Approximation processes of quasi-positive linear operators

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APPROXIMATION PROCESSES OF QUASI-POSITIVE LINEAR OPERATORS

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Abstract. The convergence on approximation processes of quasi-positive linear operators is discussed. The results are then applied to obtain several Korovkin test systems.

1. Introduction

Let X be a compact Hausdorff space and let E be a normed linear space over the scalar field $\mathbb{K}$ which is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $B(X, E)$ denote the normed linear space of all E-valued bounded functions on X, endowed with the usual pointwise addition, scalar multiplication and the supremum norm $\|\cdot\|$. We shall use the same symbol $\|\cdot\|$ for the underlying norms. $C(X, E)$ denotes the closed linear subspace of $B(X, E)$ consisting of all E-valued continuous functions on X. In the case when E is equal to $\mathbb{K}$, we simply write $B(X)$ and $C(X)$ instead of $B(X, E)$ and $C(X, E)$, respectively.

For any $a \in E$ and $v \in B(X)$, the function va is defined by $(va)(x) = v(x)a$ for all $x \in X$. Also, for any $v \in B(X)$ and $f \in B(X, E)$, we define $(vf)(x) = v(x)f(x)$ for all $x \in X$. Plainly, va and vf belong to $B(X, E)$, and $\|va\| = \|v\|\|a\|$ and $\|vf\| \leq \|v\|\|f\|$. If $a \in E$, $v \in C(X)$ and $f \in C(X, E)$, then va and vf belong to $C(X, E)$. We denote by $C(X) \otimes E$ the linear subspace of $C(X, E)$ consisting of all finite sums of functions of the form va, where $v \in C(X)$ and $a \in E$. 1_X stands for the unit function defined by $1_X(x) = 1$ for all $x \in X$.

Here motivated by the previous work of the author [11] we study the convergence on approximation processes of quasi-positive linear operators of $C(X, E)$ into E or $B(X, E)$. The results yield several Korovkin test systems, which can be useful for applications. Actually, we extend the results of [10] to the context of functions taking a value in an arbitrary normed linear space.

We refer to [1] for detailed references on the several other contributions to the area of Korovkin-type approximation theory.

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2. Auxiliary Results

We begin with the following definition:

Definition 1: Let A and B be normed linear spaces. Let $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of bounded linear operators of A into B, where D is a directed set and Λ is an index set, and let T be a bounded linear operator of A into B. Then the family $\{T_{\alpha,\lambda}\}$ is called an approximation process with respect to T on A if for every $f \in A$,

$$(1) \qquad \lim_{\alpha} \|T_{\alpha,\lambda}(f) - T(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

Obviously, if for a net $\{T_\alpha : \alpha \in D\}$ of bounded linear operators of A into B, we take $T_{\alpha,\lambda} = T_\alpha$ for all $\alpha \in D$ and all $\lambda \in \Lambda$, then the convergence behavior (1) reduces to the usual one. Several other approximation processes can be induced by various summability methods due to the author [6] (cf. [7], [8]), which include the method of almost convergence (F-summability) of Lorentz [5], as the most typical case.

Lemma 1. *Let $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ and T be as in Definition 1, and let M be a dense subset of A. If there exists an element $\alpha_0 \in D$ such that*

$$(2) \qquad \sup\{\|T_{\alpha,\lambda}\| : \alpha \geq \alpha_0, \alpha \in D, \lambda \in \Lambda\} < \infty$$

and if for all $g \in M$,

$$(3) \qquad \lim_{\alpha} \|T_{\alpha,\lambda}(g) - T(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

then $\{T_{\alpha,\lambda}\}$ is an approximation process with respect to T on A. If, moreover, A is a Banach space, then the converse is also true.

Proof: The former is a standard argument on account of (2) and (3) (cf. the theorem of Banach-Steinhaus). The latter follows from the uniform boundedness principle.

Lemma 2. *$C(X) \otimes E$ is dense in $C(X, E)$.*

Proof: This is an immediate consequence of [12; Theorem 1.15], since $C(X)$ separates the points of X.

LEMMA 3. *Let B be a normed linear space. Let $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of bounded linear operators of $C(X, E)$ into B such that there exists an element $\alpha_0 \in D$ for which (2) is fulfilled, and let T be a bounded linear operator of $C(X, E)$ into B. If for all $a \in E$ and all $v \in C(X)$,*

$$\lim_{\alpha} \|T_{\alpha,\lambda}(va) - T(va)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda,$$

then $\{T_{\alpha,\lambda}\}$ is an approximation process with respect to T on $C(X, E)$.

