x = A(X,E)$.*

PROOF: This is an immediate consequence of Lemma 4 (i).

PROPOSITION 3. *$\hat{M} = A(X,E)$ if and only if $\partial_M(X) = X$.*

PROOF: This follows from Lemma 4 (ii) and Lemma 5.

## 4. Korovkin Closures and Korovkin Spaces

Let $A$ and $B$ be normed vector lattices. Let $V$ be a linear subspace of $A$ and $T$ a positive linear operator of $A$ into $B$. Then we define $Kor(V,T)$ to be the set of all $f \in A$ such that if $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is a family of positive linear operators of $A$ into $B$ satisfying

$$\lim_{\alpha} \|T_{\alpha,\lambda}(g) - g\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $g \in V$, then

$$\lim_{\alpha} \|T_{\alpha,\lambda}(f) - f| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

$Kor(V,T)$ is called a Korovkin closure of $V$ with respect to $T$. $V$ is called a Korovkin space with respect to $T$ if $Kor(V,T)$ is identical with $A(X,E)$. If $V$ is spanned by a subset $S$ of $A$ and $Kor(V,T) = A(X,E)$, then $S$ is also called a Korovkin set with respect to $T$. Obviously, $Kor(V,T)$ is a linear subspace of $A$.

THEOREM 1. *Let $x \in X$. Then $x$ belongs to $\partial_M(X)$ if and only if $M$ is a Korovkin space with respect to $\delta_x$, i.e., $Kor(M,\delta_x) = A(X,E)$.*

PROOF: Suppose that $x \in \partial_M(X)$. Then by Lemma 5 we have $\hat{M}_x = A(X,E)$. Also, Proposition 1 implies $\hat{M}_x \subseteq Kor(M,\delta_x)$. Therefore $Kor(M,\delta_x)$ is equal to $A(X,E)$. Conversely, we assume that $Kor(M,\delta_x) = A(X,E)$ and let $\mu$ be an arbitrary element in $\mathcal{R}_x(M)$. Now for all $\alpha \in D$ and all $\lambda \in \Lambda$, we define $\mu_{\alpha,\lambda} = \mu$, which is a positive linear operator of $A(X,E)$ into $E$ satisfying

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(g) - \delta_x(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

Thus for every $f \in A(X,E)$,

$$\lim_{\alpha} \|\mu_{\alpha,\lambda}(f) - \delta_x(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

which yields $\mu = \delta_x$. Hence $x$ belongs to $\partial_M(X)$.

THEOREM 2. *If $\partial_M(X)$ is identical with $X$, then $M$ is a Korovkin space with respect to $I$, i.e., $Kor(M,I) = A(X,E)$, where $I$ denotes the identity operator on $A(X,E)$*

PROOF: By Propositions 2 and 3, we have

$$A(X,E) = \hat{M} \subseteq Kor(M,I) \subseteq A(X,E),$$

and so $Kor(M,I) = A(X,E)$.

In order to show the converse statement in Theorem 2, we assume that $X$ is a first countable, compact Hausdorff space and that $D$ is the set $\mathbb{N}$ of all natural numbers in the following arguments.

PROPOSITION 4. *$Kor(M,I)$ coincides with $\hat{M}$.*

PROOF: By Proposition 2 we have $\hat{M} \subseteq Kor(M,I)$. To show the converse inclusion, let $f$ be an arbitrary function in $Kor(M,I)$. Let $x \in X$ and $\mu \in \mathcal{R}_x(M)$. Since $X$ satisfies the first axiom of countability, there is a fundamental system $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of $x$ such that

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq V_{n+1} \supseteq \cdots .$$

For each $n \in \mathbb{N}$, by Urysohn's lemma there exists a function $f_n \in C(X)$ such that

$$0 \leq f_n(t) \leq 1 \ \ (t \in X), \ \ f_n(x) = 1, \ \text{ and } \ f_n(t) = 0 \ \ (t \in X \setminus V_n).$$

For each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, we define

$$T_{n,\lambda}(g) = f_n \mu(g) + (1 - f_n)g \qquad (g \in A(X,E)).$$

Then $\{T_{n,\lambda} : n \in \mathbb{N}, \lambda \in \Lambda\}$ is a family of positive linear operators of $A(X,E)$ into $B(X,E)$ satisfying

$$\lim_{n\to\infty} \|T_{n,\lambda}(h) - h\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $h \in M$. Therefore we have

$$\lim_{n\to\infty} \|T_{n,\lambda}(f) - f\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

In particular, there holds

$$\lim_{n\to\infty} \|T_{n,\lambda}(f)(x) - f(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

which gives $\mu(f) = \delta_x(f)$, since $T_{n,\lambda}(f)(x) = \mu(f)$ for all $n \in \mathbb{N}$ and all $\lambda \in \Lambda$. Hence, by Lemma 4 (ii) $f$ belongs to $\hat{M}$, and so we have $\hat{M} \supseteq Kor(M,I)$.

