Approximation by linear sums of bounded linear operators

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APPROXIMATION BY LINEAR SUMS OF BOUNDED LINEAR OPERATORS*

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Abstract. Quantitative results for approximation by some linear means of bounded linear operators on Banach spaces are obtained by means of moduli of continuity under certain appropriate conditions.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|_X$ and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\omega_i, i = 1, 2, \cdots r$, be non-negative functions on $X \times [0, \infty)$, which satisfy

(1) $$\omega_i(f, \delta) \leq \omega_i(f, \xi) \qquad \text{for all } f \in X,\ \xi \geq \delta \geq 0,$$

(2) $$\lim_{\delta \to +0} \omega_i(f, \delta) = 0 \qquad \text{for all } f \in X$$

and there exist constants $A_i, B_i > 0$ such that

(3) $$\omega_i(f, \xi\delta) \leq (A_i + B_i\xi)\omega_i(f, \delta) \qquad \text{for all } f \in X,\ \xi, \delta \geq 0.$$

Each function $\omega_i(f, \cdot)$ is sometimes called the modulus of continuity of f.

The purpose of this paper is to establish quantitative results for approximation by some linear means of operators in $B[X]$ associated with infinite lower triangular stochastic matrices, by using the moduli of continuity of approximating elements.

The results are applied to approximation by convolution operators, multiplier operators on Banach spaces and Cesàro-Marcinkiewicz type means of Fourier series of several variables in Orlicz spaces. Consequently, we extend the results of Firlej and Rempulska [8] (cf. [7], [34]) to the context of arbitrary Banach spaces.

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2. Main Results

Let $\mathbb{N}$ denote the set of all non-negative integers. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $B[X]$ and let $T_n, n \in \mathbb{N}$, denote the Cesàro mean operators of $\{L_n\}$, that is,

$$T_n = \frac{1}{n+1}\sum_{j=0}^{n} L_j.$$

Let $P = (p_{nj})_{n,j\in\mathbb{N}}$ be an infinite lower triangular stochastic matrix, i.e., an infinite matrix of non-negative real numbers satisfying

$$\sum_{j=0}^{\infty} p_{nj} = 1 \qquad (n \in \mathbb{N}), \qquad p_{nj} = 0 \qquad (j > n),$$

and we define

$$W_n = \sum_{j=0}^{n} p_{nj} L_j \qquad (n \in \mathbb{N}).$$

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be sequences of positive real numbers, and let $f \in X$.

Now we consider the following conditions:

(L) There exist constants $C_i > 0, i = 1, \cdots r$, such that

$$\|L_n(f) - f\|_X \leq \sum_{i=1}^{r} C_i \omega_i(f, \lambda_n) \qquad \text{for all } n \in \mathbb{N}.$$

(T) There exist constants $M_i > 0, i = 1, \cdots, r$, such that

$$\|T_n(f) - f\|_X \leq \sum_{i=1}^{r} M_i \omega_i(f, \mu_n) \qquad \text{for all } n \in \mathbb{N}.$$

(P) $A = \sup\{\sum_{j=0}^{n} |p_{nj}^{\sim}| : n \in \mathbb{N}\} < \infty$,

where

$$p_{nj}^{\sim} = \begin{cases} (j+1)(p_{nj} - p_{nj+1}) & (0 \leq j \leq n-1), \\ (n+1)p_{nn} & (j = n), \\ 0 & (j > n). \end{cases}$$

It follows from (2) that if Condition (L) holds and $\{\lambda_n\}$ converges to zero, then

$$\lim_{n\to\infty} \|L_n(f) - f\|_X = 0,$$

and so

$$\lim_{n\to\infty} \|T_n(f) - f\|_X = 0.$$

REMARK 1: We have

$$\sum_{j=0}^{n} |\widetilde{p_{nj}}| = \begin{cases} 1 & (p_{nj} \geq p_{nj+1},\ 0 \leq j \leq n-1), \\ 2(n+1)p_{nn} - 1 & (p_{nj} \leq p_{nj+1},\ 0 \leq j \leq n-1), \end{cases}$$

and

$$\sum_{j=0}^{n} |p_{nj} - p_{nj+1}| = \begin{cases} p_{n0} & (p_{nj} \geq_{nj+1}, 0 \leq j \leq n-1), \\ 2p_{nn} - p_{n0} & (p_{nj} \leq p_{nj+1}, 0 \leq j \leq n-1) \end{cases}$$

(cf. [8; Lemmas 1 and 2]).

THEOREM 1. *Let $f \in X$. Then the following statements hold:*

(a) *If Condition (L) is satisfied, then*

$$\|W_n(f) - f\|_X \leq \sum_{i=1}^{r} C_i(A_i + B_i/\epsilon)\omega_i(f, \epsilon\xi_n) \tag{4}$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\xi_n = \sum_{j=0}^{n} p_{nj}\lambda_j \qquad (n \in \mathbb{N}).$$

(b) *If Conditions (T) and (P) are satisfied, then*

$$\|W_n(f) - f\|_X \leq \sum_{i=1}^{r} M_i(AA_i + B_i/\epsilon)\omega_i(f, \epsilon\theta_n) \tag{5}$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\theta_n = \sum_{j=0}^{n} |\widetilde{p_{nj}}|\mu_j \qquad (n \in \mathbb{N}).$$

Proof: (a) Let $\delta > 0$. Since

$$W_n(f) - f = \sum_{j=0}^{n} p_{nj}(L_j(f) - f), \tag{6}$$

it follows from (L) and (3) that

$$\|W_n(f) - f\|_X \le \sum_{j=0}^{n} p_{nj}\|L_j(f) - f\|_X$$

$$\le \sum_{i=1}^{r}\sum_{j=0}^{n} p_{nj} C_i \omega_i(f, \lambda_j) \le \sum_{i=1}^{r}\sum_{j=0}^{n} p_{nj} C_i (A_i + B_i\lambda_j/\delta)\omega_i(f,\delta)$$

$$= \sum_{i=1}^{r} C_i(A_i + B_i\xi_n/\delta)\omega_i(f,\delta).$$

Therefore putting $\delta = \epsilon\xi_n$, we obtain the inequality (4).

(b) By (6) and the Abel transformation, we have

$$W_n(f) - f = \sum_{j=0}^{n}(p_{nj} - p_{nj+1})R_j(f) + p_{nn}R_n(f),$$

where

$$R_m(f) = \sum_{k=0}^{m}(L_k(f) - f) = (m+1)(T_m(f) - f) \qquad (m \in \mathbb{N}).$$

Thus we get

$$W_n(f) - f = \sum_{j=0}^{n} \tilde{p_{nj}}(T_j(f) - f),$$

and so (T), (P) and (3) yield

$$\|W_n(f) - f\|_X \le \sum_{j=0}^{n} |\tilde{p_{nj}}| \|T_j(f) - f\|_X$$

$$\le \sum_{i=1}^{r}\sum_{j=0}^{n} |\tilde{p_{nj}}| M_i \omega_i(f, \mu_j) \le \sum_{i=1}^{r}\sum_{j=0}^{n} |\tilde{p_{nj}}| M_i (A_i + B_i\mu_j/\delta)\omega_i(f,\delta)$$

$$\leq \sum_{i=1}^{r} M_i(AA_i + (B_i/\delta)\theta_n)\omega_i(f,\delta).$$

Hence putting $\delta = \epsilon\theta_n$, we obtain the inequality (5).

