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# Saturation of iterations of multiplier operators in Banach spaces



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# SATURATION OF ITERATIONS OF MULTIPLIER OPERATORS IN BANACH SPACES

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## 1. Introduction

Let $X$ be a Banach space with norm $\|\cdot\|_X$ and let $B[X]$ denote the Banach algebra of all bounded linear operators of $X$ into itself with the usual operator norm $\|\cdot\|_{B[X]}$. A family $\{L_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ of operators in $B[X]$ is called an approximation process on $X$ if for every $f \in X$,

(1) $\lim_{\alpha}\|L_{\alpha,\lambda}(f) - f\|_X = 0$ uniformly in $\lambda \in \Lambda$,

where $D$ is a directed set and $\Lambda$ is an index set.

In order to determine the optimal rate of the convergence behavior (1), we introduce the following definition (cf. [14], [18]):

Let $\{L_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ be an approximation process on $X$ and let $\{\xi_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ be a family of positive real numbers with $\lim_{\alpha}\xi_{\alpha,\lambda} = 0$ uniformly in $\lambda \in \Lambda$. The trivial or invariant class of $\{L_{\alpha,\lambda}\}$, denoted by $\mathcal{T}[X;\{L_{\alpha,\lambda}\}]$, is defined by

$$\mathcal{T}[X;\{L_{\alpha,\lambda}\}] = \{f \in X: L_{\alpha,\lambda}(f) = f \text{ for all } \alpha \in D \text{ and all } \lambda \in \Lambda\},$$

which is a closed linear subspace of $X$ and we set

$$\mathcal{G}[X;\{L_{\alpha,\lambda}\},\{\xi_{\alpha,\lambda}\}] = \{f \in X: \|L_{\alpha,\lambda}(f) - f\|_X = O(\xi_{\alpha,\lambda}) \text{ uniformly in } \lambda \in \Lambda\},$$

which is a linear subspace of $X$. The family $\{L_{\alpha,\lambda}\}$ is said to be saturated in $X$ with order $\{\xi_{\alpha,\lambda}\}$ if $\|L_{\alpha,\lambda}(f) - f\|_X = o(\xi_{\alpha,\lambda})$ uniformly in $\lambda \in \Lambda$ implies $f \in \mathcal{T}[X;\{L_{\alpha,\lambda}\}]$ and if there exists an element $f_0 \in \mathcal{G}[X;\{L_{\alpha,\lambda}\},\{\xi_{\alpha,\lambda}\}] \setminus \mathcal{T}[X;\{L_{\alpha,\lambda}\}]$. In this case, $\{\xi_{\alpha,\lambda}\}$ and $\mathcal{G}[X;\{L_{\alpha,\lambda}\},\{\xi_{\alpha,\lambda}\}]$ are called the saturation order and the saturation class of $\{L_{\alpha,\lambda}\}$, respectively.

The saturation theory is the study on the problems of finding the saturation order $\{\xi_{\alpha,\lambda}\}$ and characterizing the saturation class $\mathcal{G}[X;\{L_{\alpha,\lambda}\},\{\xi_{\alpha,\lambda}\}]$ of a given approximation process $\{L_{\alpha,\lambda}\}$ on $X$.

The purpose of this paper is to study the saturation problem for a family $\{T_{\alpha,\lambda}^{m}: \alpha \in D, \lambda \in \Lambda\}$ of the $m$-th iterations of multiplier operators $T_{\alpha,\lambda}$ in connection with Fourier series expansions corresponding to a total, fundamental sequence of mutually orthogonal projection operators in $B[X]$, where $m$ is an arbitrary fixed positive integer (cf. [6], [13], [14], [23]).

We shall make the best use of the general approach proposed in [18], which can be motivated by our work [16] (cf. [12], [14], [15]). Also, a number of applications of the results are provided by studying certain concrete instances of Banach spaces and systems of orthogonal projections.

## 2. Multiplier Operators

Let $\mathbb{Z}$ denote the set of all integers and let $\{P_j\}_{j\in\mathbb{Z}}$ be a sequence of projection operators in $B[X]$ satisfying the following condition:

(P-1) The projection operators $P_j$, $j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_jP_n = \delta_{jn}P_n$ for all $j$, $n \in \mathbb{Z}$, where $\delta_{jn}$ denotes the Kronecker's symbol.

Instead of $\mathbb{Z}$ one may use any other denumerable

index set, for example, the set of all nonnegative integers or the set of all natural numbers.

For each $f \in X$, the series $\Sigma_{j=-\infty}^{\infty} P_j(f)$ is called the (formal) Fourier series expansion of $f$ (with respect to $\{P_j\}$), and we write

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f). \tag{2}$$

**Remark 1.** The expansion (2) can be a generalization of the concept of Fourier series in a Banach space $X$ with respect to a biorthogonal system $\{f_j, f_j^*\}_{j\in\mathbb{Z}}$. Here $\{f_j\}_{j\in\mathbb{Z}}$ and $\{f_j^*\}_{j\in\mathbb{Z}}$ are sequences of elements in $X$ and $X^*$ (the dual space of $X$), respectively such that $f_j^*(f_n) = \delta_{jn}$ for all $j, n \in \mathbb{Z}$. Then (2) reads

$$f \sim \sum_{j=-\infty}^{\infty} f_j^*(f) f_j$$

(cf. [10], [21]).

Suppose that $\{P_j\}_{j\in\mathbb{Z}}$ is total, i.e., the following condition is fulfilled:

(P-2) If $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

Let $s$ denote the set of all sequences $\tau = \{\tau_j\}_{j\in\mathbb{Z}}$ of scalars. An element $\tau \in s$ is called a multiplier sequence for $X$ (corresponding to $\{P_j\}$)

if for each $f \in X$ there exists an element $f_\tau \in X$ such that

$$f_\tau \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f). \tag{3}$$

Note that $f_\tau$ is uniquely determined by $f$, and so the mapping $f \to f_\tau$ defines an operator in $B[X]$ by the closed graph theorem. An operator $T \in B[X]$ is called a multiplier operator on $X$ if it permits an expansion of type (3), and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j. \tag{4}$$

**Remark 2.** Under the setting of Remark 1, the expansion (3) reduces to

$$f_\tau \sim \sum_{j=-\infty}^{\infty} \tau_j f_j^*(f) f_j$$

and the expansion (4) is equivalent to

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j f_j^*(f) f_j \qquad (f \in X).$$

Let $M[X]$ denote the set of all multiplier operators on $X$.

**Lemma 1.** *$M[X]$ is a commutative closed subalgebra of $B[X]$ containing the identity operator $I$.*

Let us stand for the set of all multiplier sequences for $X$ by $m[X]$. Then with the usual vector operations, coordinatewise multiplication and norm

$$\|\tau\|_{m[X]} = \sup\{\|f_\tau\|_X : f \in X,\ \|f\|_X \le 1\} \quad (\tau \in m[X]),$$

$m[X]$ becomes a commutative Banach algebra with the unit $\{1\}$. Therefore, $M[X]$ can be isometrically isomorphic to $m[X]$.

