The varieties of subspaces stable under a nilpotent transformation

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THE VARIETIES OF SUBSPACES STABLE UNDER A NILPOTENT TRANSFORMATION

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ABSTRACT. Let $f : V \to V$ be a nilpotent linear transformation of a vector space V of type $V = \lambda$, i.e. the size of Jordan blocks $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. For an f-stable subspace W of V, i.e. $f(W) \subset W$, the types of W and V/W are those of the maps $f|_W : W \to W$ and $f_{V/W} : V/W \to V/W$ induced by f, respectively. For partitions ν and μ we investigate the set $S(\lambda, \nu, \mu) = \{W \subset V; f(W) \subset W, \text{type } W = \nu, \text{type } V/W = \mu\}$ and the singular locus of the Zariski closure $X(\lambda, \nu, \mu)$ of $S(\lambda, \nu, \mu)$ in the grassmaniann of subspaces of V of dimension $|\nu|$. We show that $S(\lambda, \nu, \mu)$ is nonsingular and its connected components are rational varieties (Th.A) ; generic vectors are introduced (Def.18), which define the generic points of the irreducible components of $X(\lambda, \nu, \mu)$ whose Plücker coordinates are fairly simple to express their defining equations. We describe explicitly the coordinate ring of an affine openset of $X(\lambda, (d), \mu)$ with the singular locus of codimension two (Prop.C).

Introduction. Let $f : V \to V$ be a nilpotent linear transformation of a vector space V over $\mathbb{C}$ of type $V = \lambda$, i.e. the size of Jordan blocks $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. For an f-stable subspace W of V, i.e. $f(W) \subset W$, the types of W and V/W are those of the maps $f|_W : W \to W$ and $f_{V/W} : V/W \to V/W$ induced by f, respectively. For an integer $0 < d < n = \dim V$ let $X(\lambda, d)$ be the set of f-stable subspaces of V of dimesnion d ; $X(\lambda, d) = \{W \subset V;\ \dim\ W = d,\ f(W) \subset W\ \}$, which is a closed set in the grassmaniann $G(d, V)$ of d-dimensional subspaces of V (cf.Section 1) and partitioned by the types of W and of V/W ; $X(\lambda, d) = \coprod_{\nu,\mu} S(\lambda, \nu, \mu)$ where

$$S(\lambda, \nu, \mu) = \{\ W \in X(\lambda, d)\ ;\ \text{type } W = \nu,\ \text{type } V/W = \mu\ \}$$

Our aim in this note is to deteremine the singular loci of the Zariski closures $X(\lambda, \nu, \mu)$ of the locally closed sets $S(\lambda, \nu, \mu)$ in $G(d, V)$ and

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describe their coodinate rings. Although our results are not as explicit as one might wish, the introduction of generic vectors in Definition 18 will be useful in the description of the singularities of $X(\lambda,\nu,\mu)$.

The connected components of $S(\lambda,\nu,\mu)$ are all of the same dimension $n(\lambda)-n(\nu)-n(\mu)$ where $n(\lambda)=\sum_i(\lambda_i-1)$ and in bijection with the Littlewood-Richardson tableaux (LR-tableaux) of shape λ/μ and content ν (Definition 1) so the number of the connected components is equal to the Littlewood-Richardson coefficient $c^{\lambda}_{\nu,\mu}$ [M1,p188;L,p411,p414]. We denote by S_T and X_T the connected component of $S(\lambda,\nu,\mu)$ corresponding to an LR-tableau T and its Zariski closure, respectively. On the analogy of the well known fact that the closure of the conjugacy class of the nilpotent matrices of type λ consists of those of the types less than or equal to λ in the dominance order $\lhd$ of the partition [M1,p7], we shall show in Lemma 3 that the closure $X(\lambda,\nu,\mu)$ of $S(\lambda,\nu,\mu)$ is contained in the union of $S(\lambda,\bar{\nu},\bar{\mu})$ for partitions both $\bar{\nu}\lhd\nu$ and $\bar{\mu}\lhd\mu$, but the equality does not hold in general (Example 20). In the case $\nu=(1^d)$, i.e. f-stable subspaces W with $f(W)=0$ we observe in Lemma 9 that $S(\lambda,(1^d),\mu)$ happens to be a union of Schubert cells and $X(\lambda,(1^d),\mu)$ is a Schubert variety so it is known about the singular loci in this case (Corollary 11). The set $S(\lambda,\nu,\mu)$ is, in general, not a homogeneous space under the action of the automorphism group $A(V)=\{g\in \mathrm{GL}(V)\ ;\ f\circ g=g\circ f\}$ of (V,f) (Remark 8) so it is not clear that $S(\lambda,\nu,\mu)$ is nonsingular and the singular locus of $X(\lambda,\nu,\mu)$ is a union of $S(\lambda,\bar{\nu},\bar{\mu})$'s for some $\bar{\nu}\lhd\nu$ and $\bar{\mu}\lhd\mu$ as in the case of Schubert varieties. Combining the strucure of the automorphism group $A(V)$ of (V,f) (Lemma 6) with the homogeneity of $S(\lambda,(1^d),\mu)$ with respect to $A(V)$ (Lemma 14) we show in Section 2

Theorem A. *$S(\lambda,\nu,\mu)$ are nonsingular and X_T are rational varieties.*

Section 3 is the main section of this note where we define the generic vectors for an LR-tableau T together with their algorithms (Definition 18). The generic vector corresponds to each cell t of T filled with the least letter, contains algebraically independent parameters in the coefficients of the parameter cells of the cell t (Definition 17) and has the coefficient 1 in the cell vector of t (Notation(2)). The sum of the numbers of the parameter cells over all of the cells of T is equal to the dimension of X_T so we obtain a map from the affine space of dimension $N=\dim X_T$ to X_T which associates a subspace of V to its Plücker coordinate ; $\varphi:\mathbb{A}^N\to X_T$. Denote by $T_{\geq k}$ the LR-tableau deleting from T the cells of T filled with the letters less than k. We shall show in Section 3

Theorem B. *(i) The f-stable subspace generated by the generic vectors for T has a basis consisting of the generic vectors for $T, T_{\geq 2}, \cdots, T_{\geq \nu_1}$.*

(ii) The map $\varphi : \mathbb{A}^N \to X_T$ is a birationl morphism, i.e. the f-stable subspace in (i) defines a generic point of X_T.

The image of φ intersects with $S_{\bar{T}} \subset X_T$ iff the shape of $\bar{T}$ is the same as that of T, in particular the image of the origin of $\mathbb{A}^N$ is the subspace W_0 generated by the cell vectors of all the cells of T, which is a point of S_{T_0} where T_0 is the least LR-tableau among those of shape T. Theorem B implies that it is fairly easy to deteremine the singular locus of X_T contained in the principal openset $U_{W_0} = \{\wedge^d W_0 \neq 0\}$ because the affine coordinates of U_{W_0} coming from the coefficients of the generic vectors for $T, T_{\geq 2}, \cdots, T_{\geq \nu_1}$ are so simple as to express the defining equations of X_T in $U_{W_0} = \mathbb{A}^{d(n-d)}$. However it requires another device to examine the locus of X_T where the shape is strictly less than that of T (Example 19). While there is a dual isomorphism between $X(\lambda, \nu, \mu)$ and $X(\lambda, \mu, \nu)$ (Remark 9) it seems that if both the shape and the content of $\bar{T}$ are strictly less than those of T then X_T is always singular along $X_{\bar{T}}$. We give two simple examples of generic vectors to show

(i) the shape (resp. the content) of $\bar{T}$ is the same as (resp. less than) that of T but $X_{\bar{T}}$ is not contained in X_T (Example 20),

(ii) the singular locus of X_T is not equal to the union of $S_{\bar{T}}$'s for some LR-tableaux $\bar{T}$ (Example 21).

The simplest example of the codimension two singularity is the variety $X((22), (2), (2))$ which is isomorphic to the cone over a conic the vertex of which corresponds to the point $X((22), (11), (11))$. We generalize this example as follows. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ with $\lambda_1 > \lambda_r \geq 2$ and let $3 \leq d \leq \lambda_1$. Let T be the LR-tableau of content (d) (so the shape is a horizontal strip) and assume that the length of the last row is $k \geq 2$. The LR-tableau $\bar{T}$ is of content $(d-1, 1)$ replacing the letter i in the cell of T by $i-1$ for all $2 \leq i \leq d$, deleting the leftmost cell in the last row of T and adjoining the cell in the next-to last row and in the next-to left column of T, filled with the letter 1, e.g. if $\lambda = (86553)$ and

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ & 7 & 8 \\ \hline \ & \ & \ & \ & \ & 6 \\ \cline{1-6} \ & \ & \ & \ & \ \\ \cline{1-5} \ & \ & \ & 4 & 5 \\ \cline{1-5} 1 & 2 & 3 \\ \cline{1-3} \end{array} \quad , \text{ then } \quad \bar{T} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ & 6 & 7 \\ \hline \ & \ & \ & \ & \ & 5 \\ \cline{1-6} \ & \ & \ & \ & \ \\ \cline{1-5} \ & \ & 1 & 3 & 4 \\ \cline{1-5} \ & 1 & 2 \\ \cline{1-3} \end{array}$$

We see that the codimensions of $X_{\bar{T}}$ in X_T is equal to two. We shall show in Section 4

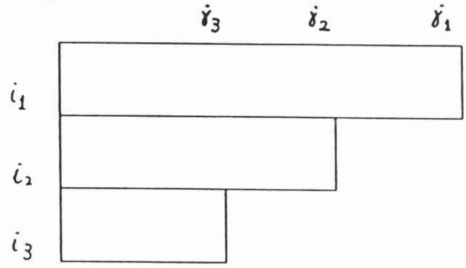
Proposition C. *X_T is singular along $X_{\bar{T}}$ and the coordinate ring of an affine openset of X_T is isomorphic to the hypersurface $\mathbb{C}[x_{ij}] \otimes \mathbb{C}[y_0, y_1, \cdots, y_k]/(f)$ with $f = y_0(y_1 + y_2 y_k + y_3 y_k^2 + y_4 y_k^3 + \cdots + y_{k-1} y_k^{k-2}) + y_k^k$. Here $X_{\bar{T}}$ is defined by $y_0 = y_1 = y_k = 0$.*

The paper is organized as follows. After the notations are introduced, the defining equations of $X(\lambda, \nu, \mu)$ and the automorphism group of (V, f) are investigated in Section 1. It is proved the nonsingualrity and the rationality of $S(\lambda, \nu, \mu)$ (Th.A) in Section 2. Generic vectors for an LR-tableau are defined in Section 3. Proposition C is proved in Section 4. The author would like to thank Professor Takeuchi for many useful discussions.