PROOF: This follows from Lemmas 1 and 2.

DEFINITION 2: Let A be a linear subspace of $B(X, E)$ which contains $C(X) \otimes E$. A bounded linear operator L of A into $B(X, E)$ is said to be quasi-positive if $v, w \in C(X)$ and $|v(x)| \leq w(x)$ for all $x \in X$, then

$$(4) \qquad \|L(va)(x)\| \leq \|L(wa)(x)\| \qquad \text{for all } a \in E \text{ and all } x \in X.$$

A bounded linear operator L of A into E is said to be quasi-positive if $v, w \in C(X)$ and $|v(x)| \leq w(x)$ for all $x \in X$, then

$$(5) \qquad \|L(va)\| \leq \|L(wa)\| \qquad \text{for all } a \in E.$$

REMARK 1: Let $E = \mathbb{K}$. Then (4) and (5) are equivalent to

$$|L(v)(x)| \leq |L(w)(x)| \qquad \text{for all } x \in X$$

and

$$|L(v)| \leq |L(w)|,$$

respectively. Furthermore, the positivity of L implies the quasi-positivity of it, but not conversely in general.

3. CONVERGENCE THEOREMS

Let $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of quasi-positive linear operators of $C(X, E)$ into E such that there exists an element $\beta_0 \in D$ for which

$$(6) \qquad \sup\{\|L_{\alpha,\lambda}\| : \alpha \geq \beta_0, \alpha \in D, \lambda \in \Lambda\} < \infty.$$

For any function $\Psi \in C(X)$ and for any element $a \in E$, we define

$$\mu_{\alpha,\lambda}(\Psi, a) = \|L_{\alpha,\lambda}(\Psi a)\| \qquad (\alpha \in D, \ \lambda \in \Lambda).$$

Let t be an arbitrary fixed point of X and let Φ_t be a non-negative function in $C(X, \mathbb{R})$ such that

(7) $\inf\{\Phi_t(x) : x \in F\} > 0$ for every closed subset F of $X \setminus \{t\}$.

REMARK 2: If there exists a non-negative function $h \in C(X, \mathbb{R})$ such that $0 < h(x) \leq \Phi_t(x)$ for all $x \in X$ with $x \neq t$, then (7) always holds. In particular, if Φ_t is a non-negative function in $C(X, \mathbb{R})$ satisfying $\Phi_t(x) > 0$ for all $x \in X$ with $x \neq t$, then (7) is automatically fulfilled.

PROPOSITION 1. *Let $a \in E$. If*

(8) $$\lim_{\alpha} \mu_{\alpha,\lambda}(\Phi_t, a) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda,$$

then for every function $\Psi \in C(X)$ satisfying $\Psi(t) = 0$,

(9) $$\lim_{\alpha} \mu_{\alpha,\lambda}(\Psi, a) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda.$$

PROOF: Let $\epsilon > 0$ be given. Then there exists an open neighborhood V_t of t such that $|\Psi(x)| < \epsilon$ for all $x \in V_t$. Let $F = X \setminus V_t$, and put

$$m = \inf\{\Phi_t(x) : x \in F\}$$

and

$$M = \sup\{|\Psi(x)| : x \in F\}.$$

Then by Condition (7), $m > 0$, and so we obtain

$$|\Psi(x)| \leq \epsilon + (M/m)\Phi_t(x)$$

for all $x \in X$. Therefore it follows that

$$\begin{aligned}\|L_{\alpha,\lambda}(\Psi a)\| &\leq \epsilon \|L_{\alpha,\lambda}(1_X a)\| + (M/m)\|L_{\alpha,\lambda}(\Phi_t a)\| \\ &\leq \epsilon \|a\| \|L_{\alpha,\lambda}\| + (M/m)\mu_{\alpha,\lambda}(\Phi_t, a),\end{aligned}$$

which together with (6) and (8) yields (9).

Let ξ be any fixed number in $\mathbb{K}$, and we define

$$L(f) = L^{(\xi,t)}(f) = \xi f(t) \qquad \text{for every } f \in C(X,E). \tag{10}$$

Evidently, L is a quasi-positive linear operator of $C(X,E)$ into E and $\|L\| = |\xi|$.