REMARK 2: The part (a) is extended to the quantitative estimate on the degree of approximation by summation processes induced by the method of summability due to the author [23] (cf. [24]) recovering that of Bell [1] (cf. [18], [29]) which includes the method of almost convergence (F-summability) and F_A-summabiliy of Lorentz [16], A_B-summability of Mazhar and Siddiqi [19] and order summability of Jurkat and Peyerimhoff [12, 13]: Let $\mathfrak{A} = \{a_{\alpha,j}^{(\lambda)} : \alpha \in D, j \in \mathbb{N}, \lambda \in \Lambda\}$ be a family of non-negative real numbers such that

$$\sum_{j=0}^{\infty} a_{\alpha,j}^{(\lambda)} = 1 \qquad \text{for each } \alpha \in D,\ \lambda \in \Lambda,$$

where D is a directed set and Λ is an index set. For examples of such families, see, for instance [23], [24] and [25]. If Condition (L) is satisfied, then

$$\sup\left\{\left\|\sum_{j=0}^{\infty} a_{\alpha,j}^{(\lambda)}(L_j(f)-f)\right\|_X : \lambda \in \Lambda\right\} \leq \sum_{i=1}^{r} C_i(A_i + B_i/\epsilon)\omega_i(f,\epsilon\xi_\alpha)$$

for all $\alpha \in D$ and all $\epsilon > 0$, where

$$\xi_\alpha = \sup\left\{\sum_{j=0}^{\infty} a_{\alpha,j}^{(\lambda)}\lambda_j : \lambda \in \Lambda\right\} < \infty \qquad (\alpha \in D).$$

As an immediate consequence of Theorem 1 (b) and Remark 1, we have the following.

COROLLARY 1. *Let $f \in X$ and suppose that Condition (T) is satisfied with $\mu_n = 1/(n+1), n \in \mathbb{N}$. Then the following assertions hold:*

(a) *If*

$$(7) \qquad p_{nj} \geq p_{nj+1} \qquad (n \geq 1,\ j = 0, 1, \cdots, n-1),$$

then

$$\|W_n(f) - f\|_X \le \sum_{i=1}^{r} M_i(A_i + B_i/\epsilon)\omega_i(f, \epsilon p_{n0}) \tag{8}$$

for all $n \in \mathbb{N}$ *and all* $\epsilon > 0$.

(b) *If*

$$p_{nj} \le p_{nj+1} \qquad (n \ge 1,\ j = 0, 1, \cdots, n-1), \tag{9}$$

and if

$$B = \sup\{(n+1)p_{nn} : n \in \mathbb{N}\} < \infty, \tag{10}$$

then

$$\|W_n(f) - f\|_X \le \sum_{i=1}^{r} M_i\{(2B-1)A_i + B_i/\epsilon\}\omega_i(f, \epsilon(2p_{nn} - p_{n0})) \tag{11}$$

for all $n \in \mathbb{N}$ *and all* $\epsilon > 0$.

REMARK 3: Let $f \in X$ and assume that Condition (L) is fulfilled. Then by Theorem 1 (a) we derive

$$\|T_n(f) - f\|_X \le \sum_{i=1}^{r} C_i(A_i + B_i/\epsilon)\omega_i(f, \epsilon\zeta_n) \tag{12}$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\zeta_n = \frac{1}{n+1}\sum_{j=0}^{n}\lambda_j \qquad (n \in \mathbb{N}).$$

In particular, if $\lambda_n = 1/(n+1)$ for all $n \in \mathbb{N}$, then

$$\zeta_n \le \frac{1}{n+1}(\gamma + \log(n+2)) \le \frac{K}{\sqrt{n+1}},$$

where γ is Euler's constant, i.e.,

$$\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right) = 0.57721566490153286060\cdots$$

and

$$K = \sup\left\{\frac{\gamma + \log(n+2)}{\sqrt{n+1}} : n \in \mathbb{N}\right\}, \tag{13}$$

and so it follows from (1) and (12) that

$$\|T_n(f) - f\|_X \le \sum_{i=1}^{r} C_i(A_i + KB_i/\epsilon)\omega_i(f, \epsilon/\sqrt{n+1})$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

Let $r \in \mathbb{N}, r \geq 1$, and set

$$\mathbb{N}^r = \{n = (n_1, n_2, \cdots, n_r) : n_j \in \mathbb{N}, j = 1, 2, \cdots, r\}$$

and for $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$, we put

$$\mathbb{N}_n^r = \{k = (k_1, k_2, \cdots, k_r) : 0 \le k_j \le n_j, j = 1, 2, \cdots, r\}.$$

Let $L_n, n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$, be operators in $B[X]$, and define

$$T_n = \frac{1}{(n_1+1)(n_2+1)\cdots(n_r+1)} \sum_{k \in \mathbb{N}_n^r} L_k,$$

which are called the Cesàro mean operators of $\{L_n\}$. Let $P^{(i)} = (p_{mj}^{(i)})_{m,j \in \mathbb{N}}, i = 1, 2, \cdots, r$, be infinite lower triangular stochastic matrices, and we define

$$W_n = \sum_{k \in \mathbb{N}_n^r} \prod_{j=1}^{r} p_{n_j k_j}^{(j)} L_k \qquad (n = (n_1, n_2 \cdots, n_r) \in \mathbb{N}^r).$$

Let $\{\lambda_m^{(i)}\}_{m \in \mathbb{N}}$ and $\{\mu_m^{(i)}\}_{m \in \mathbb{N}}, i = 1, 2, \cdots, r$, be sequences of positive real numbers and $f \in \mathbb{N}$, and we consider the following conditions:

$(L)^*$ There exist constants $C_i^* > 0, i = 1, \cdots, r$, such that

$$\|L_n(f) - f\|_X \le \sum_{i=1}^{r} C_i^* \omega_i(f, \lambda_{n_i}^{(i)}) \qquad \text{for all } n = (n_1, \cdots, n_r) \in \mathbb{N}^r.$$

$(T)^*$ There exist constants $M_i^* > 0, i = 1, \cdots, r$, such that

$$\|T_n(f) - f\|_X \le \sum_{i=1}^{r} M_i^* \omega_i(f, \mu_{n_i}^{(i)}) \qquad \text{for all } n = (n_1, \cdots, n_r) \in \mathbb{N}^r.$$

$(P)^*$ For $i = 1, 2, \cdots, r$,

$$C^{(i)} = \sup \left\{ \sum_{j=0}^{m} |\tilde{p}_{mj}^{(i)}| : m \in \mathbb{N} \right\} < \infty,$$

where

$$\tilde{p}_{mj}^{(i)} = \begin{cases} (j+1)(p_{mj}^{(i)} - p_{mj+1}^{(i)}) & (0 \le j \le m-1), \\ (m+1)p_{mm}^{(i)} & (j = m), \\ 0 & (j > m). \end{cases}$$

THEOREM 2. *Let $f \in X$. Then the following statements hold:*

(a) *If Condition $(L)^*$ is satisfied, then*

$$\|W_n(f) - f\|_X \le \sum_{i=1}^{r} C_i^*(A_i + B_i/\epsilon)\omega_i(f, \epsilon\xi_{n_i}^{(i)}), \tag{14}$$

for all $n = (n_1, n_1, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$, where

$$\xi_{n_i}^{(i)} = \sum_{j=0}^{n_i} \lambda_j^{(i)} p_{n_i j}^{(i)} \qquad (i = 1, 2, \cdots, r).$$

(b) *If Conditions $(T)^*$ and $(P)^*$ are satisfied, then*

$$\|W_n(f) - f\|_{\not{X}} \le \sum_{i=1}^{r} M_i^*(C^{(i)} A_i + B_i/\epsilon)\omega_i(f, \epsilon\theta_{n_i}^{(i)}) \tag{15}$$

for all $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$, where

$$\theta_{n_i}^{(i)} = \sum_{j=0}^{n_i} \mu_j^{(i)} |\tilde{p}_{n_i j}^{(i)}| \qquad (i = 1, 2, \cdots, r).$$

PROOF: (a) Let $\delta > 0$. Since

$$(16) \qquad W_n(f) - f = \sum_{k \in \mathbb{N}_n^r} \prod_{j=1}^{r} p_{n_j k_j}^{(j)} (L_k(f) - f),$$

it follows from $(L)^*$ and (3) that

$$\|W_n(f) - f\|_X \leq \sum_{k \in \mathbb{N}_n^r} \prod_{j=1}^{r} p_{n_j k_j}^{(j)} \|L_k(f) - f\|_X$$

$$\leq \sum_{k \in \mathbb{N}_n^r} \prod_{j=1}^{r} p_{n_j k_j}^{(j)} \sum_{i=1}^{r} C_i^* \omega_i(f, \lambda_{k_i}^{(i)})$$

$$\leq \sum_{i=1}^{r} \sum_{k_i=0}^{n_i} p_{n_i k_i}^{(i)} C_i^* (A_i + B_i \lambda_{k_i}^{(i)} / \delta) \omega_i(f, \delta)$$

$$= \sum_{i=1}^{r} C_i^* (A_i + (B_i/\delta) \xi_{n_i}^{(i)}) \omega_i(f, \delta).$$

Thus putting $\delta = \epsilon \xi_{n_i}^{(i)}$, we obtain the inequality (14).