Notations. (1) Partitions and diagrams. We identify a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ with the diagram of row-length $\lambda_1, \cdots, \lambda_l$ arranged like matrix entries, i.e. the cell (i, j) with the row index i increasing downwards and the column index j increasing to the right. The conjugate of λ is the partition $\lambda' = (\lambda'_1, \cdots, \lambda'_{\lambda_1})$ where λ'_j is the length of the j-th column of the diagram λ. We denote by

$$\lambda = [i_1, \cdots, i_m \mid j_1, \cdots, j_m]$$

if $\{i_1 < \cdots < i_m\}$ (resp. $\{j_1 > \cdots > j_m\}$) is the distinct column (resp. row) length of λ, i.e. the diagram λ consists of m rectangles with row and column sizes $(i_1, j_1), (i_2 - i_1, j_2), \cdots, (i_m - i_{m-1}, j_m)$.



The dominance order $\rhd$ on the partitions is defined by $\lambda \rhd \mu$ (μ is less than or equal to λ) iff $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

(2) The type of (V, f) and cell vectors. Let $f : V \to V$ be a nilpotent linear transformation of a vector space V over $\mathbb{C}$ of dimension n. The type of f, denoted by type V, is the partition λ of $n = \dim V$ with the size of Jordan blocks $\lambda_1 \geq \cdots \geq \lambda_l$. This means that the j-th column of λ is of length $\lambda'_j = \dim \operatorname{Ker} f^j / \operatorname{Ker} f^{j-1} = \dim f^{j-1}(V)/f^j(V)$. If we denote by x_i a generator as $\mathbb{C}[f]$-module of the i-th Jordan block of size λ_i then the cell (i, j) in the diagram λ corresponds to the vector

$f^{j-1}(x_i)$, which forms a basis of V for $1 \leq i \leq \lambda'_1$ and $1 \leq j \leq \lambda_i$. We call $f^{j-1}(x_i)$ the cell vector of the cell (i,j) of λ. We denote a linear combination of the cell vectors by the diagram λ the cells of which are filled with the coefficients of the corresponding cell vectors, e.g. if $\lambda = (21)$ then $ax_1 + bf(x_1) + cx_2 =$

a	b
c	

(3) The type of f-stable subspaces. For an f-stable subspace W of V, i.e. $f(W) \subset W$, the type of W, $\nu =$ type W (resp. the type of V/W, $\mu =$ type V/W), is the type of the map $f|_W : W \to W$ (resp. of $f|_{V/W} : V/W \to V/W$) induced by f. For $0 < d < n = \dim V$ and the diagrams ν, μ we denote by

$$X(V, f; \lambda, d) = \{\ W \in G(d, V)\ ;\ f(W) \subset W\ \}$$
$$S(V, f; \lambda, \nu, \mu) = \{\ W \in X(V, f; \lambda, d)\ ;\ \text{type } W = \nu,\ \text{type } V/W = \mu\ \}$$

and by $X(V, f; \lambda, \nu, \mu)$ the closure of $S(V, f; \lambda, \nu, \mu)$ in the grassmaniann $G(d, V)$ of d-dimensional subspaces of V. If (V, f) is fixed then they are simply written by $X(\lambda, d)$, $S(\lambda, \nu, \mu)$ and $X(\lambda, \nu, \mu)$, respectively.

(4) LR-tableaux. For an element W of $S(\lambda, \nu, \mu)$ let type $V/f^k(W) = \mu^{(k)}$ for $0 \leq k \leq \nu_1$ and $\mu^{(0)} = \mu$. Then type $f^k(W) = \nu - (kkk\cdots)$ and there is an exact sequence

$$\begin{array}{ccccccccc} 0 \to & f^{k-1}(W)/f^k(W) & \to & V/f^k(W) & \to & V/f^{k-1}(W) & \to 0 \\ \text{type} & (1^{\nu'_k}) & & \mu^{(k)} & & \mu^{(k-1)} & \end{array}$$

so that $f^{k-1}(W)/f^k(W)$ is an element of $S(V/f^k(W); \mu^{(k)}, (1^{\nu'_k}), \mu^{(k-1)})$. We define

Definition 1. *(i) The Semitic (resp. Kanji) word associated with a tableau T is the word reading the letters of T from bottom to top (resp. from left to right) in succesive columns (resp. rows) starting from the left (resp. bottom) [L2,p107]. (ii) The Littlewood-Richardson tableau (LR-tableau) associated with an element W of $S(\lambda, \nu, \mu)$ is the tableau T of shape λ/μ and content ν by filling the letter k in the cells of the vertical strip $\mu^{(k)}/\mu^{(k-1)}$ for $1 \leq k \leq \nu_1$. We denote by $T = LR(W, V) = LR(\mu \subset \mu^{(1)} \subset \cdots \subset \mu^{(\nu_1)} = \lambda)$.*

We use the following three facts.

Fact 2. *(i)[M,p186,(3.4)] The Semitic (or Kanji) word $a_1 a_2 \cdots a_d$ associated with an LR-tableau is a lattice permutation, i.e. for all $1 \leq j \leq d$*

and all $1 \leq k \leq \nu_1 - 1$ *the number of occurrences of the letter* k *in* $a_1 \cdots a_j$ *is not less than the number of occurrences of the letter* $k+1$.

(ii)[L,p411,(4.3)] The connected components of $S(\lambda, \nu, \mu)$ *correspond bijectively to the LR-tableaux of shape* λ/μ *and content* ν *so the number of the connected components of* $S(\lambda, \mu, \nu)$ *is equal to the Littlewood-Richadson coefficient* $c^{\lambda}_{\nu,\mu}$.

(iii)[L,p414,(4.4)] The dimensions of the connceted components of $S(\lambda, \nu, \mu)$ *are all equal to* $n(\lambda) - n(\mu) - n(\nu)$ *where* $n(\lambda) = \sum_{i=1}^{\nu'_1}(\lambda_i - 1) = \sum_{j=1}^{\nu_1}\binom{\lambda'_j}{2}$.

We denote by S_T the connected component of $S(\lambda, \nu, \mu)$ consisting of the f-stable subspaces with the associated LR-tableau T ; $S_T = \{W \in S(\lambda, \nu, \mu) \ ; \ LR(W, V) = T\}$, and its closure by X_T. We denote by $T_{>k}$ (resp. $T_{=k}$, $T_{\leq k}$) the LR-tableau deleting from T the cells filled with the letters $\leq k$ (resp. $\neq k$, $> k$). For a $W \in S_T$ we see that the LR-tableau $LR(f^k(W), V)$ is equal to $T_{>\mathrm{k}}$.

1. The defining equations of $X(\lambda, \nu, \mu)$ and the automorohism group of (V, f)

Let $\{v_1, \cdots, v_n\}$ be a basis of V and $A \in M_n(\mathbb{C})$ be the representation matrix of a linear transformation $f : V \to V$ with respect to $\{v_1, \cdots, v_n\}$. A basis of a d-dimensional subspace W of V is given by $(w_1, \cdots, w_d) = (v_1, \cdots, v_n)B$ for an $n \times d$ matrix B of rank d. Extend the basis $\{w_1, \cdots, w_d\}$ of W to a basis of V by $(w_1, \cdots, w_d, u_{d+1}, \cdots, u_n) = (v_1, \cdots, v_n)D$ with $D = (B \ B') \in GL(n)$ for an $n \times (n-d)$ matrix B'. Then

$$\begin{aligned}(f(w_1), \cdots, f(w_d), f(u_{d+1}), \cdots, f(u_n)) &= (f(v_1), \cdots, f(v_n)) \cdot D \\ = (v_1, \cdots, v_n) \cdot AD &= (w_1, \cdots, w_d, u_{d+1}, \cdots, u_n) \cdot D^{-1}AD\end{aligned}$$

The condition of f-stability, $f(W) \subset W$, means that

$$D^{-1}AD = \begin{pmatrix} P & R \\ O & Q \end{pmatrix} \qquad (1)$$

i.e. the lower-left $(n-d) \times d$ submatrix of $D^{-1}AD$ is the zero matrix. From the identity

$$\begin{pmatrix} I_d & X \\ 0 & I_{n-d} \end{pmatrix}^{-1} \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} \begin{pmatrix} I_d & X \\ 0 & I_{n-d} \end{pmatrix} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} P & PX+R \\ 0 & Q \end{pmatrix}$$
$$= \begin{pmatrix} P & PX - XQ + R \\ 0 & Q \end{pmatrix}$$

we see that another choice of $\{u_{d+1}, \cdots, u_n\}$, i.e. of B', alters only the submatrix R in (1). Hence $\{B \in M'(n,d)\ ;\ B$ satisfies (1) with $D = (B\ B')\}$ is a closed set of the set $M'(n,d)$ of $n \times d$ matrices of rank d, which implies that $X(\lambda, d) = \{W \in G(d,V);\ f(W) \subset W\ \}$ is a closed set of the grassmannian $G(d,V)$ of d-dimensional subspaces of V. The induced linear maps on W and on V/W are represented by the matrices P and Q in (1), respectively. Hence, an f-stabe subspace $W \in X(\lambda, d)$ is contained in $S(\lambda, \nu, \mu)$, i.e. $\dim \operatorname{Ker} f^k|_W = \dim W \cap \operatorname{Ker} f^k = \nu'_1 + \cdots + \nu'_k$ and $\dim \operatorname{Ker} f^k|_{V/W} = \mu'_1 + \cdots + \mu'_k$ iff

$$\operatorname{rank} P^k = d - (\nu'_1 + \cdots + \nu'_k), \quad \operatorname{rank} Q^k = (n-d) - (\mu'_1 + \cdots + \mu'_k)$$

for all $k \geq 1$. Hence if W is contained in the closure $X(\lambda, \nu, \mu)$ of $S(\lambda, \nu, \mu)$ in $X(\lambda, d)$ then

$$\operatorname{rank} P^k \leq d - (\nu'_1 + \cdots + \nu'_k), \quad \operatorname{rank} Q^k \leq (n-d) - (\mu'_1 + \cdots + \mu'_k) \quad (2)$$

for all $k \geq 1$. Thus we have proved

Lemma 3. *The Zariski closure $X(\lambda, \nu, \mu)$ of $S(\lambda, \nu, \mu)$ in $X(\lambda, d)$ is contained in the union of $S(\lambda, \bar{\nu}, \bar{\mu})$ for $\bar{\nu} \lhd \nu$ and $\bar{\mu}' \lhd \mu$ in the dominance order $\lhd$ of the partitions (Notation(1)).*

We will give a simple example in Section 3 that the closure $X(\lambda, \nu, \mu)$ of $S(\lambda, \nu, \mu)$ is not equal to the union of $S(\lambda, \bar{\nu}, \bar{\mu})$ for all $\bar{\nu} \lhd \nu$ and all $\bar{\mu} \lhd \mu$ (Example 20).