THEOREM 3. *The following statements are equivalent:*

(i) $\partial_M(X) = X$.

(ii) $Kor(M, I) = A(X, E)$.

(iii) $\hat{M} = A(X, E)$.

PROOF: By Theorem 2, (i) implies (ii). By Proposition 3, (ii) implies (iii). Also, by Proposition 2 (iii) implies (i).

## 5. Korovkin Sets in $C(X, E)$

Here we consider the case of $A(X, E) = C(X, E)$, and let $M$ be a linear subspace of $C(X, E)$ which contains $1_X a$ for for all $a \in E$, where $1_X$ denote the normal order unit in $C(X)$ defined by $1_X(x) = 1$ for all $x \in X$.

LEMMA 6. *$C(X) \otimes E$ is dense in $C(X, E)$.*

PROOF: This is an immediate consequence of [16; Theorem 1.15], since $C(X)$ separates of points of $X$.

LEMMA 7. *Let $x \in X$. If there exists a function $g_x \in C(X)$ such that*

$$(7) \qquad g_x \geq 0, \;\; g_x(x) = 0 \;\; and \;\; g_x(t) > 0 \quad for\ all\ t \in X\ with\ t \neq x$$

*and*

$$(8) \qquad g_x a \in M \quad for\ all\ a \in E,$$

*then $x$ belongs to $\partial_M(X)$.*

PROOF: Let $\mu \in \mathcal{R}_x(M)$. Let $\epsilon > 0$ be given and let $v$ be an arbitrary function in $C(X)$ satisfying $v(x) = 0$. Then there exists an open neighborhood $V_x$ of $x$ such that $|v(t)| \leq \epsilon$ for all $t \in V_x$. Let $F = X \backslash V_x$, and put

$$c = \min\{g(t) : t \in F\}$$

and

$$C = \max\{|v(t)| : t \in F\}.$$

Then we have

$$(9) \qquad |v(t)| \leq \epsilon + (C/c)g(t)$$

for all $t \in X$. Let $a \in E$. Let $a^+$ and $a^-$ denote the positive part and the negative part of $a$, respectively. Then it follows from (9) that

$$|va^+| \le \epsilon 1_X a^+ + (C/c)ga^+,$$

and so

$$|\mu(va^+)| \le \epsilon\mu(1_X a^+) + (C/c)\mu(ga^+) = \epsilon 1_X(x)a^+ + (C/c)g(x)a^+ = \epsilon a^+.$$

Similarly, we have

$$|\mu(va^-)| \le \epsilon a^-.$$

Therefore, we get

$$|\mu(va)| = |\mu(v(a^+ - a^-))| = |\mu(va^+) - \mu(va^-)| \le |\mu(va^+)| + |\mu(va^-)| \le \epsilon(a^+ + a^-) = \epsilon|a|,$$

which implies

$$\|\mu(va)\| \le \epsilon\|a\|.$$

Consequently, we have $\mu(va) = 0$ for all $a \in E$ and all $v \in C(X)$ satisfying $v(x) = 0$, since $\epsilon$ is an arbitrary positive real number. Now let $w \in C(X)$ and take $v = w - w(x)1_X$. Then $v$ belongs to $C(X)$ and $v(x) = 0$, and so

$$\mu((w - w(x)1_X)a) = 0 \qquad (a \in E),$$

which gives $\mu(wa) = \delta_x(wa)$. Thus we conclude that $\mu(h) = \delta_x(h)$ for all $h \in C(X) \otimes E$, and hence Lemma 6 establishes that $\mu(f) = \delta_x(f)$ for every $f \in C(X, E)$, i.e., $\mu = \delta_x$. Therefore, $x$ belongs to $\partial_M(X)$.