From now on we assume that $\{P_j\}_{j\in\mathbb{Z}}$ satisfies (P-1), (P-2) and the following additional condition:

(P-3) $\{P_j\}_{j\in\mathbb{Z}}$ is fundamental, i.e., the linear subspace of $X$ spanned by the set $\cup\{P_j(X) : j \in \mathbb{Z}\}$ is dense in $X$.

Given a sequence $\tau = \{\tau_j\}_{j\in\mathbb{Z}} \in s$, let $W[X;\tau]$ denote the set of all elements $f \in X$ for which there exists an element $f_\tau \in X$ with the expansion (3), and we define $L_\tau(f) = f_\tau$ for all $f \in W[X;\tau]$. Then we have the following lemma which can be plainly proved:

**Lemma 2.** *$W[X;\tau]$ is a dense linear subspace of $X$ and $L_\tau$ is a closed operator of $W[X;\tau]$ into $X$.*

Let $\{T_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ be a family of operators in $M[X]$ having the expansions

$$T_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} \tau_{\alpha,\lambda}(j)P_j \qquad (\alpha \in D, \lambda \in \Lambda)$$

such that there exists an element $\alpha_0 \in D$ for which

$$\sup\{\|T_{\alpha,\lambda}\|_{B[X]}: \alpha \succeq \alpha_0, \alpha \in D, \lambda \in \Lambda\} < \infty,$$

and let $m$ be any fixed positive integer. We put

$$\mathbb{Z}_m = \{j \in \mathbb{Z}: \tau_{\alpha,\lambda}^m(j) = 1 \text{ for all } \alpha \in D, \lambda \in \Lambda\}$$

and suppose that $\mathbb{Z}_m \neq \mathbb{Z}$. Furthermore, set

$$\mathcal{T}_m = \mathcal{T}[X;\{T_{\alpha,\lambda}^m\}] \quad \text{and} \quad \mathcal{G}_m = \mathcal{G}[X;\{T_{\alpha,\lambda}^m\},\{m\theta_{\alpha,\lambda}\}],$$

where $\{\theta_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ is a family of positive real numbers with $\lim_\alpha \theta_{\alpha,\lambda} = 0$ uniformly in $\lambda \in \Lambda$.

**Lemma 3.** *If $\tau_j \neq 0$ for every $j \in \mathbb{Z} \setminus \mathbb{Z}_1$, then $Ker(L_\tau) \subset \mathcal{T}_m$, where $Ker(L_\tau)$ denotes the kernel of $L_\tau$, i.e., $Ker(L_\tau) = \{f \in W[X;\tau]: L_\tau(f) = 0\}$.*

**Proof.** Let $j \in \mathbb{Z}$. Then, by Lemma 1, we have

$$(5) \quad P_j T_{\alpha,\lambda}^k = T_{\alpha,\lambda}^k P_j, \; P_j(T_{\alpha,\lambda}^k(f)) = \tau_{\alpha,\lambda}^k(j)P_j(f)$$
$$(k \geq 1, \alpha \in D, \lambda \in \Lambda, f \in X).$$

Now let $g \in Ker(L_\tau)$. If $j \in \mathbb{Z}_m$, then

$$P_j(T_{\alpha,\lambda}^m(g)) = \tau_{\alpha,\lambda}^m(j)P_j(g) = P_j(g).$$

Also, if $j \in \mathbb{Z} \setminus \mathbb{Z}_m$, then $P_j(g) = 0$, since $\tau_j P_j(g) = P_j(L_\tau(g)) = 0$ and $\tau_j \neq 0$, and so

$$P_j(T^m_{\alpha,\lambda}(g)) = T^m_{\alpha,\lambda}(P_j(g)) = 0 = P_j(g).$$

Consequently, for every $j \in \mathbb{Z}$ we have

$$P_j(T^m_{\alpha,\lambda}(g)) = P_j(g) \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

which implies $T^m_{\alpha,\lambda}(g) = g$ for all $\alpha \in D$ and all $\lambda \in \Lambda$ because of the totality of $\{P_j\}$. Therefore $g$ belongs to $\mathcal{T}_m$.

In order to achieve our purpose we need the following concept (cf. [14], [18]):

Let $\{L_{\alpha,\lambda}: \alpha \in D,\ \lambda \in \Lambda\}$ be a family of operators in $B[X]$ and let $\{\xi_{\alpha,\lambda}: \alpha \in D,\ \lambda \in \Lambda\}$ be a family of positive real numbers with $\lim_\alpha \xi_{\alpha,\lambda} = 0$ uniformly in $\lambda \in \Lambda$. Let $L$ be a linear operator with domain $D(L)$ and range in $X$, and let $E$ be a linear subspace of $D(L)$. We say that the family $\{L_{\alpha,\lambda}\}$ satisfies a Voronovskaja condition of type $(\xi_{\alpha,\lambda}; L)$ on $E$ if for every $f \in E$,

$$\lim_\alpha \| \xi^{-1}_{\alpha,\lambda}(L_{\alpha,\lambda}(f) - f) - L(f) \|_X = 0$$

uniformly in $\lambda \in \Lambda$.

**Proposition 1.** *Suppose that* $\{T_{\alpha,\lambda}\}$ *satisfies a Voronovskaja condition of type* $(\theta_{\alpha,\lambda}; L_\tau)$ *on*

$P_j(X)$ *for each* $j \in \mathbb{Z}$. *Then the following assertions hold:*

(a) $\{T^m_{\alpha,\lambda}\}$ *satisfies a Voronovskaja condition of type* $(m\theta_{\alpha,\lambda}; L_\tau)$ *on* $P_j(X)$ *for each* $j \in \mathbb{Z}$.

(b) *If* $f$ *and* $g$ *are elements in* $X$ *such that*

$$\text{(6)} \qquad \liminf_\alpha \| (m\theta_{\alpha,\lambda})^{-1} (T^m_{\alpha,\lambda}(f) - f) - g \|_X = 0$$

*uniformly in* $\lambda \in \Lambda$,

*then* $f$ *belongs to* $W[X;\tau]$ *and* $L_\tau(f) = g$.

**Proof.** (a) is shown by induction on $m$ as in the proof of [18; Proposition]. To prove (b) let $j \in \mathbb{Z}$. Then for all $\alpha \in D$ and all $\lambda \in \Lambda$ we have

$$\| P_j((m\theta_{\alpha,\lambda})^{-1}(T^m_{\alpha,\lambda}(f) - f)) - P_j(g) \|_X$$
$$\leq \| P_j \|_{B[X]} \| (m\theta_{\alpha,\lambda})^{-1} (T^m_{\alpha,\lambda}(f) - f) - g \|_X,$$

and so, by (6),

$$\text{(7)} \qquad \liminf_\alpha \| P_j((m\theta_{\alpha,\lambda})^{-1}(T^m_{\alpha,\lambda}(f) - f)) - P_j(g) \|_X = 0$$

uniformly in $\lambda \in \Lambda$.