Suppose that the $d \times d$ submatrix of the $n \times d$-matrix B consisting of the $i_1, \cdots, i_d$-th rows is of maximal rank d for $I = (i_1 < \cdots < i_d) \subset [1, n]$ and assume that it is the identity matrix. If T is the permutation matrix representing the permutation

$$\tau_I = \begin{pmatrix} 1 & \cdots & d & d+1 & \cdots & n \\ i_1 & < \cdots < & i_d & j_1 & < \cdots < & j_{n-d} \end{pmatrix} \in S_n/(S_d \times S_{n-d})$$

then an $n \times (n-d)$ matrix B' is chosen so as to satisfy the identity

$$T^{-1}D = T^{-1} \cdot (B\ B') = \begin{pmatrix} I_d & 0 \\ C & I_{n-d} \end{pmatrix}$$

with an $(n-d)\times d$ matrix C, and

$$
\begin{aligned}
D^{-1}AD &= (T^{-1}D)^{-1}\cdot T^{-1}AT\cdot T^{-1}D \\
&= \begin{pmatrix} I_d & 0 \\ C & I_{n-d} \end{pmatrix}^{-1} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} I_d & 0 \\ C & I_{n-d} \end{pmatrix} \\
&= \begin{pmatrix} I_d & 0 \\ -C & I_{n-d} \end{pmatrix} \begin{pmatrix} A_1 + A_2C & A_2 \\ A_3 + A_4C & A_4 \end{pmatrix} \qquad (3) \\
\text{where} \quad & T^{-1}AT = (a_{\tau_I(i),\tau_I(j)}) = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}
\end{aligned}
$$

Comparing (1) and (2) with (3) we see

Corollary 4. *On the principal openset* $U_I = M(n-d,d) \cong \mathbb{A}^{d(n-d)}$ *of* $G(d,V)$, *(i)* $X(\lambda,d)\cap U_I = \{C \in M(n-d,d); C(A_1+A_2C) = A_3+A_4C\}$ *(ii)* $X(\lambda,\nu,\mu)$ *consists of matries* $C \in X(\lambda,d)$ *satisfying*

$$
\begin{aligned}
&\mathit{rank}\ (A_1 + A_2C)^k \le d - (\nu_1' + \cdots + \nu_k') \\
&\mathit{rank}\ (A_4 - CA_2)^k \le (n-d) - (\mu_1' + \cdots + \mu_k')
\end{aligned}
$$

for all $k \ge 1$.

In Section 4 we will use Cor.4 to describe the coordinate ring of an affine openset of $(X,(d),\mu)$.

Remark 5. The ideal defined by the conditions of (i) and (ii) are, in general, not reduced ; In the case $\lambda = (21)$ and $d = 2$ if we take $(x_1, f(x_1), x_2)$ as an ordered basis of V and $I = (23)$ then

$$
A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} x & y & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix}
$$

$$
T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

Hence $CA_2C = (x\ y)\begin{pmatrix} 1 \\ 0 \end{pmatrix}(x\ y) = (x^2, xy)$ so $\mathrm{Rad}(x^2,xy) = (x)$. We see that $X((21),2) = S((21),(2),(1)) \coprod S((21),(11),(1)) \cong \mathbb{P}^1$ is nonsingular. □

Next we consider the automorphism group

$$A(V) = \{g \in \mathrm{GL}(V) \ ; \ f \circ g = g \circ f \ \}$$

of a nilpotent transformation f of V with the type of f equal to $\lambda = [i_1, \cdots, i_m \mid j_1, \cdots, j_m]$ (Notation (1)). Since $g \in A(V)$ induces an automorphism of all g-stable subspaces of V (for, $g(W) \subset W$ implies $g(W) = W$) there are restriction homomorphisms $A(f^k(V)) \to A(f^l(V))$ for $k < l$. Denoting by $U_j = \mathrm{Ker}\ \{A(V) \to A(f^j(V))\}$ we obtain a normal series $A(V) \triangleright U_1 \triangleright U_2 \triangleright \cdots \triangleright U_{j_1} \triangleright \{id\}$.

Lemma 6. *(i)* $A(f^k(V)) \to A(f^l(V))$ *is surjective for* $0 \le k < l \le j_1$.

(ii) $U_{j-1}/U_j \cong$ *Ker* $\{A(f^{j-1}(V)) \to A(f^j(V))\}$ *is isomorphic to a unipotent group of dimension* i_k^2 *if* $j_k < j < j_{k-1}$ *(resp. a semidirect product of a unipotent group of dimension* $i_k^2 - (i_k - i_{k-1})^2$ *and* $GL(i_k - i_{k-1})$ *if* $j = j_k$*).*

(iii) $A(V)$ *is isomorphic to a semidirect product of a unipotent group of dimension* $\sum_{k=1}^m i_k^2(j_k - j_{k+1}) - \sum_{k=}^m (i_k - i_{k-1})^2$ *and* $\prod_{k=1}^m GL(i_k - i_{k-1})$.

Proof. (i) and (ii) follow from the two facts; (a) type $f^k(V)$ is obtained by deleting the i-th column for all $1 \le i \le k$ in $\lambda =$ type V. (b) A map $g \in A(V)$ is determined by the images $g(x_i)$ of the leftmost cell vectors x_i (Notation (2)) for $1 \le i \le \lambda_1'$, which are generators of the $\mathbb{C}[f]$-module V. Hence $g|_{f(V)} = id$ iff $g(x_i) - x_i$ are contained in the subspace of V generated by the rightmost cell vectors of λ. (iii) Let $V_k = \langle\ x_i\ ;\ i_{k-1} < i \le i_k\ \rangle$ be the subspace of dimension $i_k - i_{k-1}$ generated by the leftmost cell vectors in the k-th row-block of λ and let

$$Z_k = \langle\ \mathrm{Ker}\ f^{j_k - 1},\ \mathrm{Ker}\ f^{j_k} \cap f(V)\ \rangle, \qquad Q_k = \mathrm{Ker}\ f^{j_k}/Z_k$$

Then we see that the canonical map $V_k \subset \mathrm{Ker}\ f^{j_k} \to Q_k$ is an isomorphism and there are a canonical inclusion $\prod_{k=1}^m \mathrm{GL}(V_k) \subset A(V)$ and a surjection $\pi : A(V) \to \prod_{k=1}^m \mathrm{GL}(Q_k)$ with the restriction of π to $\prod \mathrm{GL}(V_k)$ being an isomorphism. It follows from (ii) that Ker π is a unipotent group of dimensnion equal to $\sum_{j=1}^{\lambda_1} \dim U_{j-1}/U_j = \sum_{k=1}^m i_k^2$ $(j_k - j_{k+1} - 1) + \sum_{k=1}^m \{i_k^2 - (i_k - i_{k-1})^2\} = \sum_{k=1}^m i_k^2(j_k - j_{k+1}) - \sum_{k=1}^m (i_k - i_{k-1})^2$. □

We see from Lemma 6

Corollary 7. *$A(V)$ acts on S_T for each LR-tableau T.*

Proof. For $g \in A(V)$ and $W \in S(\lambda, \nu, \mu)$ we see $g(f^k(W)) = f^k(g(W))$ so $g(W) \in S(\lambda, \nu, \mu)$ hence $S(\lambda, \nu, \mu)$ is $A(V)$-stable. Since $A(V)$ is connected by Lemma 6 $A(V)$ acts on each connected component S_T of $S(\lambda, \nu, \mu)$. □

The homomorphism $A(V) \to A(f^k(V)) = A(V/\mathrm{Ker}\, f^k)$ is surjective by Lemma 6(i) but the homomorphisms $A(V) \to A(\mathrm{Ker}\, f^k)$ and $A(V) \to A(V/f^k(V))$ are not surjective in general. The representation matrix of the generic element of $A(V)$ with respect to the Jordan basis is given explicitly in [G,p220] and the dimension $\sum_{k=1}^{m} i_k^2(j_k - j_{k+1})$ of $A(V)$ in Lemma 6(iii) also can be written by $\sum_{i=1}^{\lambda_1'}(2i-1)\lambda_i = \sum_{j=1}^{\lambda_1} \lambda_j'^2$ [ibid,p222;M1,p181].

Remark 8. If $\nu \neq (1^d)$ then $S(\lambda, \nu, \mu)$ is, in general, not a single orbit ; The LR-tableau of shape $(331)/(21)$ and content (22) is $T = \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$ so $S((331), (22), (21)) = S_T$ is given by

$$\{ \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline a \\ \cline{1-1}\end{array} , \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline b \\ \cline{1-1}\end{array} \right) ; \; a, b \in \mathbb{C}, (a,b) \neq (0,0) \} \cong \mathbb{A}^2 \backslash \{0\}$$

and is not a single $A(V)$-orbit because $S((331), (22), (21))/A(V)$ is bijective with the set

$$\{ \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} , \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline b \\ \cline{1-1}\end{array} \right) ; \; b \in \mathbb{C} \} \cup \{ \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 \\ \cline{1-1}\end{array} , \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \right) \} \cong \mathbb{P}^1$$

We end this section with a remark on the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$ and the dual map $f^* : V^* \to V^*$.

Remark 9. For an f stable subspace $W \subset V$ of type $W = \nu$ and of type $V/W = \mu$ the f^*-stable subspace $(V/W)^* \subset V^*$ is of type $(V/W)^* = \mu$ and type $V^*/(V/W)^* = \nu$. Hence the isomorphism $G(d, V) \cong G(n-d, V^*)$, $W \leftrightarrow (V/W)^*$, of the grassmanianns induces the isomorphism between $S(V, f; \lambda, \nu, \mu)$ and $S(V^*, f^*; \lambda, \mu, \nu)$, which induces the commutative diagram

$$\begin{array}{ccc} X(V, f; \lambda, \nu, \mu) & \cong & X(V^*, f^*; \lambda, \mu, \nu) \\ \cup & & \cup \\ X(V, f; \lambda, \bar{\nu}, \bar{\mu}) & \cong & X(V^*, f^*; \lambda, \bar{\mu}, \bar{\nu}) \end{array}$$

for all $\bar{\nu} \lhd \nu$ and $\bar{\mu} \lhd \mu$.

2. The nonsingularity and the rationality of $S(\lambda,\nu,\mu)$

In this section we prove that $S(\lambda,\nu,\mu)$ are nonsingular rational varieties (Theorem A). Let $U \subset V$ be an f-stable subspace such that $f(U)=0$ with type $V=\lambda$ and type $V/U=\alpha$ (where λ/α is a vertical strip since $f(U)=0$). For partitions $\alpha \subset \beta \subset \lambda$ we consider the set [M1,p188]

$$H(\alpha,\beta,\lambda)=\{\ W \subset V\ f\text{-stable}\ ;\ U \supset W,\ \text{type}\ V/W=\beta\ \}$$

If $\alpha=\lambda-(111\cdots)$ deleting the first column from λ then $U=\operatorname{Ker} f$ and $H(\alpha,\beta,\lambda)=S(\lambda,(1^{|\lambda/\beta|}),\beta)$. Let $\lambda=[i_1,\cdots,i_m \mid j_1,\cdots,j_m]$ (Notation (1)). We assume that U has a basis consisting of the (rightmost) cell vectors of λ/α, i.e. $f^{j_k-1}(x_i)$ for $\alpha'_{j_k}<i\le i_k$ and $1\le k\le m$. Since λ/α is a vertical strip the length of the j_k-th column of α satisfies $i_{k-1}\le \alpha'_{j_k}\le i_k$.