THEOREM 1. *If (8) holds for all $a \in E$ and if there exists a function $u \in C(X)$ such that $u(t) \neq 0$ and*

$$\lim_{\alpha} \|L_{\alpha,\lambda}(ua) - L(ua)\| = 0 \qquad \textit{uniformly in } \lambda \in \Lambda \tag{11}$$

for every $a \in E$, then $\{L_{\alpha,\lambda}\}$ is an approximation process with respect to L on $C(X,E)$.

PROOF: Let $v \in C(X)$ and we define

$$\Psi = v - (v(t)/u(t))u.$$

Then Ψ belongs to $C(X)$ and $\Psi(t) = 0$. Therefore, by Proposition 1, (9) implies

$$\lim_{\alpha} \|L_{\alpha,\lambda}(va) - (v(t)/u(t))L_{\alpha,\lambda}(ua)\| = 0 \quad \text{uniformly in } \lambda \in \Lambda \tag{12}$$

whenever a belongs to E. Since

$$\|L_{\alpha,\lambda}(va) - L(va)\| \leq \|L_{\alpha,\lambda}(va) - (v(t)/u(t))L_{\alpha,\lambda}(ua)\|$$

$$+ \quad |v(t)/u(t)|\|L_{\alpha,\lambda}(ua) - L(ua)\|,$$

by virtue of (11) and (12), we conclude that

$$\lim_{\alpha} \|L_{\alpha,\lambda}(va) - L(va)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $v \in C(X)$ and all $a \in E$. Thus, Lemma 3 gives the desired result.

COROLLARY 1. *If* (8) *and* (11) *with* $u = 1_X$ *hold for any* $a \in E$, *then* $\{L_{\alpha,\lambda}\}$ *is an approximation process with respect to* L *on* $C(X, E)$.

COROLLARY 2. *If* T *is a quasi-positive linear operator of* $C(X, E)$ *into* E *which satisfies* $T(\Phi_t a) = 0$ *and* $T(1_X a) = L(1_X a)$ *for all* $a \in E$, *then* $T = L$.

Let p be any fixed positive real number and let G be a subset of $C(X)$ which separates the points of X. For each $g \in G$, we define

$$\Psi_t^{(g)} = \Psi_t^{(p,g)} = |g - g(t)1_X|^p.$$

PROPOSITION 2. *Let* $a \in E$. *If for every* $g \in G$,

$$\lim_{\alpha} \mu_{\alpha,\lambda}(\Psi_t^{(g)}, a) = 0 \qquad \text{uniformly in } \lambda \in \Lambda, \tag{13}$$

then (9) *holds for every function* $\Psi \in C(X)$ *satisfying* $\Psi(t) = 0$.

PROOF: Since the original topology on X is identical with the weak topology on X induced by G, there exists a finite subset $\{g_1, \cdots, g_n\}$ of G and a constant $\eta > 0$ such that

$$|\Psi(x)| \leq \epsilon + \eta \sum_{j=1}^{n} |g_j(x) - g_j(t)|^p$$

for all $x \in X$. Therefore we obtain

$$\|L_{\alpha,\lambda}(\Psi a)\| \leq \epsilon \|L_{\alpha,\lambda}(1_X a)\| + \eta \sum_{j=1}^{n} \|L_{\alpha,\lambda}(\Psi_t^{(g_j)} a)\|$$

$$\leq \epsilon \|a\| \|L_{\alpha,\lambda}\| + \eta \sum_{j=1}^{n} \mu_{\alpha,\lambda}(\Psi_t^{(g_j)}, a),$$

which together with (6) and (13) proves (9).

THEOREM 2. *If* (13) *holds for all* $a \in E$ *and all* $g \in G$ *and if there exists a function* $u \in C(X)$ *satisfying* $u(t) \neq 0$ *and* (11) *for all* $a \in E$, *then* $\{L_{\alpha,\lambda}\}$ *is an approximation process with respect to* L *on* $C(X, E)$.

PROOF: With the aid of Proposition 2, the proof is exactly the same as that given for Theorem 1.

