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CONVERGENCE OF POSITIVE LINEAR FUNCTIONALS

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1. Introduction

Let X be a compact Hausdorff space. Let B(X) denote the Banach lattice of all real-valued bounded functions on X, endowed with the supremum norm and the canonical order. C(X) denotes the closed sublattice of B(X) consisting of all real-valued continuous functions on X. For a given point $t \in X$, we denote by $C_t(X)$ the closed sublattice of B(X) consisting of all real-valued bounded functions on X which are continuous at t, and by δ_t the point evaluation functional at t, defined by $\delta_t(f) = f(x)$ for all $f \in B(X)$. Let A(X) be a linear subspace of $C_t(X)$ which contains the unit function 1_X defined by $1_X(y) = 1$ for all $y \in X$.

The purpose of this paper is to establish a convergence theorem in the context of positive linear functionals on $A(X)$. Furthermore, several applications can be provided and actually, we shall obtain a generalization of the classical Korovkin theorem (cf. [6]) for sequences of positive linear functionals on $C([a, b])$, where $[a, b]$ is a bounded closed interval in the real line $\mathbb{R}$.

For other researches of Korovkin type convergence theorems in various directions, see, e.g., [1], [2], [3], [4], [5], [8], [10], [14], [17] and [18].

2. Convergence Theorems

Let t be any fixed point of X and let Φ_t be a non-negative function in $A(X)$ such that

(1) $\inf\{\Phi_t(x); x \in F\} > 0$ for every closed subset F of $X \setminus \{t\}$.

Let $\{L_{\alpha,\lambda}; \alpha \in D, \lambda \in \Lambda\}$ be a family of positive linear functionals on $A(X)$, where D is a directed set and Λ is an index set. Let ξ be any fixed non-negative real number.

Theorem 1. If

(2) $$\lim_{\alpha} L_{\alpha,\lambda}(\Phi_t) = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

and if there exists a function $u \in A(X)$ such that

(3) $$\inf\{u(x);\ x \in X\} > 0$$

and

(4) $$\lim_{\alpha} L_{\alpha,\lambda}(u) = \xi\delta_t(u) \qquad \text{uniformly in } \lambda \in \Lambda,$$

then for every $f \in A(X)$,

(5) $$\lim_{\alpha} L_{\alpha,\lambda}(f) = \xi\delta_t(f) \qquad \text{uniformly in } \lambda \in \Lambda.$$

Proof. Let Ψ be a function in $A(X)$ with $\Psi(t) = 0$, and let $\varepsilon > 0$ be given. Then there exists a neighborhood $V(t)$ of t such that $|\Psi(x)| < \varepsilon$ for all $x \in V(t)$. Let $F = X \setminus V(t)$, and set

$$m = \inf\{\Phi_t(x);\ x \in F\} \quad \text{and} \quad n = \sup\{|\Psi(x)|;\ x \in F\}.$$

In view of (1), we have $m > 0$, and so

$$|\Psi| < \varepsilon 1_X + (n/m)\Phi_t,$$

which, by the positivity and linearity of $L_{\alpha,\lambda}$, yields

(6) $$|L_{\alpha,\lambda}(\Psi)| \leqq \varepsilon L_{\alpha,\lambda}(1_X) + (n/m)L_{\alpha,\lambda}(\Phi_t)$$

for all $\alpha \in D$, $\lambda \in \Lambda$. By (3), there exists a constant $C > 0$ such that $u(x) \geqq C$ for every $x \in X$, and thus

$$L_{\alpha,\lambda}(1_X) \leqq (1/C)L_{\alpha,\lambda}(u) \qquad (\alpha \in D,\ \lambda \in \Lambda),$$

which together with (4) shows that there exists an element $\alpha_0 \in D$ such that

(7) $$\sup\{L_{\alpha,\lambda}(1_X);\ \alpha \geqq \alpha_0,\ \alpha \in D,\ \lambda \in \Lambda\} < \infty.$$

Therefore, by (2), (6) and (7) we conclude

(8) $$\lim_{\alpha} L_{\alpha,\lambda}(\Psi) = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

Now we take

$$\Psi = f - (f(t)/u(t))u$$

for any $f \in A(X)$. Then Ψ belongs to $A(X)$ and $\Psi(t) = 0$. Hence, (8) implies

(9) $$\lim_{\alpha} \{L_{\alpha,\lambda}(f) - \delta_t(f/u)L_{\alpha,\lambda}(u)\} = 0 \quad \text{uniformly in } \lambda \in \Lambda.$$

Consequently, the equality

$$L_{\alpha,\lambda}(f) - \xi\delta_t(f) = L_{\alpha,\lambda}(f) - \delta_t(f/u)L_{\alpha,\lambda}(u)$$

$$+ \delta_t(f/u)\{L_{\alpha,\lambda}(u) - \xi\delta_t(u)\}$$

establishes the desired result (5) by virtue of (4) and (9).