Lemma 10. *$H(\alpha,\beta,\lambda)$ is a union of Schubert cells in the grassmaniann $G(|\lambda/\beta|,U)$ and its Zariski closure is a Schubert variety of dimension $\sum_{k=1}^m(i_k-\beta'_{j_k})(\beta'_{j_k}-\alpha'_{j_k})$. Hence $H(\alpha,\beta,\lambda)$ is a nonsingular rational variety.*

Proof. Let $a_k=\sum_{l=1}^k(i_l-\alpha'_{j_l})\ge b_k=\sum_{l=1}^k(i_l-\beta'_{j_l})$ with $a_m=|\lambda/\alpha|$ and $b_m=|\lambda/\beta|$, and for $a_{k-1}<s\le a_k$ denote by U_s the s-dimensional subspace of U generated by the rightmost cell vectors $f^{j_k-1}(x_i)$ for $\alpha'_{j_l}<i\le i_l$ and $1\le l\le k-1$ together with $f^{j_k-1}(x_i)$ for $\alpha'_{j_k}<i\le\alpha'_{j_k}+(s-a_{k-1})$. Let $K(b_m,a_m)=\{\ k=(1\le k_1<\cdots<k_{b_m}\le a_m)\ \}$ be the set of b_m-subsets choosing from the a_m-set $\{1,2,\cdots,a_m\}$ with the partial order defined by $k\le l$ iff $k_s\le l_s$ for all $1\le s\le b_m$. The Schubert cell and the Schubert variety assosiated to $k\in K(b_m,a_m)$ are defined by

$$C(k)=\{\ W\in G(b_m,U)\ ;\ \dim W\cap U_s=t \text{ for } k_t\le s<k_{t+1},\ 1\le t<b_m\ \}$$
$$X(k)=\bar{C}(k)=\{\ W\in G(b_m,U)\ ;\ \dim W\cap U_{k_t}\ge t \text{ for } 1\le t\le b_m\ \}$$
$$=\coprod_l C(l),\quad l\in K(b_m,a_m),\ l\le k$$

with the associated partition $\zeta=(k_{b_m}-b_m,\cdots,k_1-1)$ [G-L,p93]. The dimension of $C(k)$ is equal to $|\zeta|=\sum_{t=1}^{b_m}(k_t-t)$ [ibid,p94]. Now $H(\alpha,\beta,\lambda)$ is written by $H(\alpha,\beta,\lambda)=\{\ W\subset U\ ;\ \dim W\cap U_{a_k}=$

b_k for $1 \leq k \leq m$ }. Thus $H(\alpha,\beta,\lambda)$ is a union of the Schubert cells ; $H(\alpha,\beta,\lambda) = \coprod_k C(k)$ where $k \in K(b_m, a_m)$ satisfies

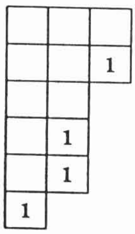
$$k_{b_1} \leq a_1 < k_{b_1+1},\ k_{b_2} \leq a_2 < k_{b_2+1},\ \cdots,\ k_{b_{m-1}} \leq a_{m-1} < k_{b_{m-1}+1} \quad (*)$$

Let $k_0 := (\ a_1 - b_1 + 1, \cdots, a_1\ ;\ a_2 - (b_2 - b_1) + 1, \cdots, a_2\ ; \cdots ;\ a_m - (b_m - b_{m-1}) + 1, \cdots, a_m\ ) \in K(b_m, a_m)$. Then k_0 is greater than any other elements of $K(b_m, a_m)$ satisfying (*) with respset to the partial order defined above. This implies that $H(\alpha,\beta,\lambda)$ is irreducible and its Zariski closure is the Schubert variety associated to k_0 ; $\bar{H}(\alpha,\beta,\lambda) = X(k_0) = \coprod_k C(k)$, $k \in K(b_m, a_m)$ with $k \leq k_0$. □

The Schubert variety $\bar{C}(k_0)$ is nonsingular iff the associatd diagram is rectangle [G-L,p156], i.e. $a_1 - b_1 = a_2 = b_2 = \cdots = a_m - b_m$. It follows from this

Corollary 11. *The closure of $H(\alpha,\nu,\mu)$ is nonsingular iff $\alpha'_{j_k} = \beta'_{j_k}$ for all $2 \leq k \leq m$.*

Example 12. $\lambda = (332221) = [256;321] \supset \beta = (32211) \supset \alpha = \lambda - (1^6) = (22111)$.



Then we see that $(a_1, a_2, a_3) = (256)$ and $(b_1, b_2, b_3) = (134)$ so $k = (k_1, k_2, k_3, k_4) \in K(4,6)$ must satisfy $k_1 \leq 2 < k_2$ and $k_3 \leq 5 < k_4$. Hence $H(\alpha,\beta,\lambda) = S(\lambda,(1^4),\beta) = \coprod_k C(k)$ with $k = (2456)$, (2356), (2346), (1456), (1356), (1346) and

$$\begin{aligned} X(\lambda,(1^4),\beta) = \bar{C}(2456) &= \coprod_l C(l), \qquad l \leq (2456) \\ &= S(\lambda,(1^4),(32211)) \coprod S(\lambda,(1^4),(321111)) \\ &\quad \coprod S(\lambda,(1^4),(22221)) \coprod S(\lambda,(1^4),(222111)) \end{aligned}$$

where $S(\lambda,(1^4),(321111)) = C(2345) \coprod C(1345)$

$$S(\lambda,(1^4),(22221)) = C(1256) \coprod C(1246) \coprod C(1245) \coprod C(1236)$$

$$S(\lambda,(1^4),(222111)) = C(1235) \coprod C(1234)$$

We see that $\bar{C}(2456) \supset \bar{C}(2345) \supset \bar{C}(1235)$ and $\bar{C}(2456) \supset \bar{C}(1256) \supset \bar{C}(1235)$ with the associatd partitions $(1222) \supset (1111) \supset (0001)$ and $(1222) \supset (0022) \supset (0001)$. The diagram of the singular loci of $\bar{C}(2456)$ is obtained by deleting the hook from the diagram (2221) of $\bar{C}(2456)$ [G-L,p156], i.e. (22). Hence we see Sing $X(\lambda, (1^4), \mu) = \bar{C}(1256) = X(\lambda, (1^4), (22221))$. □

We investigate the structure of $H(\alpha, \beta, \lambda)$ more closely. Let $A(U, V) = \{g \in \mathrm{GL}(V); f \circ g = g \circ f,\ g(U) = U\}$ be the automorphism group (V, U, f). Recall Lemma 5 that the automorphism group $A(V)$ is a semidirect product of a unipotent group and $\prod_{k=1}^{m} \mathrm{GL}(Q_k)$ where $Q_k = \mathrm{Ker}\, f^{j_k} / \langle\ \mathrm{Ker}\, f^{j_k - 1},\ \mathrm{Ker}\, f^{j_k} \cap f(V)\ \rangle$ isomprphic to the subspace V_k of dimension $i_k - i_{k-1}$ generated by the leftmost cell vectors x_i for $i_{k-1} < i \le i_k$. If we denote by V'_k (resp. Q'_k) the subspace of V_k (resp. Q_k) generated by the cell vectors (resp. the residue classes of cell vectors) x_i for $\alpha'_{j_k} < i \le i_k$ then $V'_k \cong Q'_k$ is isomorphic to $(U \cap f^{j_k - 1}(V))/(U \cap f^{j_k}(V))$. We see that $g \in A(U, V)$ induces an automorphism of Q'_k and the canonical map $A(U, V) \to \prod_{k=1}^{m} \mathrm{GL}(Q_k) \cong \prod_{k=1}^{m} \mathrm{GL}(V'_k)$ is well-defined and split surjective. Let

$$V^{(j)} = \langle\ f^{\lambda_i - j}(x_i)\ ;\ 1 \le i \le \lambda'_1\ \rangle \cong \mathrm{Ker}\, f^j / \mathrm{Ker}\, f^{j-1}$$
$$V^{[j]} = \langle\ f^{j-1}(x_i)\ ;\ 1 \le i \le \lambda'_1\ \rangle\ \cong f^{j-1}(V)/f^j(V)$$

be the subspaces of dimension λ'_j generated by the cell vectors of the j-th columns of λ from right and left, respectively. Denote the projections by $\pi_j : V \to V^{(j)}$, $\pi_j(x) = x^{(j)}$ and $\tau_j : V \to V^{[j]}$, $\tau_j(x) = x^{[j]}$. Then there are canonical maps

$$f : V^{(j)} \hookrightarrow V^{(j-1)}, \qquad f(x)^{(j-1)} = f(x^{(j)})$$
$$f : V^{[j-1]} \twoheadrightarrow V^{[j]}, \qquad f(x)^{[j]} = f(x^{[j-1]})$$

Definition 13. *The right (resp. left) degree of a nonzero vector $x \in V$ is defined by $R(x) = j$ (resp. $L(x) = j$) if $x \in \mathrm{Ker}\, f^j$ and $x \not\subset \mathrm{Ker}\, f^{j-1}$ (resp. $x \in f^{j-1}(V)$ and $x \not\subset f^j(V)$).*

Then $R(f^k(x)) = R(x) - k$ for $0 \le k \le R(x)$ and $L(f^{k-1}(x)) = L(x) + k$ for $0 \le k \le R(x^{[L(x)]})$ and a vector $x \in V$ is written by

$$x = x^{(R(x))} + x^{(R(x)-1)} + \cdots + x^{(2)} + x^{(1)}, \qquad x^{(j)} \in V^{(j)}$$
$$x = x^{[L(x)]} + x^{[L(x)+1]} + \cdots + x^{[\lambda_1 - 1]} + x^{[\lambda_1]}, \qquad x^{[j]} \in V^{[j]}$$

Denote by $\lambda = [i_1, \cdots, i_m \mid j_1, \cdots, j_m]$ (Notation (1)).

Lemma 14. *$H(\alpha,\beta,\lambda)$ is an $A(U,V)$-homogeneous space $H(\alpha,\beta,\lambda) = A(U,V)/G$, which fits in the commutative diagram with exact rows*

$$\begin{array}{ccccc}
1 \to N \cap G & \longrightarrow & G & \longrightarrow & P \to 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 \to N & \longrightarrow & A(U,V) & \longrightarrow & \prod_{k=1}^{m} GL(Q'_k) \to 1 \\
\downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow \\
N/N \cap G & \longrightarrow & H(\alpha,\beta,\lambda) & \xrightarrow{\ \pi\ } & \prod_{k=1}^{m} G(\lambda'_{j_k} - \alpha'_{j_k}, \lambda'_{j_k} - \beta'_{j_k})
\end{array}$$

where G is the stabilaizer at an element W of $H(\alpha,\beta,\lambda)$, τ is the orbit map and π is a vector bundle in the Zariski topology of rank equal to $\dim N/N \cap G = \sum_{k>s}(\lambda'_{j_k} - \beta'_{j_k})(\beta'_{j_s} - \alpha'_{j_s})$.