COROLLARY 3. *If (13) and (11) with $u = 1_X$ hold for all $a \in E$ and all $g \in G$, then $\{L_{\alpha,\lambda}\}$ is an approximation process with respect to L on $C(X,E)$.*

COROLLARY 4. *If T is a quasi-positive linear operator of $C(X,E)$ into E which satisfies $T(\Psi_t^{(g)}a) = 0$ and $T(1_X a) = L(1_X a)$ for all $a \in E$ and all $g \in G$, then $T = L$.*

REMARK 3: In view of Remark 1, in the special case $E = \mathbb{R}$, the positivity of functionals required in [10] can be weakened by the quasi-positivity of them.

4. KOROVKIN TEST SYSTEMS

In this section we give several applications of the results obtained in the previous section. For this in view of the classical Korovkin theory (cf. [4]) on the convergence of sequences of positive linear functionals on $C([a,b],\mathbb{R})$, we make the following definition:

DEFINITION 3: Let T be a quasi-positive linear operator of $C(X,E)$ into E. A subset S of $C(X,E)$ is called a Korovkin test system (or, briefly,KTS) with respect to T in $C(X,E)$ if for any family $\{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ of quasi-positive linear operators of $C(X,E)$ into E such that there exists an element $\beta_0 \in D$ satisfying (6), the relation

$$\lim_{\alpha} \|L_{\alpha,\lambda}(g) - T(g)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $g \in S$ implies that

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f) - T(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $f \in C(X,E)$.

The similar concept is defined in the setting of quasi-positive linear operators of $C(X,E)$ into $B(X,E)$ (cf. [11]). For a given subset V of $C(X)$, we define

$$VE = \{va : v \in V, a \in E\}.$$

If $B = E$ or $B = B(X,E)$, then it follows from Lemma 3 that the set $C(X)E$ is a KTS with respect to T in $C(X,E)$.

Let L be a quasi-positive linear operator of $C(X,E)$ into E defined by (10).

THEOREM 3. *Let w be a function in $C(X)$ with $w(t) \neq 0$. Then the following statements hold: (1°) Suppose that $\Phi_t(t) = 0$. Then $\{w, \Phi_t\}E$ is a KTS with respect to L in $C(X, E)$. In particular, $\{1_X, \Phi_t\}E$ is a KTS with respect to L in $C(X, E)$. (2°) Let G be a subset of $C(X)$ separating the points of X, and let p be a positive real number. Then $(\{w\} \cup \{\Psi_t^{(p,g)} : g \in G\})E$ is a KTS with respect to L in $C(X, E)$. In particular, $(\{1_X\} \cup \{\Psi_t^{(p,g)} : g \in G\})E$ is a KTS with respect to L in $C(X, E)$.*

PROOF: (1°) and (2°) immediately follow from Theorems 1 and 2, respectively.

COROLLARY 5. *(1°) Let $\{u_1, u_2, \cdots, u_m\}$ be a finite subset of $C(X)$, and let*

$$(14) \qquad \Phi_t = \sum_{k=1}^{m} a_k(t) u_k,$$

where each $a_k(t)$ is a number in $\mathbb{K}$ such that

$$\Phi_t(t) = 0, \quad \Phi_t(x) \geq 0 \quad \text{for all } x \in X$$

and (7) is satisfied. Let $u_0 \in C(X)$ and $u_0(t) \neq 0$. Then

$$\{u_0, u_1, \cdots, u_m\}E$$

is a KTS with respect to L in $C(X, E)$. In particular,

$$\{1_X, u_1, \cdots, u_m\}E$$

is a KTS with respect to L in $C(X, E)$. (2°) Let p be an even positive integer. If G is a subset of $C(X, \mathbb{R})$ which separates the points of X, then

$$\{g^k : g \in G, k = 0, 1, 2, \cdots, p\}E,$$

where $g^0 = 1_X$, is a KTS with respect to L in $C(X, E)$. If G is a subset of $C(X, \mathbb{C})$ which separates the points of X, then

$$(\{1_X\} \cup G \cup \overline{G} \cup |G|^2)E$$

is a KTS with respect to L in $C(X,E)$, where

$$\overline{G} = \{\overline{g} : g \in G\} \quad \text{and} \quad |G|^2 = \{|g|^2 : g \in G\}.$$