On the other hand, since $T^m_{\alpha,\lambda} P_j = P_j T^m_{\alpha,\lambda}$ by (5) with $k = m$, we obtain

$$\| P_j((m\theta_{\alpha,\lambda})^{-1}(T^m_{\alpha,\lambda}(f) - f)) - L_\tau(P_j(f)) \|_X$$
$$= \| (m\theta_{\alpha,\lambda})^{-1} (T^m_{\alpha,\lambda}(P_j(f)) - P_j(f)) - L_\tau(P_j(f)) \|_X,$$

which, by (a), implies

$\lim_{\alpha}\|P_j((m\theta_{\alpha,\lambda})^{-1}(T^{m}_{\alpha,\lambda}(f) - f)) - L_{\tau}(P_j(f))\|_X = 0$

uniformly in $\lambda \in \Lambda$, which together with (7) yields $P_j(g) = L_{\tau}(P_j(f)) = \tau_j P_j(f)$. Therefore, $f \in W[X;\tau]$ and $L_{\tau}(f) = g$.

**Remark 3.** $\{T_{\alpha,\lambda}\}$ satisfies a Voronovskaja condition of type $(\theta_{\alpha,\lambda};L_{\tau})$ on $P_j(X)$ for each $j \in \mathbb{Z}$ if and only if for each $j \in \mathbb{Z}$,

(8) $\lim_{\alpha}\theta^{-1}_{\alpha,\lambda}(\tau_{\alpha,\lambda}(j) - 1) = \tau_j$ uniformly in $\lambda \in \Lambda$.

**Proposition 2.** *Assume that* (8) *holds for each* $j \in \mathbb{Z}$. *Then the following statements hold:*

(a) *If* $\tau_j \neq 0$ *for all* $j \in \mathbb{Z} \setminus \mathbb{Z}_1$, *then the following assertions are equivalent for an element* $f \in X$:

(i) $\liminf_{\alpha}(m\theta_{\alpha,\lambda})^{-1}\|T^{m}_{\alpha,\lambda}(f) - f\|_X = 0$ *uniformly in* $\lambda \in \Lambda$.

(ii) $f$ *belongs to* $Ker(L_{\tau})$.

(iii) $f$ *belongs to* $\mathcal{T}_m$.

(b) *There exists an element* $f_0 \in \mathcal{G}_m \setminus \mathcal{T}_m$.

**Proof.** (a) In view of Remark 3, taking $g = 0$ in Proposition 1 (b), (i) implies (ii). It follows from Lemma 3 that (ii) implies (iii). It is clear that (iii) implies (i).

(b) Let $n \in \mathbb{Z} \setminus \mathbb{Z}_m$, and take $f_0 \in P_n(X)$ with $f_0 \neq 0$. Then, by (5) with $k = m$, we have

$$\|T_{\alpha,\lambda}^m(f_0) - f_0\|_X = |\tau_{\alpha,\lambda}^m(n) - 1| \|f_0\|_X \tag{9}$$

for all $\alpha \in D$ and all $\lambda \in \Lambda$. By Propositioin 1 (a) and Remark 3, we obtain

$$\lim_\alpha (m\theta_{\alpha,\lambda})^{-1}(\tau_{\alpha,\lambda}^m(n) - 1) = \tau_n$$

uniformly in $\lambda \in \Lambda$, which together with the equality (9) establishes the claim (b).

Now Lemma 2 asserts that $W[X;\tau]$ is a normalized Banach subspace of $X$ under the graph norm

$$\|\cdot\|_{W[X;\tau]} = \|\cdot\|_X + \|L_\tau(\cdot)\|_X,$$

and so we denote by $W[X;\tau]^{\sim X}$ the relative completion of $W[X;\tau]$ in $X$. For basic properties of relative completions of Banach subspaces in Banach spaces, we refer to [1], [3; Sec.2.2] and [5; Sec. 10.4].

A sequence $\{R_n\}_{n\geq 0}$ of operators in $B[X]$ is called a regularization process on $X$ for $\{T_{\alpha,\lambda}\}$ and $L_\tau$ if $R_n(X) \subset W[X;\tau]$ for each $n \geq 0$, $R_n T_{\alpha,\lambda} = T_{\alpha,\lambda} R_n$ for all $n \geq 0$, $\alpha \in D$, $\lambda \in \Lambda$ and $\lim_{n\to\infty} \|R_n(f) - f\|_X = 0$ for every $f \in X$.

**Theorem 1.** *Suppose that* $\{T_{\alpha,\lambda}\}$ *satisfies a Voronovskaja condition of type* $(\theta_{\alpha,\lambda};L_\tau)$ *on* $W[X;\tau]$. *Then the following statementds hold:*

(a) *If* $\tau_j \neq 0$ *whenever* $j \in \mathbb{Z} \setminus \mathbb{Z}_1$, *then* $\{T_{\alpha,\lambda}^m\}$ *is saturated in* $X$ *with order* $\{m\theta_{\alpha,\lambda}\}$ *and* $\mathcal{I}_m = Ker(L_\tau)$.

(b) $$W[X;\tau]^{\sim X} \subset \mathcal{I}_m.$$

(c) *If there exists a regularization process* $\{R_n\}$ *on* $X$ *for* $\{T_{\alpha,\lambda}\}$ *and* $L_\tau$, *then*

$$\mathcal{I}_m = \{f \in X: \|(L_\tau R_n)(f)\|_X = O(1)\} = W[X;\tau]^{\sim X}$$

*and if, furthermore,* $X$ *is reflexive, then*

$$\mathcal{I}_m = \{f \in X: \|(L_\tau R_n)(f)\|_X = O(1)\} = W[X;\tau].$$

**Proof.** By Lemma 2 and [18; Proposition], $\{T_{\alpha,\lambda}^m\}$ is an approximation process on $X$ and $L_\tau$ is a closed operator of $W[X;\tau]$ into $X$. Now (a) follows from Proposition 2. Also, (b) and (c) follow from [18; Theorem 1 (c)] and [18; Corollary 1 (b)], respectively.

Let $\{U_n\}$ be a sequence of operators in $M[X]$ having the expansions

$$U_n \sim \sum_{j=-\infty}^{\infty} \xi_n(j)P_j \tag{10}$$

with $\xi_n(j) = 0$ whenever $|j| > n$, and we set

$$V = V[X;\{U_n\},\tau]$$

$$= \{f \in X: \left\| \sum_{j=-n}^{n} \xi_n(j)\tau_j P_j(f) \right\|_X = O(1)\},$$

which is a normalized Banach subspace of $X$ under the norm $\|\cdot\|_V$ defined by

$$\|f\|_V = \|f\|_X + \sup_n \left\| \sum_{j=-n}^{n} \xi_n(j)\tau_j P_j(f) \right\|_X \qquad (f \in V).$$

**Corollary 1.** *Let $\{U_n\}$ be a uniformly bounded sequnece of operators in $M[X]$ having the expansions* (10) *with $\xi_n(j) = 0$ whenever $|j| > n$ and $\lim_{n\to\infty}\xi_n(j) = 1$ for each $j \in \mathbb{Z}$. Then under the hypothesis of Theorem* 1, *we have*

$$(11) \qquad \mathcal{Y}_m = W[X;\tau]^{\sim X} = V[X;\{U_n\},\tau]$$

*and in particular, if $X$ is reflexive, then* (11) *reduces to*

$$(12) \qquad \mathcal{Y}_m = W[X;\tau] = V[X;\{U_n\},\tau].$$

**Proof.** $\{U_n\}$ becomes a regularization process on $X$ for $\{T_{\alpha,\lambda}\}$ and $L_\tau$, and there holds

$$(L_\tau U_n)(f) = \sum_{j=-n}^{n} \xi_n(j)\tau_j P_j(f) \qquad (f \in X).$$

Hence the desired result immediately follows from Theorem 1 (c).