(b) By (16) and the Abel transformation, we conclude

$$W_n(f) - f = \sum_{k \in \mathbb{N}_n^r} \prod_{j=1}^{r} \tilde{p}_{n_j k_j}^{(j)} (T_k(f) - f),$$

and so arguing as in the proof of Part (a), $(T)^*$, $(P)^*$ and (3) establish

$$\|W_n(f) - f\|_X \leq \sum_{i=1}^{r} M_i^* (C^{(i)} A_i + (B_i/\delta) \theta_{n_i}^{(i)}) \omega_i(f, \delta) \qquad (\delta > 0).$$

Hence putting $\delta = \epsilon \theta_{n_i}^{(i)}$, we get the inequality (15).

From Theorem 2 (b) and Remark 1 we have the following.

Corollary 2. *Let $f \in X$ and suppose that Condition $(T)^*$ is satisfied with the case where*

$$\mu_m^{(i)} = \frac{1}{m+1} \qquad (m \in \mathbb{N},\ i = 1, 2, \cdots, r).$$

Then the following assertions hold:

(a) *If*

$$(17) \quad p_{mj}^{(i)} \geq p_{mj+1}^{(i)} \qquad (m \geq 1,\ j = 0, \cdots, m-1,\ i = 1, \cdots, r),$$

then

$$\|W_n(f) - f\|_X \leq \sum_{i=1}^{r} M_i^*(A_i + B_i/\epsilon)\omega_i(f, \epsilon p_{n_i 0}^{(i)})$$

for all $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$.

(b) *If*

$$(18) \quad p_{mj}^{(i)} \leq p_{mj+1}^{(i)} \qquad (m \geq 1,\ j = 0, \cdots, m-1,\ i = 1, \cdots, r),$$

and if

$$(19) \quad M^{(i)} = \sup\{(m+1)p_{mm}^{(i)} : m \in \mathbb{N}\} < \infty \qquad (i = 1, \cdots, r),$$

then

$$\|W_n(f) - f\|_X \leq \sum_{i=1}^{r} M_i^*\{(2M^{(i)} - 1)A_i + B_i/\epsilon\}\omega_i(f, \epsilon(2p_{n_i n_i}^{(i)} - p_{n_i 0}^{(i)}))$$

for all $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$.

Remark 4: Let $f \in X$ and assume that Condition $(L)^*$ is fulfilled. Then by Theorem 2 (a) we derive

$$(20) \qquad \|T_n(f) - f\|_X \leq \sum_{i=1}^{r} C_i^*(A_i + B_i/\epsilon)\omega_i(f, \epsilon\zeta_{n_i}^{(i)})$$

for all $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$, where

$$\zeta_{n_i}^{(i)} = \frac{1}{n_i + 1}\sum_{j=0}^{n_i} \lambda_j^{(i)} \qquad (i = 1, 2, \cdots, r).$$

In particular, if $\lambda_m^{(i)} = 1/(m+1)$ for all $m \in \mathbb{N}$ and for $i = 1, \cdots, r$, then

$$\zeta_{n_i}^{(i)} \le \frac{1}{n_i+1}(\gamma + \log(n_i+2)) \le \frac{K}{\sqrt{n_i+1}},$$

and so it follows from (1) and (20) that

$$\|T_n(f) - f\|_X \le \sum_{i=1}^{r} C_i^*(A_i + KB_i/\epsilon)\omega_i(f, \epsilon/\sqrt{n_i+1})$$

for all $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ and all $\epsilon > 0$, where K is a positive constant given by (13).

3. Linear Sums of Convolution Operators and Multiplier Operators

Let $\mathbb{R}$ denote the set of all real numbers and let $\{S(t) : t \in \mathbb{R}\}$ be a family of operators in $B[X]$ with $S(0) = I$, the identity operator, such that for each $f \in X$ the mapping $t \mapsto S(t)(f)$ is strongly continuous on $\mathbb{R}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. If k is a function in $L_{2\pi}^1$ having the Fourier series expansion

$$k(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k}(j)e^{ijt} \qquad (t \in \mathbb{R})$$

with its Fourier coefficients

$$\hat{k}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)e^{-ijt}\,dt \qquad (j \in \mathbb{Z}),$$

where $\mathbb{Z}$ stands for the set of all integers and if $L \in B[X]$, then we define the convolution operators $(k * L)(\varphi; \cdot)$ by

$$(k * L)(\varphi; f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t)S(\varphi(t))(L(f))\,dt \qquad (f \in X),$$

which exists as a Bochner integral (cf. [22]). Obviously, $(k * L)(\varphi; \cdot)$ belongs to $B[X]$ and

$$\|(k * L)(\varphi; \cdot)\|_{B[X]} \le B\|k\|_1\|L\|_{B[X]},$$

where

$$B = \sup\{\|S(\varphi(t))\|_{B[X]} : |t| \leq \pi\}.$$

For a given $a \in \mathbb{R}$, we define

$$\varphi_a(t) = at \qquad \text{for all } t \in \mathbb{R},$$

and put

$$(k * L)_a(f) = (k * L)(\varphi_a; f) \qquad (f \in X).$$

Let $\{k_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $L^1_{2\pi}$ and $r \in \mathbb{N}, r \geq 1$. The linear combination of the convolution operators $(k * I)_j, j = 1, \cdots, r$, which is given by

$$\Phi_{n,r} = \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} (k_n * I)_j \tag{21}$$

plays an important role in the study of direct problems of Jackson type on estimating the degree of the best approximation in Banach spaces ([28]). Here we restrict ourselves to the case where $r = 1$ and each k_n is a non-negative function with $\hat{k}(0) = 1$. Put $L_n = \Phi_{n,1}, n \in \mathbb{N}$, and thus we have

$$L_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t) S(t)(f)\, dt \qquad (n \in \mathbb{N}, f \in X).$$

If $f \in X$ and $\delta \geq 0$, then we define

$$\omega(f, \delta) = \omega(X; f, \delta) = \sup\{\|S(t)(f) - f\|_X : |t| \leq \delta\},$$

which is called the modulus of continuity of f with respect to the family $\{S(t)\}$ (cf. [22; Def. 3]). Let $\omega_1 = \omega$. Clearly, ω_1 satisfies (1) and (2) for $r = 1$. Suppose that

$$\|S(t)(f) - S(u)(f)\|_X \leq C\|S(t-u)(f) - f\|_X \tag{22}$$

for all $t, u \in \mathbb{R}$ and all $f \in X$. Note that if $\{S(t) : t \in \mathbb{R}\}$ is a uniformly bounded strongly continuous group of operators in $B[X]$, then (22) holds with

$$C = \sup\{\|S(t)\|_{B[X]} : t \in \mathbb{R}\} < \infty. \tag{23}$$

For further properties of semigroups of operators on Banach spaces, we refer to [3], [5], [6], [9] and [10]. By [22; Lemma 2 (ii)], ω_1 satisfies (3) with $A_1 = 1$ and $B_1 = C$.