Proof. For a subspace $W \in H(\alpha,\beta,\lambda)$ we see

$$\dim \frac{W \cap f^{j-1}(V)}{W \cap f^{j}(V)} = \begin{cases} \lambda'_{j_k} - \beta'_{j_k} & \text{for} \quad j = j_k,\ 1 \leq k \leq m \\ 0 & \text{otherwise} \end{cases}$$

so we can choose a basis $\{g_{k,l}\}$ of W as $L(g_{k,l}) = j_k$ for $1 \leq k \leq m$ and $1 \leq l \leq \lambda'_{j_k} - \beta'_{j_k}$. Since $g_{k,l}^{[j_k]} \in U$ the element $g_{k,l} - g_{k,l}^{[j_k]}$ is contained in $U \cap f^{j_k}(V)$ so $\{g_{k,l}^{[j_k]} = g_{j,k} - (g_{j,k} - g_{k,l}^{[j_k]})\ ;\ 1 \leq l \leq \lambda'_{j_k} - \beta'_{j_k}\}$ generates the $\lambda'_{j_k} - \beta'_{j_k}$-dimensional subspace

$$\frac{W \cap f^{j_k-1}(V)}{W \cap f^{j_k}(V)} \cong \frac{\langle W \cap f^{j_k-1}(V), U \cap f^{j_k}(V)\rangle}{U \cap f^{j_k}(V)}$$

of $(U \cap f^{j_k-1}(V))/(U \cap f^{j_k}(V)) \cong \mathbb{C}^{j'_k - \alpha'_{j_k}}$. If we take another vector $g \in W$ with $L(g) = j_k$ and $g^{[j_k]} = g_{k,l}^{[j_k]}$ then $g - g_{k,l} = (g - g^{[j_k]}) - (g_{j,k} - g_{j,k}^{[j_k]})$ is contained in $W \cap f^{j_k}(V)$. Therefore the map

$$\pi : H(\alpha,\beta,\lambda) \to \prod_{k=1}^{m} G(\lambda'_{j_k} - \beta'_{j_k}, \frac{U \cap f^{j_k-1}(V)}{U \cap f^{j_k}(V)})$$

$$\pi(W) = (\frac{W \cap f^{j_k-1}(V)}{W \cap f^{j_k}(V)})_{1 \leq k \leq m}$$

is a vector bundle with the fibre $\pi^{-1}\pi(W)$ isomorphic to $\oplus_{k=1}^{m}\oplus^{\lambda'_{j_k}-\beta'_{j_k}}$ $(U\cap f^{j_k-1}(V)/(W\cap f^{j_k}(V))$ whose dimension N is equal to

$$\begin{aligned}&\sum_{k=1}^{m}(\lambda'_{j_k}-\beta'_{j_k})\dim\frac{U\cap f^{j_k}(V)}{W\cap f^{j_k}(V)}\\&=\sum_{k=1}^{m}(\lambda'_{j_k}-\beta'_{j_k})\cdot\sum_{s=1}^{k-1}\{(\lambda'_{j_s}-\alpha'_{j_s})-(\lambda'_{j_s}-\beta'_{j_s})\}\\&=\sum_{k>s}(\lambda'_{j_k}-\beta'_{j_k})(\beta'_{j_s}-\alpha'_{j_s})\end{aligned}$$

Denote the rightmost cell vectors of λ by $y_t=f^{j_k-1}(x_t)$ for $1\le t\le\lambda'_1$. Let $W_0\in H(\alpha,\beta,\lambda)$ be the subspace of V generated by the cell vectors of λ/β, i.e. by $y_t=f^{j_k-1}(x_t)$ for $\beta'_{j_k}<t\le\lambda'_{j_k}$ and $1\le k\le m$. Then the above discussion shows that for any $W\in H(\alpha,\beta,\lambda)$ there are constants $a_{t,u}$ for $\beta'_{j_k}<t\le\lambda'_{j_k}$ and $\alpha'_{j_s}<u\le\beta'_{j_s}$ and $g_k\in\mathrm{GL}(f^{j_k-1}(V_k))$ with $g_k(f^{j_k-1}(V_k)\cap U)=f^{j_k-1}(V_k)\cap U$, such that W is generated by the elements

$$z_t=g_k(y_t)+\sum_{s=1}^{k-1}\sum_{u=\alpha'_{j_s}+1}^{\beta'_{j_s}}a_{t,u}g_s(y_u)$$

for $\beta'_{j_k}<t\le\lambda'_{j_k}$ and $1\le k\le m$. By the inclusion $\mathrm{GL}(V_k)\cong\mathrm{GL}(Q_k)\subset A(V)$ there is a unique $\varphi_k\in A(U,V)$ such that $\varphi_k|_{V_k}=g_k$ and $\varphi_k|_{V_l}=id$ for $k\ne l$. If the automorphism $\varphi\in A(U,V)$ is defined by $\varphi(x_t)$ equal to

$$\begin{aligned}&\varphi_k(x_t) && \text{for } \lambda'_{j_k+1}<t\le\beta'_{j_k},\ 1\le k\le m\\&\varphi_k(x_t)+\sum_{s=1}^{k-1}\sum_{u=\alpha'_{j_s}+1}^{\beta'_{j_s}}a_{t,u}f^{j_s-j_k}(\varphi_s(x_u)) && \\& && \text{for } \beta'_{j_k}<t\le\lambda'_{j_k},\ 1\le k\le m\end{aligned}$$

then $\varphi(y_t)=z_t$ and $\varphi(W_0)=W$ so $H(\alpha,\beta,\lambda)$ is a homogeneous space with respect to $A(U,V)$. The stabilizer G at W_0 in $A(U,V)$ is a semidirect product of a unipotent group and a parabolic subgroup P of $\prod_{k=1}^{m}$ $\mathrm{GL}(Q'_k)$. The projection $A(U,V)\to\prod\mathrm{GL}(Q'_k)$ induces the commutative diagram

$$\begin{array}{ccc}A(U,V) & \longrightarrow & \prod_{k=1}^{m}\mathrm{GL}(Q'_k)\\ \downarrow & & \downarrow\\ H(\alpha,\beta,\lambda) & \xrightarrow{\ \pi\ } & \prod_{k=1}^{m}G(\lambda'_{j_k}-\alpha'_{j_k},\lambda'_{j_k}-\beta'_{j_k})\end{array}$$

where the vertical maps are the orbit maps at W_0 and $\pi(W_0)$, respectively. $\square$

In the case $\alpha = \tilde{\lambda} = \lambda - (111\cdots)$, i.e. $U = \text{Ker } f$ let t_i $(1 \leq i \leq d := |\lambda/\beta|)$ be the cell of the vertical strip λ/β which is in the l_i-th row with $l_1 < \cdots < l_d$. If $\beta'_{j_k} < l_i \leq \lambda'_{j_k} = i_k$ then the number of the rows not containing the cells of λ/β is equal to $\sum_{s=1}^{k}(\beta'_{j_s} - \lambda'_{j_{s-1}}) = l_i - i$ (cf.Definition 18), from which we have the identity

$$\begin{aligned}\sum_{k\geq s}(\lambda'_{j_k} - \beta'_{j_k})(\beta'_{j_s} - \lambda'_{j_{s-1}}) &= \sum_{k=1}^{m}(\lambda'_{j_k} - \beta'_{j_s})\sum_{s=1}^{k}(\beta'_{j_s} - \lambda'_{j_{s-1}}) \\ &= \sum_{k=1}^{m}(\lambda'_{j_k} - \beta'_{j_k})\cdot(l_i - i) = \sum_{i=1}^{d}(l_i - i) \\ &= n(\lambda) - n(\beta) - \binom{d}{2}\end{aligned}$$

Since $A(\text{Ker } f, V) = A(V)$ because Ker f is $A(V)$-stable we see

Corollary 15. *For partitions $\lambda \supset \beta \supset \tilde{\lambda} = \lambda - (111\cdots)$ the set $H(\tilde{\lambda}, \beta, \lambda) = \{W \; ; \; V \supset \text{Ker} f \supset W, \; \text{type } V/W = \beta\ \}$ is an $A(V)$-homogeneous rational variety of dimension $n(\lambda) - n(\beta) - \binom{|\lambda/\beta|}{2}$.*

Let $T = (\mu \subset \mu^{(1)} \subset \mu^{(2)} \subset \cdots \subset \mu^{(r-1)} \subset \mu^{(r)} = \lambda)$ be an *LR*-tabluaux of shape λ/μ and content ν and set $S_T = \{W \in S(\lambda, \nu, \mu) \; ; \; LR(W, V) = T\}$. Let W_T be an element of S_T. We show

Theorem A. *(i) S_T is birationally isomorphic to the product $S_1 \times S_2 \times \cdots \times S_r$ where $S_k = S(V/f^k(W_T); \mu^{(k)}, (1^{\nu'_k}), \mu^{(k-1)})$, i.e.*

$$S_k = \{U \subset V/f^k(W_T) \; ; \; f(U) = 0, \; \textit{type } (V/f^k(W_T))/U = \mu^{(k-1)}\ \}$$

(ii) S_T is a nonsingular rational variety of dimension $n(\lambda) - n(\mu) - n(\nu)$.

Proof. Except nonsingularity, (ii) follows from (i) and Cor.15. We shall show (i) and the nonsingularity of S_T by the induction on $r := \nu_1$. If $r = 1$ then $\nu = (1^{\nu'_1})$ and $S_T = S(\lambda, (1^{\nu'_1}), \mu) = \{W \; ; \; V \supset \text{Ker } f \supset W, \text{ type } V/W = \mu\}$, which is nonsingular rational of dimension $n(\mu^{(1)}) - n(\mu) - \binom{\nu'_1}{2}$ by Cor.15. Suppose $r > 1$. Then type $f^{r-1}(W_T) = (1^{\nu'_r})$ and consider the map

$$\pi : S_T \to S_r = S(V; \lambda, (1^{\nu'_r}), \mu^{(r-1)}), \quad \pi(W) = f^{r-1}(W)$$

which we see is a surjective morphism. Let $U \in S_r = S(\lambda, (1^{\nu'_r}), \mu^{(r-1)})$ be the subspace generated by the cell vectors of $\lambda/\mu^{(r-1)}$. Then $\pi^{-1}(U) = \{W \in S_T \ ; \ f^{r-1}(W) = U\}$ is a subset of

$$\{W \in X(V,d) \ ; \ W \supset U, \text{ type } V/\langle f^k(W), U\rangle = \mu^{(k)} \text{ for } 0 \leq k \leq r-1\}$$
$$\cong \{\bar{W} \in X(V/U, d-\nu'_1) \ ; \ LR(\bar{W}, V/U) = T_{<r} \} = S_{T_{<r}}(V/U) \qquad (*)$$

where $T_{<r} = (\mu \subset \mu^{(1)} \subset \cdots \subset \mu^{(r-1)})$. Since type $V/\langle f^{r-1}(W), U\rangle = \mu^{(r-1)} =$ type V/U implies $f^{r-1}(W) \subset U$ we see that W in (*) is contained in $\pi^{-1}(U)$ iff $U = f^{r-1}(W)$, i.e. $\dim f^{r-1}(W) = \dim U = \nu'_r$. Hence $\pi^{-1}(U)$ is a Zariski openset of (*) isomorphic to $S_{T_{<r}}(V/U)$, which is nonsingular by the inductive hypothesis, so $\pi^{-1}(U)$ is nonsingular. Since $\pi : S_T \to S_r = S(\lambda, (1^{\nu'_r}), \mu^{(r-1)})$ is $A(V)$-equivariant surjective and $S(\lambda, (1^{\nu'_r}), \mu^{(r-1)})$ is $A(V)$-homogeneous by Cor.15 we see that all of the fibres of π are nonsingular, which implies that S_T is nonsingular. Since $S(\lambda, (1^{\nu'_r}), \mu^{(r-1)})$ is a Schubert cell by Lemma 10 there is a subgroup G' of $A(V)$ such that the orbit map $G' \to S(\lambda, (1^{\nu'_r}), \mu^{(r-1)})$ at U is an open immersion. We see from this that the product map $G' \times \pi^{-1}(U) \to S_T$ is an open immersion so S_T is birationally isomorphic to the product $G' \times \pi^{-1}(U) \simeq S(\lambda, (1^{\nu'_r}), \mu^{(r-1)}) \times S_{T_{<r}}$. Then (i) follows from the inductive hypothesis on $S_{T_{<r}}$. □