Now we mention some concrete examples of $\{U_n\}$ having the expansion (10) with $\xi_n(j) = 0$ whenever $|j| > n$ and $\lim_{n\to\infty}\xi_n(j) = 1$ for each $j \in \mathbb{Z}$:

$(1^{O})$ The $n$-th Cesàro mean operator $\sigma_n^{(\beta)}$ of order $\beta > -1$ is defined by

$$\sigma_n^{(\beta)} = (1/A_n^{(\beta)}) \sum_{j=-n}^{n} A_{n-|j|}^{(\beta)} P_j,$$

where

$$A_n^{(\beta)} = \binom{n+\beta}{n} = \frac{(\beta+1)(\beta+2)\cdots(\beta+n)}{n!}.$$

Obviously, we have

$$\sigma_n^{(0)} = S_n = \sum_{j=-n}^{n} P_j,$$

which is the $n$-th partial sum operator associated with the series (2), and

$$\sigma_n^{(1)} = \sigma_n = \sum_{j=-n}^{n} (1 - |j|/(n+1))P_j$$

$$= \frac{S_0 + S_1 + \cdots + S_n}{n+1},$$

which is the standard Cesàro mean operator.

$(2^{O})$ The typical mean operator $R_n^{(\beta)}$ of order $\beta > 0$ is defined by

$$R_n^{(\beta)} = \sum_{j=-n}^{n} (1 - (|j|/(n+1))^{\beta})P_j,$$

which coincides with $\sigma_n$ for $\beta = 1$.

(3°) Let $\delta = \{\delta_n\}$ be a sequence of non-negative real numbers and let $\beta > 0$. We define

$$S_n^{(\delta;\beta)} = (1/(\delta_n + 1))(\delta_n S_n + R_n^{(\beta)}),$$

which reduces to the arithmetic mean operator $(S_n + \sigma_n)/2$ of $S_n$ and $\sigma_n$ for $\delta = \{1\}$ and $\beta = 1$. It is easily seen that

$$S_n^{(\delta;\beta)} = \sum_{j=-n}^{n} \{1 - |j|^{\beta}/((\delta_n + 1)(n + 1)^{\beta})\}P_j,$$

which coincides with $R_n^{(\beta)}$ for $\delta = \{0\}$.

(4°) Let $\beta, \lambda > 0$ and we define

$$R_n^{(\beta,\lambda)} = \sum_{j=-n}^{n} (1 - (|j|/(n + 1))^{\beta})^{\lambda} P_j,$$

which is called the Riesz mean operator. Also, for $\beta = 2$ this is known as the Bochner-Riesz mean operator.

(5°) The $n$-th de La Vallée-Poussin mean operator is defined by

$$V_n = \sum_{j=-n}^{n} \frac{(n!)^2}{(n - |j|)!(n + |j|)!} P_j.$$

We consider the following condition which will ensure that $\{T_{\alpha,\lambda}\}$ satisfies a Voronovskaja condition of type $(\theta_{\alpha,\lambda}; L_{\tau})$ on $W[X;\tau]$:

(13) There exists an approximation process $\{Q_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ on $X$ of operators in $M[X]$ having the expansions

$$Q_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} \gamma_{\alpha,\lambda}(j)P_j \qquad (\alpha \in D, \lambda \in \Lambda)$$

such that

$$\theta_{\alpha,\lambda}^{-1}(\tau_{\alpha,\lambda}(j) - 1) = \tau_j\gamma_{\alpha,\lambda}(j)$$

for all $\alpha \in D$, $\lambda \in \Lambda$ and all $j \in \mathbb{Z}$.

**Lemma 4.** *If Condition* (13) *is fulfilled, then* $\{T_{\alpha,\lambda}\}$ *satisfies a Voronovskaja condition of type* $(\theta_{\alpha,\lambda}; L_\tau)$ *on* $W[X;\tau]$.

**Proof.** Let $f \in W[X;\tau]$. Then, by (13), for every $j \in \mathbb{Z}$ we have

$$P_j(\theta_{\alpha,\lambda}^{-1}(T_{\alpha,\lambda}(f) - f)) = \theta_{\alpha,\lambda}^{-1}(\tau_{\alpha,\lambda}(j) - 1)P_j(f)$$

$$= \tau_j\gamma_{\alpha,\lambda}(j)P_j(f) = \tau_j P_j(Q_{\alpha,\lambda}(f)).$$

Thus $Q_{\alpha,\lambda}(f)$ belongs to $W[X;\tau]$ and

$$L_\tau(Q_{\alpha,\lambda}(f)) = \theta_{\alpha,\lambda}^{-1}(T_{\alpha,\lambda}(f) - f),$$

which implies

(14) $$\theta_{\alpha,\lambda}^{-1}(T_{\alpha,\lambda}(f) - f) = Q_{\alpha,\lambda}(L_\tau(f)),$$

since $L_\tau$ commutes with all operators in $M[X]$.

Therefore, since $\{Q_{\alpha,\lambda}\}$ is an approximation process on $X$, (14) yields

$$\lim_{\alpha} \| \theta_{\alpha,\lambda}^{-1}(T_{\alpha,\lambda}(f) - f) - L_{\tau}(f) \|_X = 0$$

uniformly in $\lambda \in \Lambda$ and the lemma is proved.

**Theorem 2.** *If Condition* (13) *is satisfied, then the assertions* (a), (b) *and* (c) *of Theorem* 1 *hold.*

**Proof.** This immediately follows from Lemma 4 and Theorem 1.

**Corollary 2.** *Suppose that* (13) *is fulfilled. If* $\{U_n\}$ *is a sequence as in Corollary* 1, *then* (11) *holds and in particular, if* $X$ *is reflexive, then* (12) *holds.*

It should be noted that Corollaries 1 and 2 are applied to the case where $\{U_n\}$ is taken as any one of the sequences of operators given in $(1^{\circ})$ - $(5^{\circ})$ under the assumption of the uniform boundedness of them. In particular, in the study of multiplier sequences and summation processes of Fourier series expansions in Banach spaces it can be a standard condition that $\{\sigma_n^{(\beta)}\}$ is uniformly bounded for some $\beta \geq 0$ (cf. [6], [23]).