COROLLARY 6. *Let u be a strictly positive function in $C(X,\mathbb{R})$. Then the following statements hold: (1°) Let $\{u_1, u_2, \cdots, u_m\}$ be a finite subset of $C(X,\mathbb{R})$ such that for every $x \in X$ with $x \neq t$, there exists an integer $j \in \{1, 2, \cdots, m\}$ for which $u_j(x) \neq u_j(t)$. Then*

$$\{u, uu_1, uu_2, \cdots, uu_m, u \sum_{k=1}^{m} u_k^2\}E$$

is a KTS with respect to L in $C(X,E)$. (2°) Let $\{v_1, v_2, \cdots, v_m\}$ be a finite subset of $C(X,\mathbb{C})$ such that for every $x \in X$ with $x \neq t$, there exists an integer $j \in \{1, 2, \cdots, m\}$ for which $v_j(x) \neq v_j(t)$. Then

$$\{u, uv_1, uv_2, \cdots, uv_m, u\overline{v_1}, u\overline{v_2}, \cdots, u\overline{v_m}, u \sum_{k=1}^{m} |v_k|^2\}E$$

is a KTS with respect to L in $C(X,E)$.

Here we restrict ourselves to the case where X is a compact subset of $\mathbb{K}^m$ and for each $k = 1, 2, \cdots, m$, p_k denotes the k-th coordinate function defined by

$$p_k(x) = x_k \qquad \text{for every } x = (x_1, x_2, \cdots, x_m) \in X.$$

Then by Corollary 5 (2°) and Corollary 6 we have the following some Korovkin test systems:

(1°) Let $\mathbb{K} = \mathbb{R}$. Then

$$S_1 = \{1_X, p_1, \cdots, p_m, p_1^2, \cdots, p_m^2\}E$$

and

$$S_1' = \{1_X, p_1, p_2, \cdots, p_m, \sum_{k=1}^{m} p_k^2\}E$$

are Korovkin test systems with respect to L in $C(X,E)$.

(2°) Let $\mathbb{K} = \mathbb{C}$. Then

$$S_2 = \{1_X, p_1, \cdots, p_m, \overline{p_1}, \cdots, \overline{p_m}, |p_1|^2, \cdots, |p_m|^2\}E$$

and

$$S_2' = \{1_X, p_1, \cdots, p_m, \overline{p_1}, \cdots, \overline{p_m}, \sum_{n=1}^{m} |p_n|^2\}E$$

are Korovkin test systems with respect to L in $C(X, E)$.

(3°) Let X be the m-dimensional torus $\mathbb{T}^m$, i.e.,

$$\mathbb{T}^m = \{x = (x_1, x_2, \cdots, x_m) \in \mathbb{C}^m : |x_k| = 1, k = 1, 2, \cdots, m\}.$$

Then

$$S_3 = \{1_X, p_1, \cdots, p_m, \overline{p_1}, \cdots, \overline{p_m}\}E$$

is a KTS with respect to L in $C(\mathbb{T}^m, E)$.

(4°) Let $\mathbb{K} = \mathbb{C}$ and for each $k = 1, 2, \cdots, m$, we define

$$q_k(x) = \mathrm{Re}(x_k) \quad \text{and} \quad r_k(x) = \mathrm{Im}(x_k)$$

for every $x = (x_1, x_2, \cdots, x_m) \in X$, where $\mathrm{Re}(x_k)$ and $\mathrm{Im}(x_k)$ stand for the real part of x_k and the imaginary part of x_k, respectively. Then

$$S_4 = \{1_X, q_1, \cdots, q_m, r_1, \cdots, r_m, q_1^2, \cdots, q_m^2, r_1^2, \cdots, r_m^2\}E$$

and

$$S_4' = \{1_X, q_1, \cdots, q_m, r_1, \cdots, r_m, \sum_{n=1}^{m} (q_n^2 + r_n^2)\}E$$

are Korovkin test systems with respect to L in $C(X, E)$.

(5°) Let $X = \mathbb{T}^m$, and let q_k and r_k ($k = 1, 2, \cdots, m$) be as in (4°). Then

$$S_5 = \{1_X, q_1, \cdots, q_m, r_1, \cdots, r_m\}E$$

is a KTS with respect to L in $C(X, E)$.