For a given $p > 0$, we define

$$\mu(k_n; p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |t|^p k_n(t)\, dt,$$

which is called the p-th moment of k_n. Set

$$\lambda_{n,p} = \mu(k_n : p)^{1/p} \qquad (n \in \mathbb{N},\ p \geq 1).$$

Let $f \in X$ and $\tau > 0$. Then by [27; Theorem 3], we have

$$\|L_n(f) - f\|_X \leq C(p, \tau)\omega_1(f, \tau\lambda_{n,p}) \tag{24}$$

for all $n \in \mathbb{N}$, where

$$C(p, \tau) = 1 + C \min\{\tau^{-p},\ \tau^{-1}\}.$$

Therefore, Condition (L) holds for

$$r = 1, \quad C_1 = C(p, \tau), \quad \lambda_n = \tau\lambda_{n,p}.$$

In particular, if each k_n is even, then it follows from [27; Corollary 3] that

$$\|L_n(f) - f\|_X \leq C(\tau)\omega_1(f, \tau\nu_n)$$

for all $n \in \mathbb{N}$, where

$$C(\tau) = 1 + \frac{C\pi}{\sqrt{2}} \min\{\tau^{-2}\pi/\sqrt{2},\ \tau^{-1}\}$$

and

$$\nu_n = \left(1 - \hat{k}_n(1)\right)^{1/2} = \left\{1 - \frac{1}{\pi}\int_0^{\pi} k_n(t) \cos t\, dt\right\}^{1/2} \qquad (n \in \mathbb{N}).$$

Since

$$T_n = \left(\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right) * I\right)_1 \qquad (n \in \mathbb{N}),$$

we have

$$\|T_n(f) - f\|_X \leq C(p,\tau)\omega_1(f, \tau\mu_{n,p})$$

for all $n \in \mathbb{N}$, where

$$\mu_{n,p} = \left(\frac{1}{n+1} \sum_{j=0}^{n} \lambda_{j,p}^{p} \right)^{1/p} \qquad (n \in \mathbb{N}).$$

Therefore, Condition (T) holds for

$$r = 1, \quad M_1 = C(p,\tau), \quad \mu_n = \tau\mu_{n,p}.$$

In particular, if each k_n is even, then

$$\|T_n(f) - f\|_X \leq C(\tau)\omega_1(f, \tau\gamma_n),$$

where

$$\gamma_n = \left(\frac{1}{n+1} \sum_{j=0}^{n} \nu_j^2 \right)^{1/2} \qquad (n \in \mathbb{N}).$$

Hence by Theorem 1 we obtain the following.

THEOREM 3. *Let $f \in X$. Then the following statements hold:*

(a) *For all $n \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_X \leq C(p,\tau)(1 + C/\epsilon)\omega(f, \epsilon a_{n,p}), \tag{25}$$

where

$$a_{n,p} = \tau \sum_{j=0}^{n} p_{nj}\lambda_{j,p} \qquad (n \in \mathbb{N}).$$

In particular, if each k_n is even, then (25) reduces to

$$\|W_n(f) - f\|_X \leq C(\tau)(1 + C/\epsilon)\omega(f, \epsilon b_n),$$

where

$$b_n = \tau \sum_{j=0}^{n} \nu_j p_{nj} \qquad (n \in \mathbb{N}).$$

(b) *If Condition (P) is satisfied, then*

$$\|W_n(f) - f\|_X \le C(p,\tau)(A + C/\epsilon)\omega(f, \epsilon x_{n,p}) \tag{26}$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$, where

$$x_{n,p} = \tau \sum_{j=0}^{n} |p_{nj}^{\sim}| \mu_{j,p} \qquad (n \in \mathbb{N}).$$

In particular, if each k_n is even, then (26) reduces to

$$\|W_n(f) - f\|_X \le C(\tau)(A + C/\epsilon)\omega(f, \epsilon y_n),$$

where

$$y_n = \tau \sum_{j=0}^{n} |p_{nj}^{\sim}| \gamma_j \qquad (n \in \mathbb{N}).$$

Corollary 3. *Let $f \in X$ and suppose that k_n is even and $\mu_n = 1/(n+1)$ for every $n \in \mathbb{N}$. Then the following assertions hold:*

(a) *If P satisfies (7), then*

$$\|W_n(f) - f\|_X \le (1 + C\pi/\sqrt{2})(1 + C/\epsilon)\omega(f, \epsilon p_{n0})$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

(b) *If P satisfies (9) and (10), then*

$$\|W_n(f) - f\|_X \le (1 + C\pi/\sqrt{2})(2B - 1 + C/\epsilon)\omega(f, \epsilon(2p_{nn} - p_{n0}))$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

Remark 5: In view of Remark 2, Theorem 3 (a) holds for the methods of $\mathbb{A}$-summability.

Let $\{P_j\}_{j\in\mathbb{Z}}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

- $(P-1)$ The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.
- $(P-2)$ $\{P_j\}_{j\in\mathbb{Z}}$ is fundamental, i.e., the linear span of the set $\cup_{j\in\mathbb{Z}} P_j(X)$ is dense in X.
- $(P-3)$ $\{P_j\}_{j\in\mathbb{Z}}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

For any $f \in X$, we associate its (formal) Fourier series expansion (with respect to $\{P_j\}$)

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f). \tag{27}$$

An operator $L \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\alpha_j\}_{j\in\mathbb{Z}}$ of scalars such that for every $f \in X$,

$$L(f) \sim \sum_{j=-\infty}^{\infty} \alpha_j P_j(f),$$

and the following notation is used:

$$L \sim \sum_{j=-\infty}^{\infty} \alpha_j P_j \tag{28}$$

(cf. [4], [22], [23], [35]).

REMARK 6: The expansion (27) is a generalization of the concept of Fourier series in a Banach space X with respect to a fundamental, total, biorthogonal system $\{f_j, f_j^*\}_{j\in\mathbb{Z}}$. Here $\{f_j\}_{j\in\mathbb{Z}}$ and $\{f_j^*\}_{j\in\mathbb{Z}}$ are sequences of X and X^* (the dual space of X), respectively such that the linear span of $\{f_j : j \in \mathbb{Z}\}$ is dense in X (fundamental), $f_j^*(f) = 0$ for all $j \in \mathbb{Z}$ implies $f = 0$ (total), and $f_j^*(f_n) = \delta_{j,n}$ for all $j, n \in \mathbb{Z}$ (biorthogonal). Then (27) reads

$$f \sim \sum_{j=-\infty}^{\infty} f_j^*(f) f_j$$

(cf. [2], [20], [32]).

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\{S(t) : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying (23) and having the expansions

$$S(t) \sim \sum_{j=-\infty}^{\infty} \exp(\beta_j t) P_j \qquad (t \in \mathbb{R}), \tag{29}$$

where $\{\beta_j\}_{j\in\mathbb{Z}}$ is a sequence of scalars. Then $\{S(t) : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ with its infinitesimal generator G with domain $D(G)$ and there holds

$$G^r(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \beta_j^r P_j(f) \qquad (f \in D(G),\ i = 1, 2, \cdots)$$

(cf. [22; Proposition 2], [26; Proposition 3]).

If $k \in L^1_{2\pi}$ and if L is an operator in $M[X]$ having the expansion (28), then $(k * L)(\varphi;\cdot)$ belongs to $M[X]$ and

$$(30) \qquad (k * L)(\varphi;\cdot) \quad \sim \quad \sum_{j=-\infty}^{\infty} c_j(\varphi; k) P_j(\cdot),$$

where

$$c_j(\varphi; k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(t) \exp(\beta_j \varphi(t))\, dt \qquad (j \in \mathbb{Z})$$

([28; Lemma 2], cf. [26; Proposition 4]). For each $n \in \mathbb{N}, a \in \mathbb{R}$, we set

$$\Pi_{n,a} = \{k \in L^1_{2\pi} : c_j(\varphi_a; k) = 0 \quad \text{whenever} \quad |j| > n\},$$

which is a closed linear subspace of $L^1_{2\pi}$.

For each $m \in \mathbb{N}$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^m = (S(t) - I)^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S(jt) \quad (m \geq 1),$$

which stands for the m-th iteration of $S(t) - I$. Clearly, Δ_t^m belongs to $B[X]$ and

$$\|\Delta_t^m\|_{B[X]} \leq K_m,$$

where

$$K_m = \min\{(C+1)^m,\ 2^m C\}.$$

If $f \in X, m \in \mathbb{N}$ and $\delta \geq 0$, then we define

$$\omega^{(m)}(f, \delta) = \omega^{(m)}(X; f, \delta) = \sup\{\|\Delta_t^m(f)\|_X : |t| \leq \delta\},$$

which is called the m-th modulus of continuity of f with respect to the family $\{S(t)\}$. In particular, $\omega^{(1)}(f,\delta)$ is the modulus of continuity $\omega(f,\delta)$.