## 3. Convolution Operators

Let $L^1_{2\pi}$ denote the Banach space of all $2\pi$-periodic, Lebesgue measurable functions $k$ on the real line $\mathbb{R}$ with the norm

$$\|k\|_1 = (1/2\pi)\int_{-\pi}^{\pi} |k(t)|dt < \infty.$$

Let $\{T(t): t \in \mathbb{R}\}$ be a family of operators in $B[X]$ such that for each $f \in X$, the mapping $t \to T(t)(f)$ is strongly continuous on $\mathbb{R}$. With $k \in L^1_{2\pi}$ and $L \in B[X]$, we define the convolution operator $k*L$ (associated with the family $\{T(t)\}$) by

$$(k*L)(f) = (1/2\pi)\int_{-\pi}^{\pi} k(t)T(t)(L(f))dt \qquad (f \in X),$$

which exists as a Bochner integral. Obviously, $k*L$ belongs to $B[X]$ and

$$\|k*L\|_{B[X]} \le C\|k\|_1\|L\|_{B[X]}, \tag{15}$$

where

$$C = \sup\{\|T(t)\|_{B[X]}: |t| \le \pi\},$$

which is finite by the uniform boundedness principle. The direct theory for approximation processes of convolution operators of the form $k*I$ is studied by the author [13, 17] under certain appropriate conditions.

Here we first state the relation between multiplier operators and convolution operators. For this let $\{S(t)\colon t \in \mathbb{R}\}$ be a strongly continuous group of operators in $B[X]$. We define

$$G(f) = \lim_{t\to 0} (1/t)(S(t)(f) - f), \tag{16}$$

whenever the limit exists in the sense of strong convergence, and let $D(G)$ denote the set of all elements $f \in X$ for which the strong limit in (16) exists. Evidently, $D(G)$ is a linear subspace of $X$ and $G$ is a linear operator of $D(G)$ into $X$. This operator $G$ is called the infinitesimal generator of the group $\{S(t)\}$. For $r = 0, 1, 2, \cdots$, the operator $G^r$ is defined inductively by the relations $G^0 = I$, $G^1 = G$,

$$D(G^r) = \{f\colon f \in D(G^{r-1}) \text{ and } G^{r-1}(f) \in D(G)\}$$

and

$$G^r(f) = G(G^{r-1}(f)) \quad (f \in D(G^r),\ r = 1, 2, \cdots).$$

Then for each $r = 1, 2, \cdots$, $D(G^r)$ is a dense linear subspace of $X$ and $G^r$ is a closed operator (cf. [4; Propositions 1.1.4 and 1.1.6]). For further extensive properties of semigroups of operators on Banach spaces, one may consult [4] and [8].

Now suppose that $\{T(t)\colon t \in \mathbb{R}\}$ is a uniformly bounded family of operators in $M[X]$ having the expansions

$$T(t) \sim \sum_{j=-\infty}^{\infty} \exp(\mu_j t) P_j \qquad (t \in \mathbb{R}), \tag{17}$$

where $\{\mu_j\}_{j\in\mathbb{Z}}$ is a sequence in $s$.

**Proposition 3.** (a) *$\{T(t)\colon t \in \mathbb{R}\}$ is a strongly continuous group of operators in $B[X]$ and if $G$ is its infinitesimal generator, then for each $r = 1, 2, 3, \cdots$, there holds*

$$G^r(f) \sim \sum_{j=-\infty}^{\infty} \mu_j^r P_j(f) \qquad (f \in D(G^r)). \tag{18}$$

(b) *If there exists a uniformly bounded sequence $\{U_n\}$ of operators in $M[X]$ having the expansions (10) with*

$$\xi_n(j) = 0 \qquad \text{whenever } |j| > n \tag{19}$$

*and*

$$\lim_{n\to\infty} \xi_n(j) = 1 \quad \text{for each } j \in \mathbb{Z}, \tag{20}$$

*then we have*

$$D(G^r) = \{f \in X\colon g \sim \sum_{j=-\infty}^{\infty} \mu_j^r P_j(f) \text{ for some } g \in X\}.$$

**Proof.** (a) In view of [13; Proposition 2], this follows by induction on $r$.

(b) Let $f$ be an element of $X$ such that

$$g \sim \sum_{j=-\infty}^{\infty} \mu_j^r P_j(f)$$

for some $g \in X$. Then, since $G^r(P_j(f)) = \mu_j^r P_j(f)$ for all $j \in \mathbb{Z}$, we have

$$G^r(U_n(f)) = \sum_{j=-n}^{n} \xi_n(j) G^r(P_j(f))$$

$$= \sum_{j=-n}^{n} \xi_n(j) \mu_j^r P_j(f) = \sum_{j=-n}^{n} \xi_n(j) P_j(g) = U_n(g).$$

Consequently, we obtain

$$\lim_{n\to\infty} \|U_n(f) - f\|_X = 0 \text{ and } \lim_{n\to\infty} \|G^r(U_n(f)) - g\|_X = 0,$$

since $\{U_n\}$ strongly converges to $I$ on $X$. Hence, $f \in D(G^r)$ and $G^r(f) = g$, since $G^r$ is a closed operator. This fact and (18) yield the desired assertion.

**Proposition 4.** *Let $k \in L^1_{2\pi}$ and $L \in B[X]$. Then there holds*

$$P_j((k*L)(f)) = c_j P_j(L(f)) \qquad (j \in \mathbb{Z},\ f \in X),$$

*where*

$$c_j = (1/2\pi)\int_{-\pi}^{\pi} k(t)\exp(\mu_j t)dt \qquad (j \in \mathbb{Z}).$$

*In particular, if $L \in M[X]$ and*

(21) $$L \sim \sum_{j=-\infty}^{\infty} \gamma_j P_j,$$

*then* $k*L \in M[X]$ *and*

(22) $$k*L \sim \sum_{j=-\infty}^{\infty} c_j \gamma_j P_j.$$

**Proof.** Let $j \in \mathbb{Z}$ and $f \in X$. By Proposition 3, the mapping $t \to T(t)(f)$ is strongly continuous on $\mathbb{R}$, and thus we have

$$P_j((k*L)(f)) = (1/2\pi)\int_{-\pi}^{\pi} k(t)P_j(T(t)(L(f)))dt$$

$$= (1/2\pi)\int_{-\pi}^{\pi} k(t)\exp(\mu_j t)P_j(L(f))dt = c_j P_j(L(f)).$$

Let $\{k_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1_{2\pi}$ such that there exists an element $\alpha_0 \in D$ for which

(23) $K = \sup\{\|k_{\alpha,\lambda}\|_1: \alpha \succeq \alpha_0, \alpha \in D, \lambda \in \Lambda\} < \infty$

and we put

$$T_{\alpha,\lambda} = k_{\alpha,\lambda}*I \quad (\alpha \in D, \lambda \in \Lambda) \quad \text{and} \quad \tau = \{\mu_j^r\}_{j\in\mathbb{Z}}$$

for some positive integer $r$. In addition, in the rest of this section we suppose that there exists a uniformly bounded sequence $\{U_n\}$ of operators in $M[X]$ having the expansions (10) with (19) and (20) and that $\mathbb{Z}_m \neq \mathbb{Z}$.

**Theorem 3.** *Assume that* $\{T_{\alpha,\lambda}\}$ *satisfies a Voronovskaja condition of type* $(\theta_{\alpha,\lambda}; G^r)$ *on* $D(G^r)$. *Then the following statements hold:*

(a) *If* $\mu_j \neq 0$ *whenever* $j \in \mathbb{Z} \setminus \mathbb{Z}_1$, *then* $\{T_{\alpha,\lambda}^m\}$ *is saturated in* $X$ *with order* $\{m\theta_{\alpha,\lambda}\}$ *and* $\mathcal{T}_m = Ker(G^r)$.