Corollary 1. If (2) is fulfilled and if

$$\lim_{\alpha} L_{\alpha,\lambda}(1_X) = \xi \qquad \text{uniformly in } \lambda \in \Lambda,$$

then (5) holds for all $f \in A(X)$.

Corollary 2. If L is a positive linear functional on $A(X)$ satisfying $L(\Phi_t) = 0$, then $L(f) = L(1_X)\delta_t(f)$ for every $f \in A(X)$.

Remark 1. Theorem 1, Corollaries 1 and 2 even hold for arbitrary topological spaces X.

Let p be any fixed positive real number and let G be a subset of $C(X)$ separating the points of X. Suppose that $A(X)$ contains the set $\{|g - g(t)1_X|^p;\ g \in G\}$. For each $g \in G$, we define

$$\mu_{\alpha,\lambda}^{(p,t)}(g) = L_{\alpha,\lambda}(|g - g(t)1_X|^p) \qquad (\alpha \in D,\ \lambda \in \Lambda).$$

Theorem 2. If for all $g \in G$,

$$\lim_{\alpha} \mu_{\alpha,\lambda}^{(p,t)}(g) = 0 \qquad \text{uniformly in } \lambda \in \Lambda \tag{10}$$

and if there exists a function $u \in A(X)$ satisfying (3) and (4), then (5) holds for every $f \in A(X)$.

Proof. Let h be an arbitrary function in $A(X)$. Since the original topology on X is identical with the weak topology on X induced by G, given $\varepsilon > 0$, we can choose a finite subset $\{g_1, g_2, \cdots, g_m\}$ of G and a constant $C > 0$ such that

$$|h(x) - h(t)| < \varepsilon + C \sum_{i=1}^{m} |g_i(x) - g_i(t)|^p$$

for all $x \in X$. Thus, for any $\alpha \in D$, $\lambda \in \Lambda$ we obtain

$$|L_{\alpha,\lambda}(h) - h(t)L_{\alpha,\lambda}(1_X)| \leqq \varepsilon L_{\alpha,\lambda}(1_X) + C \sum_{i=1}^{m} \mu_{\alpha,\lambda}^{(p,t)}(g_i),$$

which together with (7) and (10) gives

$$\text{(11)}\quad \lim_{\alpha} \{L_{\alpha,\lambda}(h) - \delta_t(h)L_{\alpha,\lambda}(1_X)\} = 0 \quad \text{uniformly in } \lambda \in \Lambda.$$

Now, for all $\alpha \in D$, $\lambda \in \Lambda$ we have

$$\begin{aligned} L_{\alpha,\lambda}(f) - \xi\delta_t(f) &= L_{\alpha,\lambda}(f) - \delta_t(f)L_{\alpha,\lambda}(1_X) \\ &+ \delta_t(f/u)\{\delta_t(u)L_{\alpha,\lambda}(1_X) - L_{\alpha,\lambda}(u)\} \\ &+ \delta_t(f/u)\{L_{\alpha,\lambda}(u) - \xi\delta_t(u)\}, \end{aligned}$$

which yields the desired assertion (5) on account of (4) and (11).

Corollary 3. Suppose that $A(X)$ contains the set $V = \{g^i;\ g \in G,\ i = 0, 1, 2, \cdots, p\}$, where p is an arbitrary fixed even positive integer. If for all $h \in V$,

$$\lim_{\alpha} L_{\alpha,\lambda}(h) = \xi\delta_t(h) \qquad \text{uniformly in } \lambda \in \Lambda,$$

then (5) holds for any $f \in A(X)$.

Corollary 4. Let L be a positive linear functional on $A(X)$. If $L(|g - g(t)1_X|^p) = 0$ for each $g \in G$, then $L(f) = L(1_X)\delta_t(f)$ for every $f \in A(X)$. Also, under the hypothesis of Corollary 3, if $L(h) = \xi\delta_t(h)$ for all $h \in V$, then $L = \xi\delta_t$ on $A(X)$.