Let $m \in \mathbb{N}$ and $f \in X$. Assume that

$$\rho_m = \sup\left\{\sum_{j=0}^{m+1}\binom{m+1}{j}\lambda_n^{-j}\mu(k_n;j) : n \in \mathbb{N}\right\} < \infty. \tag{31}$$

Then it follows from [28; Lemma 1 (c), Lemma 3] that

$$\|\Phi_{n,m+1}(f) - f\|_X \le C\rho_m\omega^{(m+1)}(f,\lambda_n) \le C\rho_m K_m\omega^{(1)}(f,\lambda_n) \tag{32}$$

for all $n \in \mathbb{N}$.

REMARK 7: If $m = 0$, then (32) becomes

$$\|\Phi_{n,1}(f) - f\|_X \le C\sup\{1 + \mu(k_i;1)/\lambda_i : i \in \mathbb{N}\}\omega^{(1)}(f,\lambda_n),$$

and so for $\tau > 0$, taking $\lambda_j = \tau\mu(k_j;1)$ for all $j \in \mathbb{Z}$ we have

$$\|\Phi_{n,1}(f) - f\|_X \le C(1+\tau^{-1})\omega^{(1)}(f,\tau\mu(k_n;1)),$$

which should be compared with the estimate (24) for $p = 1$.

Now let

$$\omega_1 = \omega^{(1)}, \quad L_n = \Phi_{n,m+1} \quad (n \in \mathbb{N}).$$

Then (32) implies that Condition (L) holds for

$$r = 1, \quad C_1 = C\rho_m K_m.$$

Suppose that

$$\delta_m = \sup\left\{\frac{1}{n+1}\sum_{i=0}^{n}\sum_{j=1}^{m+1}\binom{m+1}{j}\mu_n^{-j}\mu(k_i;j) : n \in \mathbb{N}\right\} < \infty.$$

Then since

$$T_n = \sum_{j=1}^{m+1}(-1)^{j+1}\binom{m+1}{j}\left(\left(\frac{1}{n+1}\sum_{i=0}^{n}k_i\right) * I\right)_j \qquad (n \in \mathbb{N}),$$

it follows again from [28; Lemma 1 (c), Lemma 3] that

$$\|T_n(f) - f\|_X \le C\delta_m\omega^{(m+1)}(f,\mu_n) \le C\delta_m K_m\omega_1(f,\mu_n) \tag{33}$$

for all $n \in \mathbb{N}$, and so Condition (T) holds for

$$r = 1, \quad M_1 = C\delta_m K_m.$$

Thus Theorem 1 gives:

THEOREM 4. *Let $m \in \mathbb{N}$ and $f \in X$. Then the following statements hold:*

(a) *For all $n \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_X \leq C\rho_m K_m(1 + C/\epsilon)\omega(f, \epsilon\xi_m).$$

(b) *For all $n \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_X \leq C\delta_m K_m(1 + C/\epsilon)\omega(f, \epsilon\theta_n).$$

COROLLARY 4. *Let $m \in \mathbb{N}$ and $f \in X$. Suppose that $\mu_n = 1/(n+1)$ for all $n \in \mathbb{N}$. Then the following assertions hold:*

(a) *If P satisfies (7), then*

$$\|W_n(f) - f\|_X \leq C\delta_m K_m(1 + C/\epsilon)\omega(f, \epsilon p_{n0})$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

(b) *If P satisfies (9) and (10), then*

$$\|W_n(f) - f\|_X \leq C\delta_m K_m(2B - 1 + C/\epsilon)\omega(f, \epsilon(2p_{nn} - p_{n0}))$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

Here we consider the generalized Jackson kernel given by

$$J_{n,s}(t) = c_{n,s}\left\{\frac{\sin((n+1)t/2)}{\sin(t/2)}\right\}^{2s} \qquad (n, s \in \mathbb{N}, s \geq 1),$$

where the normalizing constant $c_{n,s} > 0$ is taken in such a way that

$$\hat{J}_{n,s}(0) = \frac{1}{\pi}\int_0^\pi J_{n,s}(t)\,dt = 1$$

(cf. [17]). Note that

$$J_{n,1}(t) = F_n(t) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}$$

is the Fejér kernel, and so

$$J_{n,s}(t) = c_{n,s}(n+1)^s F_n^s(t)$$

is a non-negative, even trigonometric polynomial of degree ns with $\hat{J}_{n,s}(0) = 1$. Also, we have

$$J_{n,2}(t) = J_n(t) = \frac{3}{(n+1)(2(n+1)^2+1)} \left\{ \frac{\sin((n+1)t/2)}{\sin(t/2)} \right\}^4,$$

which is the Jackson kernel (cf. [11], [21]).

Now let $m \in \mathbb{N}$ and

$$s(m) = \left[\frac{m+1}{2} \right],$$

where $[\xi]$ denotes the largest integer not exceeding $\xi \geq 0$. Then we have the following.

THEOREM 5. *Let* $k_n = J_{n,s(m)}, n \in \mathbb{N}$ *and* $f \in X$. *Then the following statements hold:*

(a) *For all* $n \in \mathbb{N}$ *and all* $\epsilon > 0$,

$$\|W_n(f) - f\|_X \leq C \rho_{m,s(m)} K_m (1 + C/\epsilon) \omega(f, \epsilon \xi_n),$$

where

$$\rho_{m,s(m)} = \sup \left\{ \sum_{j=0}^{m+1} \binom{m+1}{j} (n+1)^j \mu(J_{n,s(m)}; j) : n \in \mathbb{N} \right\}$$

and

$$\xi_n = \sum_{j=0}^{n} \frac{p_{nj}}{j+1} \qquad (n \in \mathbb{N}).$$

(b) *If*

$$\delta_{m,s(m)} = \sup_{n \in \mathbb{N}} \frac{1}{n+1} \sum_{i=0}^{n} \sum_{j=0}^{m+1} \binom{m+1}{j} \mu_n^{-j} \mu(J_{i,s(m)}; j) < \infty,$$

then

$$\|W_n(f) - f\|_X \leq C\delta_{m,s(m)}K_m(1 + C/\epsilon)\omega(f, \epsilon\theta_n)$$

for all $n \in \mathbb{N}$ and all $\epsilon > 0$.

PROOF: This follows from [28; Lemma 6] and Theorem 4.

REMARK 8: For $m = 0$, we have

$$\rho_{0,2} = \sup\{1 + (n+1)\mu(J_n; 1) : n \in \mathbb{N}\} \leq \frac{\pi^3\sqrt{\pi}}{2\sqrt{2}}$$

and

$$\delta_{0,2} = \sup\left\{\frac{1}{n+1}\sum_{i=0}^{n}(1 + \mu(J_i; 1)/\mu_n) : n \in \mathbb{N}\right\}$$

$$\leq \sup\left\{1 + \frac{\pi^3\sqrt{\pi}}{2\sqrt{2}}\frac{1}{n+1}\sum_{i=0}^{n}\frac{i+1}{\mu_n} : n \in \mathbb{N}\right\}$$

(cf. [28; Lemma 6]). In particular, if we take

$$\mu_n = \left(\frac{1}{n+1}\right)^\alpha \qquad (0 < \alpha < 1, n \in \mathbb{N}),$$

then

$$\delta_{0,2} \leq \sup\left\{1 + \frac{\pi^3\sqrt{\pi}}{2\sqrt{2}}\frac{\gamma + \log(n+2)}{(n+1)^{1-\alpha}} : n \in \mathbb{N}\right\}.$$

For each $n \in \mathbb{N}$, we set

$$M_n[X] = \bigoplus_{j=-n}^{n} P_j(X),$$

which stands for the direct sum of $\{P_j(X) : |j| \leq n\}$. Note that $M_n[X]$ is a closed linear subspace of X. For a given $f \in X$, we define

$$E_n(f) = E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n[X]\},$$

which is called the best approximation of degree n to f with respect to $M_n[X]$. Obviously,

$$E_0(f) \geq E_1(f) \geq \cdots \geq E_n(f) \geq E_{n+1}(f) \geq \cdots \geq 0,$$

and Condition $(P-2)$ implies that

$$\lim_{n\to\infty} E_n(f) = 0 \qquad \text{for every } f \in X.$$

In [28] we related the rapidity with which $E_n(f)$ approaches zero to certain smoothness properties of f, which can be described in terms of its moduli of continuity $\omega^{(m)}(f,\cdot), m \in \mathbb{N}, m \geq 1$.