(6°) Let $C_{2\pi}(\mathbb{R}^m, E)$ denote the normed linear space of all E-valued continuous functions f on $\mathbb{R}^m$ which are periodic with period 2π in each variable with the norm

$$\|f\| = \sup\{\|f(x)\| : x \in \mathbb{R}^m\}.$$

Then $C(\mathbb{T}^m, E)$ is isometrically isomorphic to $C_{2\pi}(\mathbb{R}^m, E)$. For each $k = 1, 2, \cdots, m$ we define

$$c_k(x) = \cos x_k, \quad s_k(x) = \sin x_k$$

for all $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ and

$$U(g) = \xi g(y) \qquad \text{for every } g \in C_{2\pi}(\mathbb{R}^m, E),$$

where ξ is a fixed number in $\mathbb{K}$ and y is a fixed point of $\mathbb{R}^m$. Then U is a quasi-positive linear operator of $C_{2\pi}(\mathbb{R}^m, E)$ into E and

$$S_6 = \{1_{\mathbb{R}^m}, c_1, \cdots, c_m, s_1, \cdots, r_m\}E$$

is a KTS with respect to U in $C_{2\pi}(\mathbb{R}^m, E)$.

THEOREM 4. *Let X be a compact subset of an inner product space $(H, <,>)$ over $\mathbb{K}$, and we define*

$$h(x) =< x, x > \quad \text{and} \quad h_t(x) =< x, t > \qquad \text{for every } x \in X.$$

Then the following statements hold: (1°) *If* $\mathbb{K} = \mathbb{R}$, *then* $\{1_X, h, h_t\}E$ *is a KTS with respect to L in $C(X, E)$.* (2°) *If* $\mathbb{K} = \mathbb{C}$, *then* $\{1_X, h, h_t, \overline{h_t}\}E$ *is a KTS with respect to L in $C(X, E)$.*

PROOF: Let

$$\Phi_t(x) =< x - t, x - t > \qquad \text{for all } x \in X.$$

If $\mathbb{K} = \mathbb{R}$, then

$$\Phi_t(x) = h(x) - 2h_t(x) + h(t).$$

If $\mathbb{K} = \mathbb{C}$, then

$$\Phi_t(x) = h(x) - h_t(x) - \overline{h_t}(x) + h(t).$$

Therefore Φ_t is represented in the form (14), and so the desired results follow from Corollary 5 (1°).

For example, if we take $H = \mathbb{R}^m$ with the usual inner product

$$< x, y >= x_1y_1 + x_2y_2 + \cdots + x_my_m$$

for $x = (x_1, x_2, \cdots, x_m), y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, then

$$h = \sum_{k=1}^{m} p_k^2 \quad \text{and} \quad h_t = \sum_{k=1}^{m} p_k(t)p_k.$$

Also, if we take $H = \mathbb{C}^m$ with the inner product

$$< x, y >= x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_m\overline{y_m}$$

for $x = (x_1, x_2, \cdots, x_m), y = (y_1, y_2, \cdots, y_m) \in \mathbb{C}^m$, then

$$h = \sum_{k=1}^{m} |p_k|^2 \quad \text{and} \quad h_t = \sum_{k=1}^{m} \overline{p_k}(t) p_k.$$

Consequently, S_1' and S_2' again become Korovkin test systems with respect to L in $C(X, E)$.

REMARK 4: The results obtained in Sections 3 and 4 can also be applied to the case where

$$G = \{g\,|_X : g \in F^*\},$$

where X is a compact subset of a locally convex Hausdorff vector space F over $\mathbb{K}$ with its dual space F^* and $g\,|_X$ denotes the restriction of g to X (cf. [2], [3], [7], [9], [13]).

REMARK 5: In view of [11; Remarks 1 and 2], the sets $S_1, S_1', S_2, S_2', S_3, S_4, S_4', S_5$ and S_6 are also Korovkin test systems with respect to a multiplication operator T given by

$$T(f) = uf \qquad \text{for every } f \in C(X, E),$$

where u is an arbitrary fixed function in $B(X)$, in the setting of quasi-positive linear operators of $C(X, E)$ into $B(X, E)$.

Finally, we give the concrete examples of approximation processes of the integral operators: Suppose that E is a Banach space and

$$X = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : a \leq x_n \leq b, n = 1, \cdots, m\}.$$