Let M be a linear subspace of $C(X)$ which contains 1_X. For any $x \in X$, we denote by $R_x(M)$ the set of all positive linear functionals L on $C(X)$ such that $L(f) = f(x)$ for all $f \in M$ and by ε_x the restriction of δ_x to $C(X)$. Obviously, ε_x belongs to $R_x(M)$. We set

$$\partial_M(X) = \{x \in X;\ R_x(M) = \{\varepsilon_x\}\},$$

which is called the Choquet boundary of X with respect to M. For any $f \in B(X)$, we define

$$f^{\#} = \inf\{g;\ f \leqq g,\ g \in M\}$$

and

$$f_{\#} = \sup\{g;\ g \leqq f,\ g \in M\}$$

which are called the upper and lower M-envelopes of f, respectively. Evidently, we have

$$f_{\#} \leqq f \leqq f^{\#}, \qquad f_{\#} = -(-f)^{\#} \qquad (f \in B(X)).$$

Moreover, $f^{\#}$ and $f_{\#}$ are upper and lower semi-continuous functions in $B(X)$, respectively. Also, we get

$$(f + g)^{\#} \leqq f^{\#} + g^{\#} \qquad (f, g \in B(X))$$

and

$$(\beta f)^{\#} = \beta f^{\#} \qquad (f \in B(X), \beta \in \mathbb{R}, \beta \geqq 0).$$

Note that the set

$$M^{\#}(X) = \{f \in C(X);\ f^{\#} = f_{\#}\} = \{f \in C(X);\ f^{\#} = f = f_{\#}\}$$

is a linear subspace of $C(X)$ containing M. Define

$$\partial_M^{\#}(X) = \{x \in X;\ f^{\#}(x) = f_{\#}(x) \text{ for all } f \in C(X)\}$$

$$= \{x \in X;\ f^{\#}(x) = f(x) \text{ for all } f \in C(X)\}.$$

Then we have

$$\partial_M(X) = \partial_M^{\#}(X)$$

and

$$\partial_M(X) = X \quad \text{if and only if} \quad M^{\#}(X) = C(X)$$

(cf. [2]).

In view of these observations and Corollaries 2 and 4, we make the following remarks.

Remark 2. (i) Let M be a linear subspace of C(X) containing 1_X and Φ_t. If $\Phi_t(t) = 0$, then t belongs to $\partial_M(X)$.

(ii) Let M be a linear subspace of C(X) which contains 1_X and $\{|g - g(t)1_X|^p;\ g \in G\}$. Then t belongs to $\partial_M(X)$.

(iii) Let V be as in Corollary 3 and let M be a linear subspace of C(X) containing V. Then we have $\partial_M(X) = X$, and so $M^{\#}(X) = C(X)$.

Remark 3. Suppose that $M \subset A(X)$. If for all $g \in M$,

$$\lim_\alpha L_{\alpha,\lambda}(g) = \xi\delta_t(g) \qquad \text{uniformly in } \lambda \in \Lambda,$$

then for all $f \in M^{\#}(X) \cap A(X)$,

$$\lim_\alpha L_{\alpha,\lambda}(f) = \xi\delta_t(f) \qquad \text{uniformly in } \lambda \in \Lambda.$$

Also, the statement analogous to this result can be formulated in the context of positive linear operators of a linear subspace of C(X) into B(X). These results extend [2; Proposition 1.4] (cf. [3; Theorem 1]).

3. Applications

Let E_1 and E_2 be normed vector lattices with norms $\|\cdot\|_{E_1}$ and $\|\cdot\|_{E_2}$, respectively. Let Y be a linear subspace of E_1 and let T be a positive linear operator of Y into E_2. A subset S of Y is called a Korovkin test system (or, briefly, KTS) with respect to $\{T, Y, E_1, E_2\}$ if for any family $\{T_{\alpha,\lambda};\ \alpha \in D,\ \lambda \in \Lambda\}$ of positive linear operators of Y into E_2, the relation

$$\lim_{\alpha} \| T_{\alpha,\lambda}(g) - T(g) \|_{E_2} = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $g \in S$ implies that

$$\lim_{\alpha} \| T_{\alpha,\lambda}(f) - T(f) \|_{E_2} = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $f \in Y$ (cf. [10]). In particular, if $E_1 = E_2$ or $E_2 = \mathbb{R}$, then such a subset S is called a Korovkin test system with respect to T in Y.

With this notion Corollary 3 claims that V is a KTS with respect to $\xi\delta_t$ in A(X). As an immediate consequence of Theorems 1 and 2, we have the following:

Theorem 3. Let w be a function in A(X) such that $\inf\{w(x);\ x \in X\} > 0$.