For each $n \in \mathbb{N}$, we denote by Π_n the set of all trigonometric polynomials of degree at most n. Suppose that

$$(34) \qquad \Pi_n \subseteq \bigcap_{j=1}^{\infty} \Pi_{n,j} \qquad \text{for each } n \in \mathbb{N}.$$

REMARK 9: Let $\{\beta_j\}_{j\in\mathbb{N}} = \{-ij\}_{j\in\mathbb{N}}$. Then we have:

(a) For every $n \in \mathbb{N}$,

$$\Pi_n \subseteq \bigcap_{j\in\mathbb{Z}\setminus\{0\}} \Pi_{nj},$$

and so (34) always holds.

(b) If $\varphi = \varphi_q, q \in \mathbb{Z}\setminus\{0\}$, then (27) reduces to

$$(k * L)_q \sim \sum_{j=-\infty}^{\infty} \hat{k}(jq)\alpha_j P_j,$$

and in particular if $k \in \Pi_n$, then

$$(k * L)_q = \sum_{|j|\leq[n/|q|]} \hat{k}(jq)\alpha_j P_j.$$

Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $B[X]$ satisfying

$$(35) \qquad \alpha = \sup\{\|U_n\|_{B[X]} : n \in \mathbb{N}\} < \infty$$

and $U_n(g) = g$ for every $g \in M_n[X]$. Let $L_n = U_{n+1}, n \in \mathbb{N}, m \in \mathbb{N}$, $m \geq 1$ and $f \in X$. Then it follows from [28; Lemma 1 (c), Theorem 4] that

$$\|L_n(f) - f\|_X \leq C(\alpha+1)\eta_m\omega^{(m+1)}(f, 1/(n+1))$$

$$\leq C(\alpha+1)\eta_m K_m \omega^{(1)}(f, 1/(n+1))$$

for all $n \in \mathbb{N}$, where η_m is a positive constant depending only on m, and so Condition (L) holds for

$$r = 1, \;\; \omega_1 = \omega^{(1)}, \;\; C_1 = C(\alpha+1)\eta_m K_m, \;\; \lambda_n = \frac{1}{n+1}.$$

Since

$$\beta = \sup\{\|T_n\|_{B[X]} : n \in \mathbb{N}\} \leq \alpha < \infty,$$

if $T_n(g) = g$ for all $g \in M_n[X]$, then we have also

$$\|T_n(f) - f\|_X \leq C(\beta+1)\eta_m \omega^{(m+1}(f, 1/(n+1))$$

$$\leq C(\beta+1)\eta_m K_m \omega^{(1)}(f, 1/(n+1)),$$

and so Condition (T) holds for

$$r = 1, \;\; \omega_1 = \omega^{(1)}, \;\; M_1 = C(\beta+1)\eta_m K_m, \;\; \mu_n = \frac{1}{n+1}.$$

Hence Theorem 1 yields the following.

THEOREM 6. *Let $f \in X$. Then the following statements hold:*

(a) *For all $n \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_X \leq C(\alpha+1)\eta_m K_m (1 + C/\epsilon)\omega(f, \epsilon\xi_n),$$

where

$$\xi = \sum_{j=0}^{n} \frac{p_{nj}}{j+1} \qquad (n \in \mathbb{N}).$$

(b) *Suppose that $T_n(g) = g$ for all $g \in M_n[X]$. Then for all $n \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_X \leq C(\beta+1)\eta_m K_m (1 + C/\epsilon)(f, \epsilon\theta_n), \tag{36}$$

where

$$\theta_m = \sum_{j=0}^{n} |p_{nj} - p_{nj+1}| \qquad n \in \mathbb{N}).$$

In particular, if P satisfies (7), *then* (36) *reduces to*

$$\|W_n(f) - f\|_X \leq C(\beta + 1)\eta_m K_m(1 + C/\epsilon)\omega(f, \epsilon p_{n0}),$$

and if P satisfies (9) *and* (10), *then* (36) *reduces to*

$$\|W_n(f) - f\|_X \leq C(\beta + 1)\eta_m K_m(1 + C/\epsilon)\omega(f, \epsilon(2p_{nn} - p_{n0})).$$

Let $\{S_n\}_{n\in\mathbb{N}}$ be the sequence of the n-th partial sum operators associated the Fourier series (27), that is,

$$S_n = \sum_{j=-n}^{n} P_j \qquad (n \in \mathbb{N}),$$

and let $\sigma_n, n \in \mathbb{N}$, be the n-th Cesàro mean operators, that is,

$$\sigma_n = \frac{1}{n+1}\sum_{j=0}^{n} S_j = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right) P_j.$$

Then for each $n \in \mathbb{N}$, we define

$$V_n = \frac{1}{n+1}\sum_{j=n+1}^{2n+1} S_j = 2\sigma_{2n+1} - \sigma_n,$$

which is called the de la Vallée-Poussin operator.

COROLLARY 5. *Let $f \in X$. Then the following assertions hold:*

(a) *Let $U_n = S_n, n \in \mathbb{N}$. If* (35) *is fulfilled, then the statement* (a) *in Theorem* 6 *holds.*

(b) *Let $U_n = V_n, n \in \mathbb{N}$. If*

$$\sigma = \sup\{\|\sigma_n\|_{B[X]} : n \in \mathbb{N}\} < \infty,$$

then the statement (a) *in Theorem* 6 *holds for $\alpha = 3\sigma$.*

In the rest of this section we restrict ourselves to the case where X is a homogeneous Banach space, i.e.,

$(H-1)$ X is a Banach space with norm $\|\cdot\|_X$.

$(H-2)$ X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $C_0 > 0$ such that $\|f\|_1 \le C_0 \|f\|_X$ for all $f \in X$.

$(H-3)$ The translation operator $S(t)$ defined by

$$S(t)(f)(\cdot) = f(\cdot - t) \qquad (f \in X),$$

is isometric on X for each $t \in \mathbb{R}$.

$(H-4)$ For each $f \in X$, the mapping $t \mapsto S(t)(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$, the Banach space of all 2π-periodic, continuous functions f defined on $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \le \pi\}$$

and $L^p_{2\pi}$, the Banach space of all 2π-periodic, p-th power Lebesgue integrable functions f defined on R with the norm

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right\}^{1/p} \qquad (1 \le p < \infty).$$

For other examples see [22] (cf. [14], [31]).

Now we define

$$P_j(f)(\cdot) = \hat{f}(j) e^{ij\cdot} \qquad (f \in X),$$

which satisfies Conditions $(P-1), (P-2)$ and $(P-3)$ just as in Section 3 (cf. [14], [22]). Note that $S(t)$ has the expansion (26) with $\beta_j = -ij$, and so for $\varphi = \varphi_m, m \in \mathbb{Z}$, the expansion (27) reduces to

$$(k * L)_m \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k}(jm) \alpha_j P_j,$$

and

$$M_n[X] \;=\; \Pi_n \;\subseteq\; \bigcap_{j \in \mathbb{Z} \setminus \{0\}} \Pi_{n,j}$$

for each $n \in \mathbb{N}$ (cf. Remark 7). Furthermore, for $f \in X$ and $t \in \mathbb{R}$ we have

$$\Delta_t^0(f) = f, \;\; \Delta_t^m(f)(\cdot) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\cdot - jt) \qquad (m \ge 1).$$

Consequently, in the above setting the results obtained in this section hold with $C = 1$.