As an immediate of consequence of Lemma 7, Theorems 1 and 2, we have the following.

COROLLARY 2. (a) Let $x$ be a fixed point of $X$. If there exists a function $g_x \in C(X)$ satisfying (7) and (8), then $M$ is a Korovkin space with respect to $\delta_x$. (b) If for each point $x \in X$, there exists a function $g_x \in C(X)$ satisfying (7) and (8), then $M$ is a Korovkin space with respect to $I$.

For a given subset $S$ of $C(X)$, we define

$$SE = \{va : v \in S, a \in E\}.$$

REMARK 2: Let $x \in X$ and let $g_x$ be a function in $C(X)$ which satisfies (7) and (8). Then $\{1_X, g_x\}E$ is a Korovkin set with respect to $\delta_x$. In fact, this follows immediately from Corollary 2 (a).

THEOREM 4. *Let* $\{u_1, u_2, \cdots, u_m\}$ *be a finite subset of* $C(X)$ *and let*

$$U = \{1_X, u_1, u_2, \cdots, u_m\}E.$$

*Then the following assertions hold:* (a) Let $x$ be a fixed point of $X$. If there exists a finite subset $\{a_1(x), a_2(x), \cdots, a_m(x)\}$ of $\mathbb{R}$ such that

$$g_x = \sum_{i=1}^{m} a_i(x) u_i \tag{10}$$

satisfies (7), then $U$ is a Korovkin space with respect to $\delta_x$. (b) If for each point $x \in X$, there exists a finite subset $\{a_1(x), a_2(x), \cdots, a_m(x)\}$ of $\mathbb{R}$ such that the function $g_x$ defined by (10) satisfies (7), then $U$ is a Korovkin set with respect to $I$.

PROOF: (a) and (b) follows from Corollary 2 (a) and (b), respectively.

From now on let $p$ be any fixed even positive integer.

COROLLARY 3. *Let* $\{v_1, v_2, \cdots, v_n\}$ *be a finite subset of* $C(X)$ *separating the points of* $X$ *and let*

$$V = \{1_X, v_1, \cdots, v_n, v_1^2, \cdots, v_n^2, \cdots, v_1^{p-1}, \cdots, v_n^{p-1}, \sum_{i=1}^{n} v_i^p\}.$$

*Then for a fixed point* $x \in X$, $VE$ *is a Korovkin set with respect to* $\delta_x$ *and* $VE$ *is also a Korovkin set with respect to* $I$.

Indeed, with the help the function

$$g_x = \sum_{i=1}^{n} (v_i - v_i(x))^p,$$

this result follows from Theorem 4.

THEOREM 5. *Let* $G$ *be a subset of* $C(X)$ *separating the points of* $X$ *and let*

$$W = \{g^i : g \in G, i = 0, 1, 2, \cdots, p\},$$

*where* $g^0 = 1_X$. *Then* $WE$ *is a Korovkin set with respect to* $I$.

PROOF: In view of Theorem 2, it will suffice to prove that $\partial_H(X) = X$, where $H$ denotes the linear subspace of $C(X, E)$ spanned by $WE$. Let $x$ be an arbitrary point of $X$ and let $\mu$ be any element of $\mathcal{R}_x(H)$. Let $\epsilon > 0$ be given and let $v \in C(X)$. Since the original topology on $X$ is identical with the weak topology on $X$ induced by $G$, there exists a finite subset $\{g_1, g_2, \cdots, g_r\}$ of $G$ and a constant $C > 0$ such that

$$|v(t) - v(x)| \leq \epsilon + C\sum_{i=1}^{r}(g_i(t) - g_i(x))^p$$

for all $t \in X$. Let $a = a^+ - a^- \in E$. Then we have

$$|va^+ - v(x)1_X a^+| \leq \epsilon 1_X a^+ + C\sum_{i=1}^{r}(g_i - g_i(x)1_X)^p a^+,$$

and so

$$|\mu(va^+) - v(x)\mu(1_X a^+)| \leq \epsilon\mu(1_X a^+) + C\sum_{i=1}^{r}\mu((g_i - g_i(x)1_X)^p a^+),$$

which gives

$$|\mu(va^+) - \delta_x(va^+)| \leq \epsilon a^+.$$

Similarly, we have

$$|\mu(va^-) - \delta_x(va^-)| \leq \epsilon a^-.$$

Thus we get

$$|\mu(va) - \delta_x(va)| \leq \epsilon|a|,$$

which implies

$$||\mu(va) - \delta_x(va)|| \leq \epsilon||a||.$$

Consequently, we conclude that $\mu(va) = \delta_x(va)$ for all $v \in C(X)$ and all $a \in X$, since $\epsilon$ is an arbitrary positive real number. Therefore Lemma 6 yields that $\mu = \delta_x$, and thus $x$ belongs to $\partial_H(X)$. This proves $\partial_H(X) = X$.