Let $\{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of non-negative functions in $C([a, b], \mathbb{R})$ and for each $\alpha \in D$ and $\lambda \in \Lambda$, we define the operator $L_{\alpha,\lambda}$ by

$$L_{\alpha,\lambda}(f) = \int_a^b \cdots \int_a^b k_{\alpha,\lambda}(x_1) \cdots k_{\alpha,\lambda}(x_m) f(x_1, \cdots, x_m)\, dx_1 \cdots dx_m$$

for every $f \in C(X,E)$. Then each $L_{\alpha,\lambda}$ is a quasi-positive linear operator of $C(X,E)$ into E and the the following assertion is true: If

$$\lim_{\alpha} \int_a^b k_{\alpha,\lambda}(y)\,dy = \eta_0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

$$\lim_{\alpha} \int_a^b y\,k_{\alpha,\lambda}(y)\,dy = \eta_1 \qquad \text{uniformly in } \lambda \in \Lambda,$$

$$\lim_{\alpha} \int_a^b y^2 k_{\alpha,\lambda}(y)\,dy = \eta_2 \qquad \text{uniformly in } \lambda \in \Lambda$$

and $\eta_0\eta_2 = \eta_1^2$, then

$$\text{(15)} \qquad \lim_{\alpha} \|L_{\alpha,\lambda}(f) - L^{(\xi,t)}(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for all $f \in C(X,E)$, where $L^{(\xi,t)}$ is the operator defined by (10) with

$$\xi = \eta_0^m \qquad \text{and} \qquad t = (\eta_1/\eta_0, \eta_1/\eta_0, \cdots, \eta_1/\eta_0).$$

Let $\{k_n : n \in \mathbb{N}\}$ be a sequence of non-negative functions in $C([a,b],\mathbb{R})$, where $\mathbb{N}$ denotes the set of all positive integers. In view of the concept of the almost convergence (F-summability) introduced by Lorentz [4], we define

$$k_{n,r} = \frac{1}{n}\sum_{q=r}^{n+r-1} k_q \qquad (n,\ r \in \mathbb{N})$$

and

$$\nu_n^{(j)} = \int_a^b y^j k_n(y)\,dy \qquad (n \in \mathbb{N},\ j = 0,1,2).$$

Then it follows from the above observation that if for each $j = 0$, 1 and 2, $\{\nu_n^{(j)} : n \in \mathbb{N}\}$ is almost convergent to η_j with $\eta_0\eta_2 = \eta_1^2$, then (15) holds for every $f \in C(X,E)$, where

$$k_{\alpha,\lambda} = k_{n,r} \qquad \text{and} \qquad D = \Lambda = \mathbb{N}.$$

For instance, we take $[a,b] = [0,1]$. If

$$k_n(y) = (n+1)y^n \qquad (n \in \mathbb{N},\ a \le y \le b),$$

then for every $f \in C(X, E)$,

$$\lim_{n\to\infty} \|L_{n,r}(f) - L^{(\xi,t)}(f)\| = 0 \qquad \text{uniformly in } r \in \mathbb{N} \tag{16}$$

with $\xi = 1$ and $t = (1, 1, \cdots, 1)$. Also, if

$$k_n(y) = 2(1 - y^2)^n / \rho_n \qquad (n \in \mathbb{N},\ a \leq y \leq b),$$

where

$$\rho_n = \int_{-1}^{1} (1 - y^2)^n \, dy = \Gamma(1/2)\Gamma(n + 1)\Gamma(n + 3/2)$$

$$= \frac{2^{(2n+1)}(n!)^2}{(2n + 1)!},$$

then for every $f \in C(X, E)$, (16) holds with $\xi = 1$ and $t = (0, 0, \cdots, 0)$.

In the light of the above arguments, we close with the following remark:

REMARK 6: Recall that X is a compact Hausdorff space and E is a normed linear space over $\mathbb{K}$. Let $\{T_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of bounded linear operators of $C(X, E)$ into E and $\{W_{\alpha,\lambda} : \alpha \in D, \lambda \in\in \Lambda\}$ a family of bounded linear functionals on $C(X)$. Let T be a bounded linear operator of $C(X, E)$ into E and W a bounded linear functional on $C(X)$. Suppose that there exists an element $\alpha_0 \in D$ satisfying (2) and that

$$T_{\alpha,\lambda}(va) = W_{\alpha,\lambda}(v)a \quad \text{and} \quad T(va) = W(v)a$$

for all $\alpha \in D, \lambda \in \Lambda, v \in C(X)$ and all $a \in E$. Then $\{T_{\alpha,\lambda}\}$ is an approximation process with respect to T on $C(X, E)$ if and only if $\{W_{\alpha,\lambda}\}$ is an approximation process with respect to W on $C(X)$. The similar claim also remains valid in the setting of bounded linear operators of $C(X, E)$ into $B(X, E)$ and bounded linear operators of $C(X)$ into $B(X)$ (cf. [11]).

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