4. Linear Sums of Cesàro-Marcinkiewicz Type Operators

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space with the usual inner product

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$$

for $x = (x_1, x_2, \cdots, x_d), y = (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d$. Let $\mathbb{T}^d$ be the cube given by

$$\mathbb{T}^d = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : -\pi \le x_j < \pi, j = 1, \cdots, d\}.$$

Let φ be a non-decreasing, continuous convex function on $[0, \infty)$ satisfying

$$\varphi(0) = 0, \ \ \varphi(t) > 0 \ \ (t > 0), \ \ \lim_{t \to 0} \frac{\varphi(t)}{t} = 0, \ \ \lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty.$$

The function φ is said to satisfy Condition (Δ_2) if there exist constants $c > 0$ and $t_0 \ge 0$ such that

$$\varphi(2t) \le c\varphi(t) \quad \text{for all } t \ge t_0$$

(cf. [15], [30]).

Let $L_\varphi(\mathbb{T}^d)$ be the set of all measurable functions f on $\mathbb{R}^d$ having period 2π in each variable such that

$$\int_{\mathbb{T}^d} \varphi(|f(x)|)\, dx < \infty,$$

and $L_\varphi^*(\mathbb{T}^d)$ denotes the set of all measurable functions f on $\mathbb{R}^d$ such that $\alpha f \in L_\varphi(\mathbb{T}^d)$ for some $\alpha > 0$. Let ψ be the complementary function to φ in the sense of Young, i.e.,

$$\psi(u) = \sup\{tu - \varphi(t) : t \ge 0\} \qquad (u \ge 0),$$

and so evidently, the pair (φ, ψ) satisfies Young's inequality:

$$tu \le \varphi(t) + \psi(u) \quad \text{for all } t, u \ge 0.$$

For each $f \in L_\varphi^*(\mathbb{T}^d)$, we define

$$\|f\|_\varphi = \sup\left\{\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}|f(x)g(x)|\,dx : \varrho(g,\psi) \le 1\right\},$$

where

$$\varrho(g,\psi) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\psi(|g(x)|)\,dx,$$

which is called the Orlicz norm of f with respect to φ. Then $L_\varphi^*(\mathbb{T}^d)$ becomes a Banach space with the norm $\|\cdot\|_\varphi$, which can be equivalent to the Luxemburg's norm defined by

$$\|f\|_{(\varphi)} = \inf\left\{\lambda > 0 : \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\varphi\left(\frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}$$

(cf. [15], [30]). Let $\mathbb{Z}^d$ be the set of all lattice points in $\mathbb{R}^d$, i.e.,

$$\mathbb{Z}^d = \{m = (m_1, \cdots, m_d) : m_j \in \mathbb{Z}, j = 1, \cdots, d\}.$$

For a given $f \in L_\varphi^*(\mathbb{T}^d)$, we define the Fourier coefficient of f by

$$\hat{f}(m) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x)e^{-im\cdot x}\,dx \qquad (m \in \mathbb{Z}^d),$$

and then the Fourier series of f is defined by

$$(37) \qquad f(x) \quad \sim \quad \sum_{m\in\mathbb{Z}^d}\hat{f}(m)e^{im\cdot x} \qquad (x \in \mathbb{R}^d).$$

For a point $n = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$, we denote the n-th partial sum of the Fourier series (37) of f by

$$S_n(f)(x) = \sum_{|m_j|\le n_j, j=1,\cdots,d}\hat{f}(m)e^{im\cdot x},$$

and let σ_n be the n-th Cesàro mean operator of $\{S_n\}$, i.e.,

$$\sigma_n = \frac{1}{(n_1+1)(n_2+1)\cdots(n_d+1)}\sum_{k\in\mathbb{N}_n^d} S_k.$$

For each $m \in \mathbb{N}$, we define

$$\sigma_m^* = \frac{1}{m+1} \sum_{j=0}^{m} S_{(j,j,\cdots,j)},$$

which is called the m-th Cesàro-Marcinkiewicz mean operator, and let V_m^* be the m-th de la Vallée-Poussin-Marcinkiewicz operator, that is,

$$V_m^* = \frac{1}{m+1} \sum_{j=m+1}^{2m+1} S_{(j,j,\cdots,j)} = 2\sigma_{2m+1}^* - \sigma_m^*.$$

For a point $n = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$, we denote by Π_n^d the set of all d-dimensional trigonometric polynomials of degree n, i.e., all functions $g(x_1, x_2, \cdots, x_d)$ which are trigonometric polynomials of degree n_j with respect to $x_j, j = 1, 2, \cdots, d$. For a given $f \in L_\varphi^*(\mathbb{T}^d)$, we define

$$E_n(\varphi; f) = \inf\{\|f - g\|_\varphi : g \in \Pi_n^d\},$$

which is called the best approximation of degree n to f with respect to Π_n^d (cf. [17], [33]). If $f \in L_\varphi^*(\mathbb{T}^d)$ and $\delta \geq 0$, then we define

$$\omega_j(f, \delta) = \omega_j(\varphi; f, \delta) = \sup\{\|\Delta_{t,j}(f)\|_\varphi : |t| \leq \delta\} \quad (j = 1, \cdots, d),$$

where

$$\Delta_{t,j}(f)(x_1, \cdots, x_d) = f(x_1, \cdots, x_j - t, \cdots, x_d) - f(x_1, \cdots, x_j, \cdots, x_d).$$

The quantities $\omega_j(f, \delta), j = 1, 2, \cdots, d$, are called the j-th partial moduli of continuity of f, and (1), (2) and (3) hold for $A_j = B_j = 1$ (cf. [17], [33]).

From now on we suppose that $L_\varphi^*(\mathbb{T}^d)$ is reflexive, which can be equivalent to that φ and ψ satisfy Condition (Δ_2) (cf. [15], [30]). Furthermore, for simplicity, we consider the case $d = 2$; The case where $d \geq 3$ is similar.

Hereafter, let $M_j(\varphi)(j = 1, 2, \cdots, 4)$ denote the suitable positive constants depending only on φ. Then the following results are obtained by Firlej [7]: Let $f \in L_\varphi^*(\mathbb{T}^2)$ and $n = (n_1, n_2) \in \mathbb{N}^2$.

(38) $\quad E_n(\varphi; f) \leq M_1(\varphi)\{\omega_1(f, 1/(n_1 + 1)) + \omega_2(f, 1/(n_2 + 1))\};$

$$\|S_n(f) - f\|_\varphi \le M_2(\varphi) E_n(\varphi; f); \tag{39}$$

$$\|\sigma_n(f) - f\|_\varphi \le M_3(\varphi)\{\omega_1(f, 1/(n_1+1)) + \omega_2(f, 1/(n_2+1))\}. \tag{40}$$

It follows from (35) and (36) that

$$\|S_n(f) - f\|_\varphi \le M_1(\varphi) M_2(\varphi) \times \{\omega_1(f, 1/(n_1+1)) + \omega_2(f, 1/(n_2+1))\} \tag{41}$$

for all $n = (n_1, n_2) \in \mathbb{N}^2$, and there holds

$$\|\sigma^*_m(f) - f\|_\varphi \le M_4(\varphi)\{\omega_1(1, 1/(m+1)) + \omega_2(f, 1/(m+1))\} \tag{42}$$

for all $m \in \mathbb{N}$ ([8; Theorem 1]). Now we take

$$L_m = S_{(m,m)}, \quad \lambda_m = \mu_m = \frac{1}{m+1} \qquad (m \in \mathbb{N}).$$

Then in view of (41) and (42), applying Theorem 1 we have the following.