(i) Suppose that $\Phi_t(t) = 0$. Then $\{w, \Phi_t\}$ is a KTS with respect to $\xi\delta_t$ in A(X). In particular, $\{1_X, \Phi_t\}$ is a KTS with respect to $\xi\delta_t$ in A(X).

(ii) $\{w\} \cup \{|g - g(t)1_X|^p;\ g \in G\}$ is a KTS with respect to $\xi\delta_t$ in A(X). In particular,

$\{1_X\} \cup \{|g - g(t)1_X|^p;\ g \in G\}$ is a KTS with respect to $\xi\delta_t$ in A(X).

Corollary 5. Let $\{f_1, f_2, \cdots, f_m\}$ be a finite subset of A(X) and let

$$\Phi_t(x) = \sum_{i=1}^{m} a_i(t) f_i(x) \qquad (x \in X),$$

where each $a_i(t)$ is a real number, such that Φ_t is a non-negative function in B(X) satisfying (1) and $\Phi_t(t) = 0$. Let w be as in Theorem 3. Then $\{w, f_1, f_2, \cdots, f_m\}$ is a KTS with respect to $\xi\delta_t$ in A(X). In particular, $\{1_X, f_1, f_2, \cdots, f_m\}$ is a KTS with respect to $\xi\delta_t$ in A(X).

We shall now mention some examples of $\Phi_t \in C(X)$ satisfying $0 = \Phi_t(t) < \Phi_t(x)$ for each $x \in X$, $x \neq t$, and so (1) is always satisfied:

(1°) Let $\{g_1, g_2, \cdots, g_r\}$ be a finite subset of C(X) such that for every $x \in X$, $x \neq t$, there exists an integer $j \in \{1, 2, \cdots, r\}$ for which $g_j(x) \neq g_j(t)$. Assume that A(X) contains the set

$$G_1 = \{1_X, g_1, g_2, \cdots, g_r, g_1^2, g_2^2, \cdots, g_r^2\}.$$

Then with the help of the function Φ_t given by

$$\Phi_t = \sum_{i=1}^{r} (g_i - g_i(t)1_X)^2,$$

it follows from Corollary 5 that the set

$$G_2 = \{1_X, g_1, g_2, \cdots, g_r, g_1^2 + g_2^2 \cdots + g_r^2\}$$

is a KTS with respect to $\xi\delta_t$ in A(X). Note that G_1 is also a KTS with respect to $\xi\delta_t$ in A(X). Furthermore, if $\{g_1, g_2, \cdots, g_r\}$ separates the points of X and if M is a linear subspace of C(X) containing G_1, then we have $\partial_M(X) = \partial_M^{\#}(X) = X$ and $M^{\#}(X) = C(X)$.

For example, we take $X = X_r$ a compact subset of $\mathbb{R}^r$ and let us denote by e_i ($i = 1, 2, \cdots, r$) the i-th coordinate function on X_r. Suppose that $A(X_r)$ contains the set

$$K_1 = \{1_X, e_1, e_2, \cdots, e_r, e_1^2, e_2^2, \cdots, e_r^2\}.$$

Then K_1 and

$$K_2 = \{1_X, e_1, e_2, \cdots, e_r, e_1^2 + e_2^2 + \cdots + e_r^2\}$$

are Korovkin test systems with respect to $\xi\delta_t$ in $A(X_r)$. Next, let $X = \mathbb{T}^r$ be the r-dimensional torus and for $i = 1, 2, \cdots, r$, we define

$$c_i(x_1, x_2, \cdots, x_r) = \cos x_i \qquad ((x_1, x_2, \cdots, x_r) \in X)$$

and

$$s_i(x_1, x_2, \cdots, x_r) = \sin x_i \qquad ((x_1, x_2, \cdots, x_r) \in X).$$

Assume that $A(\mathbb{T}^r)$ contains the set

$$M_r = \{1_X, c_1, c_2, \cdots, c_r, s_1, s_2, \cdots, s_r\}.$$

Then M_r is a KTS with respect to $\xi\delta_t$ in $A(\mathbb{T}^r)$.
In particular, the sets

$$\{1_X, e_1, e_1^2\} \quad \text{and} \quad \{1_X, c_1, s_1\}$$

are Korovkin test systems with respect to $\xi\delta_t$ in $A(X_1)$ and in $A(\mathbb{T}^1)$, respectively. Concerning the usual convergence, for the special case $\xi = 1$ this is the classical result due to Korovkin (cf. [6]). For further generalizations of the Korovkin theorem from other approaches, one may consult [1], [2], [3], [4], [5], [8], [10], [14], [17] and [18].