REMARK 3: If $E = \mathbb{R}$, then Theorem 5 reduces to [12 ; Corollary 1 (a) for $A = C(X)$ and $h = 1_X$] for the usual convergence behavior.

Corollary 4. *Let $K$ be a compact subset of a real locally convex Hausdorff vector space $F$ with its dual space $F^*$ and let*

$$H = \{(f|_K)^i : f \in F^*, i = 0, 1, \cdots, p\},$$

*where $f|_K$ denotes the restriction of $f$ to $K$ and $(f|_K)^0 = 1_X$. Then $HE$ is a Korovkin set with respect to $I$.*

Finally, we restrict ourselves to the case where $X$ is a compact subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ or the $n$-dimensional Unitary space $\mathbb{C}^n$ and $p = 2$. For each $k = 1, 2, \cdots, n$, $p_k$ denotes the $n$-th coordinate function defined by

$$p_k(x) = x_k \qquad \text{for every } x = (x_1, x_2, \cdots, x_n) \in X.$$

Then by Corollary 3 and Theorem 5 we have the following several Korovkin sets which can be the classical ones in the case of $E = \mathbb{R}$ (cf. [9], [11], [13], [14], [18]).

(1°) Let $X$ be a compact subset of $R^n$. Then

$$\{1_X, p_1, p_2, \cdots, p_n, \sum_{i=1}^{n} p_i^2\}E$$

and

$$\{1_X, p_1, p_2, \cdots, p_n, p_1^2, p_2^2, \cdots, p_n^2\}E$$

are Korovkin sets with respect to $I$.

(2°) Let $X$ be a compact subset $\mathbb{C}^n$ and for each $k = 1, 2, \cdots, n$, we define

$$q_k(x) = \mathrm{Re}(x_k) \quad \text{and} \quad r_k(x) = \mathrm{Im}(x_k)$$

for every $x = (x_1, x_2, \cdots, x_n) \in X$, where $\mathrm{Re}(x_k)$ and $\mathrm{Im}(x_k)$ stand for the real part and the imaginary part of $x_k$, respectively. Then

$$\{1_X, q_1, \cdots, q_n, r_1, \cdots, r_n, \sum_{m=1}^{n} (q_m^2 + r_m^2)\}E$$

and

$$\{1_X, q_1, \cdots, q_n, r_1, \cdots, r_n, q_1^2, \cdots, q_n^2, r_1^2, \cdots, r_n^2\}E$$

are Korovkin sets with respect to $I$.

(3°) Let $X$ be the $n$-dimensional torus $\mathbb{T}^n$, i.e.,

$$\mathbb{T}^n = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : |x_k| = 1, k = 1, 2, \cdots, n\},$$

and $q_k$ and $r_k$ $(k = 1, 2, \cdots, n)$ be as in (2°). Then

$$\{1_X, q_1, \cdots, q_n, r_1, \cdots, r_n\}E$$

is a Korovkin set with respect to $I$.

(4°) Let $C_{2\pi}(\mathbb{R}^n, E)$ denote the normed vector lattice of all $E$-valued continuous functions $f$ on $\mathbb{R}^n$ which are periodic with period $2\pi$ in each variable with the norm

$$\|f\| = \sup\{\|f(x)\| : x \in \mathbb{R}^n\}.$$

Then $C(\mathbb{T}^n, E)$ is isometrically isomorphic to $C(\mathbb{R}^n, E)$. For each $k = 1, 2, \cdots, n$, we define

$$c_k(x) = \cos x_k \quad \text{and} \quad s_k(x) = \sin x_k$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then

$$\{1_{\mathbb{R}^n}, c_1, \cdots, c_n, s_1, \cdots, s_n\}E$$

is a Korovkin set with respect to the identity operator in $C_{2\pi}(\mathbb{R}^n, E)$.

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