3. Generic vectors for an LR-tableau

In this section we define the generic vectors for an LR-tableau T and give some examples. We begin with a remark on a linear isomorphism of the vector space V with a basis consisting of the cell vectors $f^{j-1}(x_i)$ of the cells of λ for $1 \leq i \leq \lambda'_1$ and $1 \leq j \leq \lambda_i$. Let y_i be a vector contained in the subspace $\langle x_1, \cdots, x_{i-1}, f(V)\rangle$ for $1 \leq i \leq \lambda'_1$. We define a linear map $g : V \to V$ by $g(f^k(x_i)) = f^k(g(x_i)) = f^k(x_i) + f^k(y_i)$ for $1 \leq i \leq \lambda'_1$ and $0 \leq k \leq \lambda_i - 1$. The representation matrix with respect to the basis of cell vectors with the Semitic order (Definition 1) is unitriangular so g is an isomorphism.

Definition 16. *We call the isomorphism g of V above the canonical isomorphism determined by the vectors $y_1, \cdots, y_{\lambda'_1}$.*

In general g is not an automorphism of (V, f), e.g. if $\lambda = (21)$ and $y_2 = x_1$ then $fg(x_2) = f(x_2 + y_2) = f(x_1)$ while $gf(x_2) = g(0) = 0$. Let $T = (\mu \subset \mu^{(1)} \subset \cdots \subset \mu^{(\nu_1)} = \lambda)$ be an LR-tableau of shape λ/μ and content ν. We define the parameter cells and vectors, and the weight of the cells of T as follows.

Definition 17. *Let t_{ik} $(1 \le i \le \nu'_k)$ be the cell of the vertical strip $T_{=k} = \mu^{(k)}/\mu^{(k-1)}$ which is in the l_{ik}-th row with $l_{1k} < \cdots < l_{\nu'_1,k}$. Then the parameter cells and vectors of the cell t_{ik} are defined by the cells $(j, \mu_j^{(k)} - k + 1)$ and by the corresponding cell vectors $f^{\mu_j^{(k)}-k}(x_j)$ for $1 \le j \le l_{ik}$ and $j \ne l_{1,k}, \cdots, l_{i-1,k}$, respectively. The weight $w(t_{ik})$ of the cell t_{ik} is defined by the number of the parameter cells of t_{ik}, i.e. $l_{ik} - i$.*

The sum of the weights for all of the cells of T is equal to the dimension of S_T because

$$\begin{aligned}\sum_{k=1}^{\nu_1}\sum_{i=1}^{\nu'_k} w(t_{ik}) &= \sum_{k=1}^{\nu_1}\{\sum_{i=1}^{\nu'_k}(l_{ik}-i)\} \\ &= \sum_{k=1}^{\nu_1}\{n(\mu^{(k)}/\mu^{(k-1)}) - n(1^{\nu'_k})\} \\ &= n(\lambda) - n(\mu) - n(\nu)\end{aligned}$$

Now we construct vectors v_i $(1 \le i \le \nu'_1)$ for each cell of $T_{=1} = \mu^{(1)}/\mu$, called the generic vectors for T (Definition 18), such that the f-stable subspace generated by $\{v_1, \cdots, v_{\nu'_1}\}$ is a subspace which defines a generic point of S_T, by the induction on ν_1 as follows.

(i) If $\nu_1 = 1$ then T is a vertical strip of shape λ/μ filled with the letter 1. We asssociate to each cell t_i of T in the l_i-th row the vector v_i containing $w(t_i)$ parameters in the coefficients of the parameter vectors of t_i and the coefficient of the cell vector $f^{\lambda_{l_i}-1}(x_{l_i})$ of the cell t_i equal to 1 ; $v_i = f^{\lambda_{l_i}-1}(x_{l_i}) + \sum_j a_{ij} f^{\lambda_j - 1}(x_j)$ where $1 \le j < l_i$ and $j \ne l_1, \cdots, l_{i-1}$, e.g. if $\lambda = (4322)$, $\nu = (11)$, $\mu = (4221)$ and

$T =$

		1	
	1		

then $v_1 =$

0	0	0	a
0	0	1	
0	0		
0	0		

, $v_2 =$

0	0	0	c
0	0	0	
0	b		
0	1		

If $g_1 \in \mathrm{GL}(V)$ is the canonical isomorphism of V (Definition 16) determined by the vectors $y_{l_i} = \sum_j a_{ij} f^{\lambda_j - \lambda_{l_i}}(x_j)$ for $1 \le j < l_i$ and $j \ne l_1, \cdots, l_{i-1}$, and $y_k = 0$ for $k \ne l_1, \cdots, l_{\nu'_1}$ then we see $v_i = g_1(x_{l_i})$ for $1 \le i \le \nu'_1$. We see that the subspaces generated by $\{v_1, \cdots, v_{\nu'_{\nu_1}}\}$ is nothing but a Schubert cell which is an openset of S_T with the affine coordinates $\{a_{ij}\}$ (cf. Lemma 10).

(ii) Suppose $\nu_1 > 1$ and assume that the generic vectors for $T_{<\nu_1}$ have been defined by the induction. Let $t_i = t_{i,\nu_1}$ $(1 \leq i \leq \nu'_{\nu_1})$ be the cell of the vertical strip $T_{=\nu_1} = \lambda/\mu^{(\nu_1-1)}$ which is in the position (l_i, λ_{l_i}) with $l_1 < \cdots < l_{\nu'_{\nu_1}}$. We consider an element $g_{\nu_1} \in \mathrm{GL}(V)$ so as to transform the cell vector $f^{\lambda_{l_i}-1}(x_{l_i})$ of t_i to the generic vector $v_i \in \mathrm{Ker}\, f$ for $T_{=\nu_1}$ defined as in (i). As such a $g_{\nu_1} \in \mathrm{GL}(V)$ we choose the canonical isomorphism of V (Definition 16) determined by the vectors $y_{l_i} = \sum_j a_{ij} f^{\lambda_j - \lambda_{l_i}}(x_j)$ for $1 \leq j < l_i$ and $j \neq l_1, \cdots, l_{i-1}$, i.e. $f^{\lambda_j - \nu_1}(x_j)$ are the parameter vectors of the cell t_i, and $y_k = 0$ for $k \neq l_1, \cdots, l_{\nu'_{\nu_1}}$. Here the $\sum_{i=1}^{\nu'_{\nu_1}} w(t_{i,\nu_1})$ parameters a_{ij} are algebraically independent. The generic vectors for T is obtained by applying g_{ν_1} to the generic vectors for $T_{<\nu_1}$ constructed by the induction.

Now we define

Definition 18. *The generic vectors for T are the ν'_1 vectors in the above construction corresponding to the cells of T filled with the least letter.*

Example 19. For $\lambda = (5332)$, $\nu = (331)$ and $\mu = (321)$ we consider

$$T = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 3 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array}, \quad T_{\leq 2} = \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 1 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} \end{array}, \quad T_{=1} = \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 1 \\ \hline \ & \ \\ \cline{1-2} \ & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

The generic vectors for $T_{=1}$ are given by

$$v_1 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & x & * \\ \cline{1-3} 0 & 0 \\ \cline{1-2} 1 & x \\ \cline{1-2} \end{array}, \quad v_2 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & y & * \\ \cline{1-3} 0 & 1 \\ \cline{1-2} 0 & * \\ \cline{1-2} \end{array}, \quad v_3 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & * \\ \cline{1-3} 0 & 0 \\ \cline{1-2} 0 & * \\ \cline{1-2} \end{array}$$

In order to construct the generic vectors for $T_{\leq 2}$ we consider $g_2 \in \mathrm{GL}(V)$ determined by the vectors $y_1 = 0$, $y_2 = x_1$, $y_3 = 0$ and $y_4 = x_3$, i.e.

$$g_2(x_2) = \begin{array}{|c|c|c|c|} \hline 0 & a & 0 & 0 \\ \hline 1 & 0 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} 0 & 0 \\ \cline{1-2} \end{array}, \quad g_2(x_4) = \begin{array}{|c|c|c|c|} \hline 0 & c & 0 & 0 \\ \hline 0 & 0 & 0 \\ \cline{1-3} b & 0 \\ \cline{1-2} 1 & 0 \\ \cline{1-2} \end{array}$$

and $g_2(x_i) = x_i$ for $i = 1, 3$. The generic vectors for $T_{\leq 2}$ are obtained by applying g_2 to the generic vectors for $T_{=1}$:

$$g_2(v_1) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & ax+c & 0 \\ \hline 0 & x & 0 \\ \cline{1-3} b & 0 \\ \cline{1-2} 1 & 0 \\ \cline{1-2} \end{array}, \quad g_2(v_2) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & ay & 0 \\ \hline 0 & y & 0 \\ \cline{1-3} 0 & 1 \\ \cline{1-2} 0 & 0 \\ \cline{1-2} \end{array}, \quad g_2(v_3) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} 0 & 0 \\ \cline{1-2} \end{array}$$