(2°) Let (X, d) be a compact metric space. Let

$$\Phi_t(x) = \Phi(d(x, t)) \qquad (x \in X),$$

where Φ is a strictly increasing continuous function on $[0, \infty)$ with $\Phi(0) = 0$. For instance, the case where Φ is given by $\Phi(y) = y^q$, $q > 0$, may be useful (cf. (3°), (4°)).

(3°) Let X be a compact subset of a normed linear space with norm $||\cdot||$. Let

$$\Phi_t(x) = ||x - t||^q, \quad q > 0, \qquad (x \in X).$$

(4°) Let $(H, \langle\cdot, \cdot\rangle)$ be a real pre-Hilbert space and let X be a compact subset of H. Let

$$\Phi_t(x) = \langle x - t, x - t\rangle \qquad (x \in X).$$

For any $x \in X$, we define

$$h(x) = \langle x, x\rangle \quad \text{and} \quad h_t(x) = \langle x, t\rangle.$$

Then we have the following:

Theorem 4. If h and h_t belong to A(X), then $\{1_X, h, h_t\}$ is a KTS with respect to $\xi\delta_t$ in A(X).

Proof. This follows from Corollary 1 (cf. Theorem 3 (i)).

For example, one takes $H = \mathbb{R}^r$ with the usual inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ry_r$$

for $x = (x_1, x_2, \cdots, x_r)$, $y = (y_1, y_2, \cdots, y_r) \in \mathbb{R}^r$. Then we have

$$h = \sum_{i=1}^{r} e_i^2 \quad \text{and} \quad h_t = \sum_{i=1}^{r} e_i(t)e_i .$$

If $A(X)$ contains K_1 and if for $i = 1, 2, \cdots, r$,

$$\lim_{\alpha} L_{\alpha,\lambda}(e_i) = \xi\delta_t(e_i) \qquad \text{uniformly in } \lambda \in \Lambda,$$

then

$$\lim_{\alpha} L_{\alpha,\lambda}(h_t) = \xi\delta_t(h_t) \qquad \text{uniformly in } \lambda \in \Lambda.$$

Consequently, K_2 becomes again a KTS with respect to $\xi\delta_t$ in $A(X)$.

Corollaries 3 and 4, Remark 2 (ii), (iii) and Theorem 3 (ii) can be applied in the following situation (cf. [14]):

(5°) Let X be a compact subset of a real locally convex Hausdorff vector space E with its dual space E^* and $G = \{\tau|_X;\ \tau \in E^*\}$, where $\tau|_X$ denotes the restriction of τ to X.

For instance, taking $E = \mathbb{R}^r$, K_1 becomes again a KTS with respect to $\xi\delta_t$ in $A(X)$.

Let $\mathbb{N}$ denote the set of all non-negative integers. Let $B = \{b_{\alpha,n}^{(\lambda)};\ \alpha \in D,\ \lambda \in \Lambda,\ n \in \mathbb{N}\}$ be a family of non-negative real numbers with $\Sigma_{n=0}^{\infty} b_{\alpha,n}^{(\lambda)} = 1$ for each $\alpha \in D$, $\lambda \in \Lambda$. For examples of such families, see, e.g., [12] (cf. [9],

[10]). Let $\{L_n ; n \in \mathbb{N}\}$ be a sequence of positive linear functionals on $A(X)$. For any $f \in A(X)$, we define

$$(12) \qquad L_{\alpha,\lambda}(f) = \sum_{n=0}^{\infty} b_{\alpha,n}^{(\lambda)} L_n(f) \qquad (\alpha \in D, \lambda \in \Lambda),$$

which converges in $\mathbb{R}$. Plainly, each $L_{\alpha,\lambda}$ is a positive linear functional on $A(X)$, and consequently our general results can be applicable to the family $\{L_{\alpha,\lambda}\}$.