(b) $$\mathcal{G}_m = D(G^r)^{\sim X} = V[X;\{U_n\},\tau].$$

*In particular, if* $X$ *is reflexive, then*

$$\mathcal{G}_m = D(G^r) = V[X;\{U_n\},\tau].$$

**Proof.** By Proposition 4 with $k = k_{\alpha,\lambda}$ and $L = I$, we obtain

$$T_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} \tau_{\alpha,\lambda}(j) P_j \qquad (\alpha \in D,\ \lambda \in \Lambda), \tag{24}$$

where

$$\tau_{\alpha,\lambda}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{\alpha,\lambda}(t)\exp(\mu_j t)dt. \tag{25}$$

Also, the inequality (15) with $k = k_{\alpha,\lambda}$, $L = I$ and (23) imply

$$\sup\{\|T_{\alpha,\lambda}\|_{B[X]}: \alpha \geq \alpha_0,\ \alpha \in D,\ \lambda \in \Lambda\} \leq CK.$$

Furthermore, by Proposition 3, we have $W[X;\tau] = D(G^r)$ and $L_\tau = G^r$. Therefore, the assertions (a) and (b) follow from Theorem 1 (a) and Corollary 1, respectively.

**Corollary 3.** *Suppose that Condition* (13) *is satisfied with* $\tau_{\alpha,\lambda}(j)$ *given by* (25) *and* $\tau_j = \mu_j^r$. *Then the assertions* (a) *and* (b) *of Theorem* 3 *hold.*

In view of the direct approximation theory for convolution operators in Banach spaces due to the author [13] (cf. [17]), if $\{k_{\alpha,\lambda}\}$ satisfies the additional conditions that each $k_{\alpha,\lambda}$ is nonnegative and

$$\lim_{\alpha}\{k_{\alpha,\lambda}\hat{}(0) - \mathrm{Re}(k_{\alpha,\lambda}\hat{}(1))\} = 0$$

uniformly in $\lambda \in \Lambda$, where

$$k_{\alpha,\lambda}\hat{}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{\alpha,\lambda}(t)e^{-ijt}dt \qquad (j \in \mathbb{Z})$$

is the $j$-th (complex) Fourier coefficient of $k_{\alpha,\lambda}$ and $\mathrm{Re}(k_{\alpha,\lambda}\hat{}(1))$ denotes the real part of $k_{\alpha,\lambda}\hat{}(1)$, then Theorem 3 and Corollary 3 can be applied to the case where

$$\theta_{\alpha,\lambda} = k_{\alpha,\lambda}\hat{}(0) - \mathrm{Re}(k_{\alpha,\lambda}\hat{}(1)).$$

Besides, for the particular case when

$$\{\mu_j\}_{j\in\mathbb{Z}} = \{-ij\}_{j\in\mathbb{Z}},$$

(24) reduces to

$$T_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} k_{\alpha,\lambda}\hat{}(j)P_j \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

and considering the case $r = 2$, we make the following remark on Condition (8) (cf. [14; Remark 4]):

**Remark 4.** Let $\{k_{\alpha,\lambda}: \alpha \in D, \lambda \in \Lambda\}$ be a family of nonnegative, even functions in $L^1_{2\pi}$ having the Fourier series expansions

$$k_{\alpha,\lambda}(t) \sim 1 + 2\sum_{j=1}^{\infty} k_{\alpha,\lambda}\hat{}(j)\cos jt$$

with

$$\lim_{\alpha} k_{\alpha,\lambda}\hat{}(1) = 1 \quad \text{uniformly in } \lambda \in \Lambda.$$

Then the following statements are equivalent:

(i) For each $j \in \mathbb{Z}$,

$$\lim_{\alpha}(k_{\alpha,\lambda}\hat{}(j) - 1)/(1 - k_{\alpha,\lambda}\hat{}(1)) = -j^2$$

uniformly in $\lambda \in \Lambda$.

(ii) (i) holds for $j = 2$.

(iii) $$\int_0^{\pi} k_{\alpha,\lambda}(t)\sin^4(t/2)dt = o(1 - k_{\alpha,\lambda}\hat{}(1))$$

uniformly in $\lambda \in \Lambda$.

(iv) For any fixed $\delta$ satisfying $0 < \delta < \pi$,

$$\int_{\delta \le |t| \le \pi} k_{\alpha,\lambda}(t)dt = o(1 - k_{\alpha,\lambda}\hat{}(1))$$

uniformly in $\lambda \in \Lambda$.

## 4. $\mathcal{A}$-Summation Processes

Let $\mathbb{N}$ denote the set of all nonnegative integers. Let $\mathcal{A} = \{a_{\alpha,\lambda}^{(n)}: \alpha \in D, \lambda \in \Lambda, n \in \mathbb{N}\}$ be a family of scalars. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements in $X$ is said to be $\mathcal{A}$-summable to an element $f \in X$ if

$$(26)\quad \lim_{\alpha} \Big\| \sum_{n=0}^{\infty} a_{\alpha,\lambda}^{(n)} f_n - f \Big\|_X = 0 \quad \text{uniformly in } \lambda \in \Lambda,$$

where it is assumed that the series in (26) converges for each $\alpha \in D$ and $\lambda \in \Lambda$ (cf. [14], [18]). A sequence $\{L_n\}_{n\in\mathbb{N}}$ of operators in $B[X]$ is called an $\mathcal{A}$-summation process on $X$ if $\{L_n(f)\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-summable to $f$ for every $f \in X$.

Now let $\mathcal{A} = \{a_{\alpha,\lambda}^{(n)}: \alpha \in D, \lambda \in \Lambda, n \in \mathbb{N}\}$ be a family of nonnegative real numbers with

$$(27)\qquad \sum_{n=0}^{\infty} a_{\alpha,\lambda}^{(n)} = 1 \qquad (\alpha \in D, \lambda \in \Lambda).$$

For examples of such families, see, for instance, [14] and [18]. Let $\{k_n\}_{n\in\mathbb{N}}$ and $\{\ell_n\}_{n\in\mathbb{N}}$ be sequences of nonnegative integers. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $M[X]$ having the expansions

$$L_n \sim \sum_{j=-\infty}^{\infty} \omega_n(j) P_j$$

with

(28) $$\sup\{\|L_{\ell_n}^{k_n}\|_{B[X]} : n \in \mathbb{N}\} < \infty.$$

We set

$$T_{\alpha,\lambda} = \sum_{n=0}^{\infty} a_{\alpha,\lambda}^{(n)} L_{\ell_n}^{k_n} \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

which converges in the uniform operator topology on $B[X]$ because of (27) and (28). By Lemma 1, $T_{\alpha,\lambda}$ belongs to $M[X]$ and it has the expansion

$$T_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} \tau_{\alpha,\lambda}(j) P_j \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

where

$$\tau_{\alpha,\lambda}(j) = \sum_{n=0}^{\infty} a_{\alpha,\lambda}^{(n)} \omega_{\ell_n}^{k_n}(j).$$