Lastly consider $g_3 \in \mathrm{GL}(V)$ determined by the vectors $y_3 = x_2$ and $y_i = 0$ for $i = 1,2,4$, i.e. $g_3(x_3) = x_3 + px_2$ and $g_3(x_i) = x_i$ for $i = 1,2,4$. The generic vectors for T are obtained by applying g_3 to the one for $T_{\geq 2}$ constructed above :

$$v_{41} = g_3g_2(v_1) = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & ax+c & 0 & 0 \\ \hline bp & x & 0 \\ \cline{1-3} b & 0 & 0 \\ \cline{1-3} 1 & 0 \\ \cline{1-2} \end{array} \quad , \quad v_{32} = g_3g_2(v_2) = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & ay & 0 & 0 \\ \hline 0 & y+p & 0 \\ \cline{1-3} 0 & 1 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array}$$

and $v_{14} = g_3g_2(v_3) = f^3(x_1)$. Hence the images of these three vectors v_{41}, v_{32}, v_{14} under f are represented by

$$f(v_{41}) = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & c & 0 \\ \hline 0 & bp & 0 \\ \cline{1-3} 0 & b & 0 \\ \cline{1-3} 0 & 1 \\ \cline{1-2} \end{array} + x \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & a & 0 \\ \hline 0 & 0 & 1 \\ \cline{1-3} 0 & 0 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array} := v_{42} + x \cdot v_{23}$$

$$f(v_{32}) = y \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & a & 0 \\ \hline 0 & 0 & 1 \\ \cline{1-3} 0 & 0 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & p \\ \cline{1-3} 0 & 0 & 1 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array} := y \cdot v_{23} + v_{33}$$

$$f(v_{14}) = f^4(x_1) := v_{15}$$

$$f^2(v_{41}) = b \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & p \\ \cline{1-3} 0 & 0 & 1 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array} + (ax+c) \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \cline{1-3} 0 & 0 & 0 \\ \cline{1-3} 0 & 0 \\ \cline{1-2} \end{array}$$

$$:= b \cdot v_{33} + (ax+c) \cdot v_{15}$$

$$f^2(v_{32}) = ay \cdot v_{15}, \qquad f^2(v_{14}) = 0$$

Here $\{v_{42}, v_{23}\}$ and $\{v_{33}, v_{15}\}$ are the generic vectors for

$$T_{\geq 2} = \begin{array}{|c|c|c|c|c|} \hline \; & \; & \; & \; & 3 \\ \hline & & 2 \\ \cline{1-3} & & 3 \\ \cline{1-3} & 2 \\ \cline{1-2} \end{array} \quad \text{and} \quad T_{\geq 3} = \begin{array}{|c|c|c|c|c|} \hline \; & \; & \; & \; & 3 \\ \hline & & \\ \cline{1-3} & & 3 \\ \cline{1-3} & \\ \cline{1-2} \end{array}$$

respectively. □

The generic vectors $v_1, \cdots, v_{\nu'_1}$ contain the $\dim S_T$ parameters a_{ji} $(1 \leq i \leq n(\mu^{(k)}/\mu^{(k-1)}) - n(1^{\nu'_k}))$ corresponding to the parameter cells of the cells of $T_{=j}$ for $1 \leq j \leq \nu'_1$. Let W be the f-stable subspace of $V \otimes \mathbb{C}(a_{ji})$ generated by the generic vectors for T over the rational function field $\mathbb{C}(a_{ji})$ of $\dim S_T$-variables a_{ji}'s. As is observed in Example 19 we can show

Theorem B. *(i) For $1 \leq k \leq \nu_1$ the residure classes modulo $f^k(W)$ of the generic vectors for $T_{\geq k}$ is a basis of $f^{k-1}(W)/f^k(W)$. Hence the generic vectors for $T, T_{\geq 2}, \cdots, T_{\geq \nu_1}$ form a basis of W. (ii) The morphism from the affine space of dimension $N = \dim S_T$ with the affine coordinates being the parameters of the generic vectors for T to X_T ; $\varphi : \mathbb{A}^N \to X_T$, which associates a subspace of V to its Plücker coordinate, is a birational morphism, i.e. the Plücker coordinate of W is a generic point of S_T over $\mathbb{C}$.*

Proof. (i) We shall show by the induction on ν_1. If $\nu_1 = 1$ then the assertion holds as in the construction (i) preceding Definition 19. Suppose $\nu_1 > 1$ and assume that the residue classes modulo $f^k(W)$ of the generic vectors of $T_{\geq k}$ is a bsais of $f^{k-1}(W)/f^k(W)$ for $2 \leq k \leq \nu_1$, and the generic vectors for $T_{\geq 2}, \cdots, T_{\geq \nu_1}$ is a basis of $f(W)$. We shall show that the residue classes modulo $f(W)$ of the generic vectors for T is a basis of $W/f(W)$. If the cells of $T_{=1}$ are in the $l_1, \cdots, l_{\nu'_1}$-th rows then the generic vectors $v_1, \cdots, v_{\nu'_1}$ for T are represented as $v_i = g_{\nu_1} \cdots g_1(x_{l_i})$ by the construction (ii) preceding Definition 19. If the generic vector v_i has a nonzero coefficient in the cell vector of a rightmost cell of the k-th row of λ then there are no cells in the k-th row of T, which implies that $f(v_i) = f g_{\nu_1} \cdots g_1(x_{l_i}) = g_{\nu_1} \cdots g_2 f g_1(x_{l_i})$. Now $f g_1(x_{l_i})$ is a linear combination of the cell vectors of the cells of $T_{>1}$ with the coefficients $\{a_{1,i}\}$ corresponding to the parameter cells of the cells of $T_{=1}$. Hence $f(v_i) = g_{\nu_1} \cdots g_2 f g_1(x_{l_i})$ is a linear combination of the generic vectors for $T_{\geq 2}, \cdots, T_{\geq \nu_1}$ by the inductive hypothesis for all $1 \leq i \leq \nu'_1$. The generic vector v_i of the cell t_i has coefficient equal to 1 in the cell vector of the cell t_i so the residue classes of $v_1, \cdots, v_{\nu'_1}$ form a basis of $W/f(W)$. (ii) follows from (i) and the fact that the generic point $W/f(W)$ of $X_{T_{=1}}$ is the image under $g_{\nu_1} \cdots g_2$ of the Schubert cell of $S_{T_{=1}}$ with the affine coordinates being the parameter $\{a_{1,i}\}$ of the parameter cells of the cells of $T_{=1}$. □

We give the simplest examples that
(i) the shape (resp. the content) of $\bar{T}$ is the same as (resp. less than)

that of T but $X_{\bar{T}}$ is not contained in X_T (Example 20),

(ii) the singular locus of X_T is not equal to the union of $S_{\bar{T}}$'s for some LR-tableaux $\bar{T}$ (Example 21)

Example 20. Let $\lambda = (3211)$, $\nu = (22)$, $\bar{\nu} = (211)$ and $\mu = (21)$. Then $\nu \rhd \bar{\nu}$ and the LR-tableaux of shape λ/μ and content ν (resp. $\bar{\nu}$) are

$$T = \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 2 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad (\text{resp. } T_1 = \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \ )$$

The generic vectors for T, T_1 and T_2 are, respectively

$$\begin{array}{|c|c|c|}\hline 0 & b & 0 \\ \hline a & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & d & 0 \\ \hline c & 0 \\ \cline{1-2} 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & x & 0 \\ \hline 0 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & y & 0 \\ \hline 0 & 0 \\ \cline{1-2} 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & z & 0 \\ \hline 0 & 1 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & a\,x & 0 \\ \hline a & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & b\,x & 0 \\ \hline b & 0 \\ \cline{1-2} 1 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 0 & x \\ \hline 0 & 1 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1}\end{array}$$

If $z \neq 0$ then the Plücker coordinate $x_4 \wedge x_3 \wedge f(x_1) \wedge f^2(x_1)$ of W_1 is nonzero where x_i are the cell vectors of λ (Notation (2)), so W_1 is contained in the closure X_T of S_T iff $z = 0$, hence S_{T_1} is not contained in X_T. On the other hand the set S_{T_2} is contained in X_T because W_2 is defined by the condition $ad = bc$ in the parametrization of W. We see that X_T is isomorphic to $G(2,4)$, the grassmaniann of subspaces of $\mathbb{C}^4$ of dimension two so is nonsingular and $X_{T_1} \cap X_{T_2} \cong \mathbb{P}^2$. □

Example 21. Let $\lambda = (3221)$, $\nu = (22) \rhd \bar{\nu} = (211)$ and $\mu = (211)$. Then the LR-tableaux of shape λ/μ and content ν (resp. $\bar{\nu}$) are

$$T = \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 \\ \cline{1-2} \ & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad (\text{ resp. } T_1 = \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 \\ \cline{1-2} \ & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \quad \textit{and} \quad T_2 = \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 1 \\ \cline{1-2} \ & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \ )$$

The generic vectors for T and $T_{\geq 2}$ are

$$v_1 = \begin{array}{ccc} 0 & b & 0 \\ ax & 0 & \\ a & 0 & \\ 1 & & \end{array}, \quad v_2 = \begin{array}{ccc} 0 & c & 0 \\ 0 & 1 & \\ 0 & 0 & \\ 0 & & \end{array}, \quad v_3 = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & x & \\ 0 & 1 & \\ 0 & & \end{array}, \quad v_4 = \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & \\ 0 & 0 & \\ 0 & & \end{array}$$

We form the 4×7 matrix whose entries are the coefficients of the cell vectors $f(x_1), x_2, x_3, f^2(x_1), f(x_2), f(x_3), f(x_4)$ in column in the vectors v_1, v_2, v_3, v_4 in row :

$$\wedge^4 \begin{pmatrix} b & ax & a & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \equiv \wedge^3 \begin{pmatrix} b & ax & a & 0 & 0 & 1 \\ c & 0 & 0 & 1 & 0 & 0 \\ -cx & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

from which the Plücker coordinate of the generic element of S_T is given by $(1 : a : b : c : ax : cx : ac : acx : acx^2)$ in $\mathbb{P}^8$, $(x_0 : \cdots : x_8)$. We see from this that the singular locus is $\{(x_0 : x_1)\} = \mathbb{P}^1$, which corresponding to the line L joining the point $[1456] = f(x_1) \wedge f^2(x_1) \wedge f(x_2) \wedge f(x_3)$ and $[4567] = f^2(x_1) \wedge f(x_2) \wedge f(x_3) \wedge f(x_4)$. The generic vectors for $T_1, (T_1)_{\geq 2}$ and $T_2, (T_2)_{\geq 2}$ are

$$\begin{array}{ccc} 0 & a & 0 \\ 0 & 0 & \\ 0 & 0 & \\ 1 & & \end{array}, \quad \begin{array}{ccc} 0 & b & 0 \\ 0 & 0 & \\ 0 & 0 & \\ 1 & & \end{array}, \quad \begin{array}{ccc} 0 & c & 0 \\ 0 & 0 & \\ 0 & 0 & \\ 1 & & \end{array}, \quad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & \\ 0 & 0 & \\ 0 & & \end{array}$$

$$\begin{array}{ccc} 0 & ay & 0 \\ ax & 0 & \\ a & 0 & \\ 1 & & \end{array}, \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & \\ 0 & & \end{array}, \quad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & \\ 0 & 0 & \\ 0 & & \end{array}, \quad \begin{array}{ccc} 0 & 0 & y \\ 0 & x & \\ 0 & 1 & \\ 1 & & \end{array}$$

from which we have

$$\wedge^4 \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \equiv \wedge^3 \begin{pmatrix} a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$$

$$\wedge^4 \begin{pmatrix} ay & ax & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & x & 1 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \equiv \wedge^3 \begin{pmatrix} ay & ax & a & 0 & 0 & 1 \\ -cx & 0 & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We see that the line $L = \mathbb{P}^1$ is contained in the 3-dimensional subvariety S_{T_1} and $L = X_{T_1} \cap X_{T_2}$. $\square$