For this it is convenient to make the following definition: Let $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$ be normed vector lattices. Let Y be a linear subspace of E_1 and T a positive linear operator of Y into E_2. A subset S of Y is called a B-Korovkin test system (or, briefly, B-KTS) with respect to $\{T, Y, E_1, E_2\}$ if for any sequence $\{T_n ; n \in \mathbb{N}\}$ of positive linear operators of Y into E_2, the relation

$$\lim_{\alpha} \left\| \sum_{n=0}^{\infty} b_{\alpha,n}^{(\lambda)} T_n(g) - T(g) \right\|_{E_2} = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $g \in S$ implies that

$$\lim_{\alpha} \left\| \sum_{n=0}^{\infty} b_{\alpha,n}^{(\lambda)} T_n(f) - T(f) \right\|_{E_2} = 0 \qquad \text{uniformly in } \lambda \in \Lambda$$

for every $f \in Y$, where it is assumed that the above series converge for each α, λ, g and f. In particular, if $E_1 = E_2$ or $E_1 = \mathbb{R}$, then such a subset S is called a B-KTS with respect to T in Y.

With this concept we are able to derive various B-Korovkin test systems. For instance, concerning the almost convergence (F-summability) introduced by Lorentz [7], which is the case where

$$D = \mathbb{N} \setminus \{0\}, \quad \Lambda = \mathbb{N},$$

$$b_{\alpha,n}^{(\lambda)} = 1/\alpha \qquad (\lambda \leqq n < \alpha + \lambda)$$

and

$$b_{\alpha,n}^{(\lambda)} = 0 \qquad (n < \lambda,\ \alpha + \lambda \leqq n),$$

(12) reduces to

$$L_{\alpha,\lambda}(f) = (1/\alpha) \sum_{n=\lambda}^{\alpha+\lambda-1} L_n(f).$$

In this case,

$$\lim_{\alpha\to\infty} L_{\alpha,\lambda}(f) = \xi\delta_t(f) \qquad \text{uniformly in } \lambda \in \mathbb{N}$$

if and only if each Banach limit of the sequence $\{L_n(f)\}$ is equal to $\xi\delta_t(f)$ (cf. [7; Theorem 1]).

We give now the concrete examples: Let I_r be the unit r-cube, i.e.,

$$I_r = \{x = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r;\ 0 \leqq x_i \leqq 1,\ i = 1, 2, \cdots, r\}$$

and let $B_t(I_r)$ denote the linear subspace of $C_t(I_r)$

consisting of all real-valued bounded integrable functions on I_r which are continuous at $t = (y, y, \cdots, y)$, where y is an arbitrary fixed point of I_1. Let $\{k_n; n \in \mathbb{N}\}$ be a sequence of non-negative functions in $C(I_1)$. For each $n \in \mathbb{N}$, we define the functional L_n by

$$L_n(f) = \int_{I_r} \prod_{i=1}^{r} k_n(e_i(x))f(x)dx$$

for all $f \in B_t(I_r)$. Put

$$\nu_n^{(i)} = \int_0^1 x^i k_n(x)dx \qquad (n \in \mathbb{N},\ i = 0, 1, 2).$$

Then $\{\nu_n^{(i)}\}$ converges to y^i for $i = 0, 1, 2$ if and only if $\{L_n(f)\}$ converges to $\delta_t(f)$ for every $f \in B_t(I_r)$. Also, if $\{\nu_n^{(0)}\}$ converges to one and if $\{\nu_n^{(i)}\}$ is almost convergent to y^i for $i = 1, 2$, then $\{L_n(f)\}$ is almost convergent to $\delta_t(f)$ for all $f \in B_t(I_r)$.

For example, if we take

$$k_n(x) = (n + 1)x^n \qquad (n \in \mathbb{N},\ 0 \leqq x \leqq 1)$$

and $t = (1, 1, \cdots, 1)$, then for every $f \in B_t(I_r)$, $\{L_n(f)\}$ converges to $\delta_t(f)$ and it is also almost convergent to $\delta_t(f)$. Next, if we take

$$k_n(x) = 2(1 - x^2)^n/\rho_n \qquad (n \in \mathbb{N},\ 0 \leqq x \leqq 1),$$

where

$$\rho_n = \int_{-1}^{1} (1 - x^2)^n dx = \Gamma(1/2)\Gamma(n + 1)/\Gamma(n + 3/2)$$

$$= 2^{2n+1}(n!)^2/(2n + 1)!,$$

and $t = (0, 0, \cdots, 0)$, then for all $f \in B_t(I_r)$, $\{L_n(f)\}$ is almost convergent to $\delta_t(f)$.

Finally, it should be remarked that under certain appropriate conditions, Theorems 1 and 2 can be recast in a quantitative form in which we are able to estimate the rate of convergence of $\{L_{\alpha,\lambda}(f)\}$ by using a modulus of continuity of f (cf. [10], [11], [12], [13], [15], [16]). We omit the details.

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