THEOREM 7. *Let $f \in L^*_\varphi(\mathbb{T}^2)$. Then the following statements hold:*

(a) *For all $m \in \mathbb{N}$ and all $\epsilon > 0$,*

$$\|W_m(f) - f\|_\varphi \le M_1(\varphi) M_2(\varphi)(1 + \epsilon^{-1})\{\omega_1(f, \epsilon\xi_m) + \omega_2(f, \xi_m)\},$$

where

$$\xi_m = \sum_{j=0}^{m} \frac{p_{mj}}{j+1} \qquad (m \in \mathbb{N}).$$

(b) *If Condition (P) is satisfied, then*

$$\|W_m(f) - f\|_\varphi \le M_4(\varphi)(A + \epsilon^{-1})\{\omega_1(f, \epsilon\theta_m) + \omega_2(f, \epsilon\theta_m)\}$$

for all $m \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\theta_m = p_{mm} + \sum_{j=0}^{m-1} |p_{mj} - p_{mj+1}| \qquad (m \in \mathbb{N}).$$

COROLLARY 6. *Let $f \in L_\varphi^*(\mathbb{T}^2)$. Then the following assertions hold:*

(a) *If (P) satisfies* (7), *then*

$$(43)\quad \|W_m(f) - f\|_\varphi \le M_4(\varphi)(1+\epsilon^{-1})\{\omega_1(f,\epsilon p_{m0}) + \omega_2(f,\epsilon p_{m0})\}$$

for all $m \in \mathbb{N}$ and all $\epsilon > 0$.

(b) *If P satisfies* (9) *and* (10), *then*

$$(44)\qquad \|W_m(f) - f\|_\varphi \le M_4(\varphi)(2B - 1 + \epsilon^{-1})$$

$$\times \quad \{\omega_1(f,\epsilon(2p_{mm} - p_{m0})) + \omega_2(f,\epsilon(2p_{mm} - p_{m0}))\}$$

for all $m \in \mathbb{N}$ and all $\epsilon > 0$.

REMARK 10: If one take $\epsilon = 1$, then (43) reduces to

$$\|W_m(f) - f\|_\varphi \le 2M_4(\varphi)\{\omega_1(f,p_{m0}) + \omega_2(f,p_{m0})\};$$

Also, by selecting $\epsilon = 1/2$, (44) reduces to

$$\|W_m(f) - f\|_\varphi \le (2B+1)M_4(\varphi)$$

$$\times \quad \{\omega_1(f,(2p_{mm} - p_{m0})/2) + \omega_2(f,(2p_{mm} - p_{m0})/2)\}$$
$$\le (2B+1)M_4(\varphi)\{\omega_1(f,p_{mm}) + \omega_2(f,p_{mm})\}.$$

Thus, this result yields [8; Theorem 2].

Next take

$$L_n = S_n \;\; (n = (n_1,n_2) \in \mathbb{N}^2),\; \lambda_m^{(j)} = \mu_m^{(j)} = \frac{1}{m+1} \;\; (m \in \mathbb{N}, j = 1,2).$$

Then in view of (40) and (41), applying Theorem 2 we derive the following.

THEOREM 8. *Let $f \in L_\varphi^*(\mathbb{T}^2)$. Then the following statements hold:*

(a) *For all $n = (n_1,n_2) \in \mathbb{N}^2$ and all $\epsilon > 0$,*

$$\|W_n(f) - f\|_\varphi \le M_1(\varphi)M_2(\varphi)(1+\epsilon^{-1})\{\omega_1(f,\epsilon\xi_{n_1}^{(1)}) + \omega_2(f,\epsilon\xi_{n_2}^{(2)})\},$$

where

$$\xi_{n_j}^{(j)} = \sum_{k=0}^{n_j} \frac{p_{n_j k}^{(j)}}{k+1} \qquad (j = 1, 2).$$

(b) *If Condition $(P)^*$ is satisfied, then*

$$\|W_n(f) - f\|_\varphi \le M_3(\varphi)$$

$$\times \quad \{(C^{(1)} + \epsilon^{-1})\omega_1(f, \epsilon\theta_{n_1}^{(1)}) + (C^{(2)} + \epsilon^{-1})\omega_2(f, \epsilon\theta_{n_2}^{(2)})\}$$

for all $n = (n_1, n_2) \in \mathbb{N}^2$ and all $\epsilon > 0$, where

$$\theta_{n_j}^{(j)} = \sum_{k=0}^{n_j} \frac{|p_{n_j k}^{\sim}|}{k+1} \qquad (j = 1, 2).$$

COROLLARY 7. *Let $f \in L_\varphi^*(\mathbb{T}^2)$. Then the following assertions hold:*

(a) *If (17) holds for $r = 2$, then*

$$\|W_n(f) - f\|_\varphi \le M_3(\varphi)(1 + \epsilon^{-1})\{\omega_1(f, \epsilon p_{n_1 0}^{(1)}) + \omega_2(f, \epsilon p_{n_2 0}^{(2)})\} \tag{45}$$

for all $n = (n_1, n_2) \in \mathbb{N}^2$ and all $\epsilon > 0$.

(b) *If (18) and (19) hold for $r = 2$, then*

$$\|W_n(f) - f\|_\varphi \le M_3(\varphi) \tag{46}$$

$$\times \quad \{(2M^{(1)} - 1 + \epsilon^{-1})\omega_1(f, \epsilon(2p_{n_1 n_1}^{(1)} - p_{n_1 0}^{(1)}))$$

$$+ \quad (2M^{(2)} - 1 + \epsilon^{-1})\omega_2(f, \epsilon(2p_{n_2 n_2}^{(2)} - p_{n_2 0}^{(2)}))\}$$

for all $n = (n_1, n_2) \in \mathbb{N}^2$ and all $\epsilon > 0$.

REMARK 11: If one take $\epsilon = 1$, then (45) reduces to

$$\|W_n(f) - f\|_\varphi \le 2M_3(\varphi)\{\omega_1(f, p_{n_1 0}^{(1)}) + \omega_2(f, p_{n_2 0}^{(2)})\};$$

Also, by choosing $\epsilon = 1/2$, (46) reduces to

$$\|W_n(f) - f\|_\varphi \le M_3(\varphi)$$

$$\begin{aligned}
\times \quad & \{(2M^{(1)}+1)\omega_1(f,(2p^{(1)}_{n_1n_1}-p^{(1)}_{n_10})/2) \\
+ \quad & (2M^{(2)}+1)\omega_2(f,(2p^{(2)}_{n_2n_2}-p^{(2)}_{n_20})/2)\} \\
\leq M_3(\varphi)\{ & (2M^{(1)}+1)\omega_1(f,p^{(1)}_{n_1n_1})+(2M^{(2)}+1)\omega_2(f,p^{(2)}_{n_2n_2})\}.
\end{aligned}$$

Thus, this result yields [8; Theorem 3].

REMARK 12: Let $C(\mathbb{T}^d)$ be the Banach space of all continuous functions f on $\mathbb{R}^d$ which have period 2π in each variable, with the norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{T}^d\}.$$

Let $f \in C(\mathbb{T}^d), m \in \mathbb{N}$ and define

$$E_m(f) = \inf\{\|f-g\|_\infty : g \in \Pi^d_{(m+1,\cdots,m+1)}\}.$$

Then we have

$$\|V^*_m(f)-f\|_\infty \leq (\|V^*_m\|_{B[C(\mathbb{T}^d)]}+1)E_m(f) \leq 4E_m(f)$$

(cf. [28; Theorem 4]), and so [17; Chap. 6, Theorem 6] (cf. [33; Sec. 5.3]) establishes

$$\|V^*_m(f)-f\|_\infty \leq C\sum_{j=1}^{d}\omega_j(f,1/(m+1)),$$

where C is a positive constant independent of f and m, and $\omega_j(f,\cdot)$, $j=1,2,\cdots,d$, denote the jth partial moduli of continuity of f with respect to the norm $\|\cdot\|_\infty$. Let $L_m = V^*_m, m \in \mathbb{N}$. Then applying Theorem 1 (a), we obtain

$$\|W_m(f)-f\|_\infty \leq C(1+\epsilon^{-1})\sum_{j=1}^{d}\omega_j(f,\epsilon\xi_m)$$

for all $m \in \mathbb{N}$ and all $\epsilon > 0$, where

$$\xi_m = \sum_{j=0}^{m}\frac{p_{mj}}{j+1} \qquad (m \in \mathbb{N})$$

(cf. Corollary 5 (b)).

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