Furthermore, by (27) and (28), we have

$$\sup\{\|T_{\alpha,\lambda}\|_{B[X]} : \alpha \in D,\ \lambda \in \Lambda\}$$

$$\leq \sup\{\|L_{\ell_n}^{k_n}\|_{B[X]} : n \in \mathbb{N}\} < \infty.$$

Consequently, the results obtained in Section 2 are applicable to the family $\{T_{\alpha,\lambda}\}$ with $\mathbb{Z}_m \neq \mathbb{Z}$. As illustrations of these results we restrict ourselves to the following situation:

(29) There exist sequences $\{\theta_n\}_{n\in\mathbb{N}}$ of positive real numbers and $\tau = \{\tau_j\}_{j\in\mathbb{Z}} \in s$ such that $\{\theta_{\ell_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-summable to zero in $\mathbb{R}$ and

$$\omega_{\ell_n}^{k_n}(j) - 1 = \theta_{\ell_n}\tau_j$$

for all $n \in \mathbb{N}$ and all $j \in \mathbb{Z}$.

Setting

$$\mathbb{P} = \{j \in \mathbb{Z}: \omega_{\ell_n}^{k_n}(j) = 1 \text{ for all } n \in \mathbb{N}\}$$

and

$$\theta_{\alpha,\lambda} = \sum_{n=0}^{\infty} a_{\alpha,\lambda}^{(n)}\theta_{\ell_n} \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

we have the following:

**Theorem 4.** *Suppose that Condition* (29) *is satisfied. Then the following assertions hold:*

(a) *If* $\tau_j \neq 0$ *for every* $j \in \mathbb{Z} \setminus \mathbb{P}$, *then* $\{T_{\alpha,\lambda}^{m}\}$ *is saturated in* $X$ *with order* $\{m\theta_{\alpha,\lambda}\}$ *and* $\mathcal{T}_m = Ker(L_\tau)$. *Moreover, the statements* (b) *and* (c) *of Theorem* 1 *hold.*

(b) *Let* $\{U_n\}$ *be as in Corollary* 1. *Then we have* (11), *which reduces to* (12) *if* $X$ *is reflexive.*

**Proof.** By Conditions (27) and (29), we get

$$\tau_{\alpha,\lambda}(j) - 1 = \theta_{\alpha,\lambda}\tau_j \qquad (\alpha \in D,\ \lambda \in \Lambda,\ j \in \mathbb{Z}),$$

which implies Condition (13) with $\{Q_{\alpha,\lambda}\} = \{I\}$. Therefore, the desired statements follow from Theorem 2 and Corollary 2.

Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{2\pi}$ with

$$\sup\{\|g_{\ell_n}\|_1^{k_n} : n \in \mathbb{N}\} < \infty. \tag{30}$$

Let $\{T(t): t \in \mathbb{R}\}$ be a family of contractions in $M[X]$ having the expansions (17) and take $L_n = g_n * I$ for each $n \in \mathbb{N}$. Then, by Proposition 4, we have

$$L_n \sim \sum_{j=-\infty}^{\infty} \omega_n(j)P_j \qquad (n \in \mathbb{N}),$$

where

$$\omega_n(j) = (1/2\pi)\int_{-\pi}^{\pi} g_n(t)\exp(\mu_j t)dt. \tag{31}$$

**Theorem 5.** *Assume that Condition* (29) *is satisfied with* $\{\omega_n(j): n \in \mathbb{N}, j \in \mathbb{Z}\}$ *given by* (31) *and* $\{\tau_j\}_{j\in\mathbb{Z}} = \{\mu_j^r\}_{j\in\mathbb{Z}}$ *for some positive integer* $r$ *and that there exists a uniformly bounded sequence* $\{U_n\}$ *of operators in* $M[X]$ *having the expansions* (10) *with* (19) *and* (20). *Then the same statements as* (a) *and* (b) *of Theorem* 3 *hold.*

**Proof.** In view of (15), (30) implies (28). Also, by Lemma 4 and Proposition 3, $\{T_{\alpha,\lambda}\}$ satisfies a Voronovskaja condition of type $(\theta_{\alpha,\lambda}; G^r)$ on $D(G^r)$. Thus the desired result follows from Theorem 1 (a) and Corollary 1.

**Remark 5.** Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative functions in $L^1_{2\pi}$ satisfying (30) such that $\{g_{\ell_n}\hat{}(0) - \mathrm{Re}(g_{\ell_n}\hat{}(1))\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-summable to zero in $\mathbb{R}$. Then Theorem 5 holds with

$$\theta_n = g_n\hat{}(0) - \mathrm{Re}(g_n\hat{}(1)) \qquad (n \in \mathbb{N}).$$

## 5. Homogeneous Banach Spaces and Orthogonal Projections

A linear subspace $X$ of $L^1_{2\pi}$ is called a homogeneous Banach space if it is a Banach space with norm $\|\cdot\|_X$ which satisfies the following conditions (cf. [9], [13], [20]):

(H-1) $X$ is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $C > 0$ such that

$$\|f\|_1 \le C\|f\|_X$$

for all $f \in X$.

(H-2) Let $\{T(t): t \in \mathbb{R}\}$ be the family of translation operators defined by

$$T(t)(f)(u) = f(u-t) \qquad (f \in X,\ t,\ u \in \mathbb{R}).$$

Then for each $t \in \mathbb{R}$, $T(t)$ is an isometric operator in $B[X]$.

(H-3) For each $f \in X$, the mapping $t \to T(t)(f)$ is strongly continuous on $\mathbb{R}$.

The concrete examples of homogeneous Banach spaces are $C_{2\pi}$, the Banach space of all $2\pi$-periodic, continuous functions $f$ with the norm

$$\|f\|_\infty = \sup\{|f(t)|: |t| \le \pi\}$$

and $L^p_{2\pi}$, the Banach space of all $2\pi$-periodic, $p$-th power Lebesgue integrable functions $f$ with the norm

$$\|f\|_p = \left((1/2\pi)\int_{-\pi}^{\pi}|f(t)|^p dt\right)^{1/p} \qquad (1 \le p < \infty).$$

Also, for other examples of such spaces, see, [9] and [13].

Now let $X$ be a homogeneous Banach space with norm $\|\cdot\|_X$, and we define

$$P_j(f)(t) = f^\wedge(j)e^{ijt} \qquad (j \in \mathbb{Z},\ f \in X,\ t \in \mathbb{R}).$$

Then $\{P_j\}_{j\in\mathbb{Z}}$ is a total, fundamental sequence of orthogonal projection operators in $B[X]$, i.e., it satisfies Conditions (P-1), (P-2) and (P-3) just as in Section 2 (cf. [9], [13]). Note that $\lim_{n\to\infty}\|\sigma_n(f) - f\|_X = 0$ for every $f \in X$, and so $\{\sigma_n\}$ is uniformly bounded because of the uniform boundedness principle. Moreover, we have

$$T(t) \sim \sum_{j=-\infty}^{\infty} \exp(-ijt)P_j \qquad (t \in \mathbb{R}),$$

and if $k \in L^1_{2\pi}$ and $L$ is an operator in $M[X]$ having the expansion (21), then (22) reduces to

$$k*L \sim \sum_{j=-\infty}^{\infty} k^{\wedge}(j)\gamma_j P_j$$

and

$$(k*L)(f)(u) = (1/2\pi)\int_{-\pi}^{\pi} k(t)L(f)(u-t)dt \qquad (f \in X).$$

Consequently, the results obtained in the preceding sections are applied to the above situation. In particular, we obtain saturation theorems for approximatin processes of iterations of convolution operators generated via various summability kernels which are nonnegative and even trigonometric polynomials. For the classical examples of such summability kernels, see, e.g., [5].

Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ and let $1 \leq p < \infty$. Let $X = L^p(E)$ denote the Banach space of all $p$-th power Lebesgue integrable functions $f$ with the norm

$$\|f\|_p = \left( \int_E |f(t)|^p dt \right)^{1/p}.$$

Let $\{f_j\}$ be a sequence of functions in $X \cap X^*$ which are orthonormal on $E$. We define

$$P_j(f) = \left(\int_E f(t)\overline{f_j(t)}dt\right) f_j \qquad (f \in X).$$

Then $\{P_j\}$ becomes a sequence of orthogonal projection operators in $B[X]$ (cf. Remark 1). Consequently, under the additional conditions (P-2) and (P-3) the results obtained in previous sections are applied to the saturation problems in Lebesgue spaces $X$.

Finallly, we provide several examples of $\{f_j\}$ which forms a Schauder basis in $L^p(E)$.

($1^O$) *Bessel system.* Let $E = (0,1)$ and $\nu > -1$. Let

$$f_j(t) = \sqrt{2t}J_\nu(\mu_j t)/J_{\nu+1}(\mu_j) \qquad (j \geq 1),$$

where $J_\nu(t)$ is the Bessel function of the first kind of order $\nu$, i.e.,

$$J_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (t/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

and $\{\mu_j\}$ is the sequence of positive zeros of $J_\nu(t)$, arranged in ascending order of magnitude (cf. [22; Sec.1.71], [24]). The Bessel series converges in $L^p(E)$ whenever $\nu \geq -1/2$ and $1 < p < \infty$, and this result establishes the convergence of Dini series in the same spaces ([24]).

($2^{\circ}$) *Haar system.* Let $E = [0,1]$ and let $\{f_j\}$ be the sequence of Haar functions on $E$ defined as follows:

$$f_1(t) = \chi_E(t),$$

$$f_j(t) = 2^{n/2}\{\chi_{[0,1)}(2^{n+1}t - 2\ell + 2)$$

$$- \chi_{[0,1)}(2^{n+1}t - 2\ell + 1)\},$$

$$(j = 2^n + \ell,\ n = 0, 1, \cdots;\ \ell = 1, 2, \cdots, 2^n),$$

where $\chi_F(t)$ denotes the characteristic function of the interval $F$. The Haar series converges in $L^p(E)$ whenever $1 \leq p < \infty$ (cf. [21; Chap.I, §2]).

($3^{\circ}$) *Hermite system.* Let $E = \mathbb{R}$, and let

$$f_j(t) = (2^j j!\sqrt{\pi})^{-1/2}\exp(-t^2/2)H_j(t) \qquad (j \geq 0),$$

where $H_j(t)$ is the Hermite polynomial given by

$$H_j(t) = (-1)^j \exp(t^2)\frac{d^j}{dt^j}\exp(-t^2)$$

$$= j! \sum_{k=0}^{[j/2]} \frac{(-1)^k (2t)^{j-2k}}{k!(j-2k)!}$$

(cf. [22; Chap.V]). The Hermite series converges in $L^p(E)$ whenever $4/3 < p < 4$ ([2]).

($4^{O}$) *Jacobi system.* Let $E = [-1,1]$ and $\alpha > -1$, $\beta > -1$. Let

$$f_j(t) = h_j^{(\alpha,\beta)}(1-t)^{\alpha/2}(1+t)^{\beta/2}P_j^{(\alpha,\beta)}(t) \qquad (j \geq 0),$$

where

$$h_j^{(\alpha,\beta)} = \left(\frac{j!(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}\right)^{1/2}$$

and $P_j^{(\alpha,\beta)}(t)$ is the Jacobi polynomial given by

$$\begin{aligned} P_j^{(\alpha,\beta)}(t) &= \frac{(-1)^j}{2^j j!}(1-t)^{-\alpha}(1+t)^{-\beta} \\ &\quad \times \frac{d^j}{dt^j}\{(1-t)^{j+\alpha}(1+t)^{j+\beta}\} \\ &= \sum_{k=0}^{j}\binom{j+\alpha}{k}\binom{j+\beta}{j-k}\left(\frac{t-1}{2}\right)^{j-k}\left(\frac{t+1}{2}\right)^{k} \end{aligned}$$

(cf. [22; Chap.IV]). The Jacobi series converges in $L^p(E)$ whenever $\alpha, \beta \geq -1/2$ and $4/3 < p < 4$ ([24]). Note that the following particular selections of $\alpha$ and $\beta$ carry special names.

$\alpha = \beta = 0$: *Legendre system.*

$\alpha = \beta = -1/2$: *Chebyshev system of the first kind.*

$\alpha = \beta = 1/2$: *Chebyshev system of the second kind.*

$\alpha = \beta$: *Ultraspherical (Gegenbauer) system.*

(5°) *Laguerre system.* Let $E = [0, \infty)$ and $\nu > -1$. Let

$$f_j(t) = \{j!/\Gamma(\nu+j+1)\}^{1/2} t^{\nu/2} e^{-t/2} L_j^{[\nu]}(t) \qquad (j \geq 0),$$

where $L_j^{[\nu]}(t)$ is the Laguerre polynomial of order $\nu$ defined by

$$L_j^{[\nu]}(t) = \frac{1}{j!} t^{-\nu} e^t \frac{d^j}{dt^j}(t^{j+\nu} e^{-t})$$

$$= \sum_{k=0}^{j} (-1)^k \binom{j+\nu}{j-k} \frac{1}{k!} t^k$$

(cf. [22; Chap.V]). The Laguerre series converges in $L^p(E)$ whenever $\nu \geq 0$ and $4/3 < p < 4$ ([2]).

(6°) *Walsh system.* Let $E = [0,1]$. The Rademacher functions are defined by

$$r_0(t) = \chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t), \quad r_0(t+1) = r_0(t),$$

$$r_j(t) = r_0(2^j t) \qquad (j = 1, 2, \cdots).$$

Let $\{f_j\}$ be the sequence of Walsh functions on $E$ defined as follows:

$$f_0(t) = 1, \quad f_j(t) = r_{j_1}(t) r_{j_2}(t) \cdots r_{j_k}(t)$$

$$(j = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k}, \ j_1 > j_2 > \cdots > j_k \geq 0).$$

The Walsh series converges in $L^p(E)$ whenever $1 < p < \infty$ (cf. [7],[11],[19],[21; Chap.II, §14]).

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