Example 19 (*continued*). We consider the variety X_T for T in Example*. We form the 7×11 matrix whose entries are the coefficients of the cell vectors ordered by the right tableau $o(T)$

$$T = \begin{array}{|c|c|c|c|c|}\hline & & & 1 & 3 \\ \hline & & 2 \\ \cline{1-3} & 1 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2}\end{array} \quad , \quad o(T) = \begin{array}{|c|c|c|c|c|}\hline & & 1 & 4 & 8 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 3 & 6 & 10 \\ \cline{1-3} 7 & 11 \\ \cline{1-2}\end{array}$$

in row from top to bottom, in the generic vectors $(v_{41}, v_{32}, v_{14} \,; v_{42}, v_{23} \,; v_{33}, v_{15})$ for $T, T_{\geq 2}, T_{\geq 3}$ obtained in Example 19 in column from left to right :

$$\wedge^7 \begin{pmatrix} ax+c & bp & b & 0 & x & 0 & 1 & 0 & 0 & 0 & 0 \\ ay & 0 & 0 & ay & y+p & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & bp & b & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

which is simplified as

$$\begin{aligned} &\wedge^3 \begin{pmatrix} ax+c & bp & b & x & 0 & 1 & 0 \\ ay & 0 & 0 & y+p & 1 & 0 & 0 \\ 0 & 0 & 0 & bp & b & 0 & 1 \end{pmatrix} \\ \equiv &\wedge^3 \begin{pmatrix} c' & bp & b & x & 0 & 1 & 0 \\ ay & 0 & 0 & y+p & 1 & 0 & 0 \\ -aby & 0 & 0 & -by & 0 & 0 & 1 \end{pmatrix} \end{aligned} \qquad (*)$$

where $c' = ax + c$. This means that X_T is contained in $G(3,7) \subset \mathbb{P}[\wedge^3 V_0]$ where $V_0 = \langle 1, 2, 3, 5, 6, 7, 11 \rangle = \mathbb{C}^7$, and X_T is parametrized on the principal affine openset $U_{[6,7,11]} = U_{[4,6,7,8,9,10,11]} \cong \mathbb{A}^{12}$ by

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{15} \\ Y_{21} & 0 & 0 & Y_{25} \\ Y_{31} & 0 & 0 & Y_{35} \end{pmatrix} = \begin{pmatrix} ax+c & bp & b & x \\ ay & 0 & 0 & y+p \\ -aby & 0 & 0 & -by \end{pmatrix}$$

We see that $Y_{31} = -aby = -Y_{21}Y_{13}$ and $Y_{12} - Y_{35} = bp + by = b(p+y) = Y_{13}Y_{25}$, which implies that $X_T \cap U_{[6,7,11]} \cong \mathbb{A}^6$ is nonsingular. Next we consider the complement $X_T \backslash U_{[6,7,11]}$. If $S_{\bar{T}} \cap X_T$ is nonempty with shape $\bar{T} = \lambda/\bar{\mu}$ then $\bar{\mu} \subset \lambda$ is less than or equal to $\mu = (321)$, i.e. $\bar{\mu} = (321), (311), (222), (2211)$. This means that X_T is covered by four principal opensets corresponding to these $\bar{\mu}$'s ; $X_T \subset U_{[6,7,11]} \cup U_{[5,6,11]} \cup U_{[1,7,11]} \cup U_{[1,6,11]}$. We saw above $X_T \cap U_{[6,7,11]} \cong \mathbb{A}^6$.

(i) On the openset $U_{[5,6,11]} = U_{4,5,6,8,9,10,11]}$, (*) is equivalent to

$$\wedge^3 \begin{pmatrix} c' & bp & b & x & 0 & 1 & 0 \\ axy - c'(y+p) & -bp(y+p) & -b(y+p) & 0 & x & -(y+p) & 0 \\ -abxy + c'by & bpby & bby & 0 & 0 & by & x \end{pmatrix}$$

We see that on the openset $U_{[5,6,11]} = \mathbb{A}^{12}$, X_T is parametrized by

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{17} \\ Y_{21} & Y_{22} & Y_{23} & Y_{27} \\ Y_{31} & Y_{32} & Y_{33} & Y_{37} \end{pmatrix} = \begin{pmatrix} c' & bp & b & 1 \\ axy - c'(y+p) & -bp(y+p) & -b(y+p) & -(y+p) \\ -abxy + c'by & b^2py & b^2y & by \end{pmatrix}$$

The 6-dimensional variety X_T is defined on $U_{[5,6,11]}$ by

$$\operatorname{rank} \begin{pmatrix} Y_{12} & Y_{13} & Y_{17} \\ Y_{22} & Y_{23} & Y_{27} \\ Y_{32} & Y_{33} & Y_{37} \end{pmatrix} \leq 1, \qquad Y_{12} + Y_{23} + Y_{37} = 0$$

$$\begin{aligned} & Y_{12}Y_{11} + Y_{13}Y_{21} + Y_{17}Y_{31} \\ & = bp \cdot c' + b\{axy - c'(y+p)\} + (-abxy + c'by) = 0 \end{aligned}$$

We see from this that the singular locus of X_T on $U_{[5,6,11]}$ is equal to $\{(Y_{11}, Y_{21}, Y_{31})\} \cong \mathbb{A}^3$, which corresponds to $S_{T_1} \cup S_{T_2}$ for the *LR*-tableaux

$$T_1 = \begin{array}{ccccc} \square & \square & \square & 2 & 3 \\ \square & 1 & 2 & & \\ \square & 1 & 2 & & \\ \square & 1 & & & \end{array} \quad , \quad T_2 = \begin{array}{ccccc} \square & \square & \square & 1 & 1 \\ \square & 1 & 2 & & \\ \square & 1 & 2 & & \\ \square & 1 & & & \end{array}$$

(ii) On the openset $U_{[1,7,11]} = U_{[1,4,7,8,9,10,11]}$, (*) is equivalent to

$$\wedge^3 \begin{pmatrix} 0 & bp & b & x - (y+p)c'/ay & -c'/ay & 1 & 0 \\ 1 & 0 & 0 & (y+p)/ay & 1/ay & 0 & 0 \\ 0 & 0 & 0 & bp & b & 0 & 1 \end{pmatrix}$$

from which we see that $X_T \cap U_{[1,7,11]} \cong \mathbb{A}^6$ is nonsingular.

(iii) The LR-tableaux $\bar{T}$ with shape $\bar{T} = \lambda/(2211)$ and $X_{\bar{T}} \subset X_T$ are

$$T_3 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 2 & 3 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 & 2 \\ \cline{1-3} \ & 1 \\ \cline{1-2} \end{array} \quad , \quad T_4 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 2 & 3 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & 1 & 2 \\ \cline{1-3} \ & 1 \\ \cline{1-2} \end{array}$$

so $S_{T_3} (\supset S_{T_4})$ are contained in X_{T_1} and the singular locus of X_T on $U_{[1,6,11]} = U_{[1,4,6,8,9,10,11]}$ is reduced to the case (i).

Summing up the singular locus of X_T is equal to X_{T_1}. Considering the generic vectors for T_1 we see that $X_{T_1} \cong \mathbb{P}^3 \supset X_{T_3} \cong \mathbb{P}^2 \supset X_{T_4} \cong \mathbb{P}^1$ and $X_{T_2} = \{\text{one point}\}$. The embedding dimension at $W \in X_{T_1}$ is equal to 10 (resp. 11) if $W \not\subset X_{T_2}$ (resp. if $W \in X_{T_2}$). Besides $T_1, \cdots, T_4$ the LR-tableaux whose shapes and contents are less than or equal to those of T are the followings. There are two LR-rableaux of shape $\lambda/(321)$ and content (331) :

$$T_5 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 2 & 3 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array} \quad , \quad T_6 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array}$$

where X_{T_5} is not contained in X_T with $X_{T_5} \cap X_T = X_{T_7}$. Here

$$T_7 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 3 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array} \quad , \quad T_8 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array}$$

are the LR-tableaux of shape $\lambda/(321)$ and content (321) for which $X_7 \cong G(2,4)$ while $X_T \supset X_{T_6} \supset X_{T_8}$ with $\operatorname{Sing} X_{T_6} = \operatorname{Sing} X_{T_8} = X_{T_2}$. The least LR-tableau among those of shape $\lambda/(321)$ is

$$T_9 = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & 1 & 2 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & 1 & 2 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array}$$

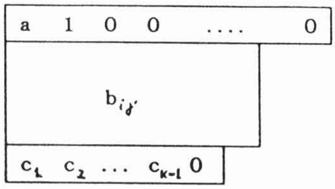
for which $X_{T_9} = \mathbb{P}^2$. There are two LR-tableaux of shape $\lambda/(222)$:

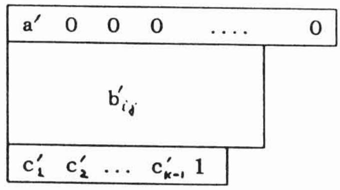
$$T_{10} = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 2 & 3 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & \ & 3 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array} \quad , \quad T_{11} = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 2 & 3 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & \ & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} \end{array}$$

where $\operatorname{Sing} X_{10} = T_4 \cong \mathbb{P}^2$ and $X_{11} \cong \mathbb{P}^2$. $\square$

4. Proof of Proposition C

In this section we prove Proposition C in Introduction. We may assume $d = \lambda_1$ by deleting the columns of λ in which there are no cell of T. We note that $S_{\bar{T}}$ is homogeneous. Let us considre the d-dimensional subspace $W = \langle\ A,\ f(A),\ f^2(A), \cdots,\ f^{d-2}(A),\ B\ \rangle$ with

$A =$ 

, $B =$ 

where $2 \le i \le r-2$ and $1 \le j \le \lambda_i$. If all of the $2(|\lambda| - d)$ variables $\{a,\ b_{ij},\ c_j,\ a',\ b'_{ij},\ c'_j\}$ are equal to zero then W is an element of $S_{\bar{T}}$. Now W is f-stable iff $f^{d-1}(W)$ and $f(B)$ are expressed by linear combinations of $\{A, f(A), \cdots f^{d-2}(A), B\}$, i.e.

$$(-a)^{d-1}A + (-a)^{d-2}f(A) + (-a)^{d-3}f^2(A) + \cdots$$
$$\cdots + a^2 f^{d-3}(A) - af^{d-2}(A) + f^{d-1}(A) + (-a)^{d-k}\beta B = 0$$

where $\quad \beta = c_1 - ac_2 + a^2c_3 - \cdots + (-a)^{k-2}c_{k-1}$

and $\quad f(B) = a'A + c'_{k-1}B$

from which we have the following identities (where $2 \le i \le r-1$) :

$$(11) \quad 0 = (-a)^{d-1}\cdot a + (-a)^{d-k}\beta a' = (-1)^{d-1}a^{d-k}\{a^k + (-1)^{k-1}\beta a'\}$$
$$(ik) \quad 0 = (-a)^{d-1}b_{ik} + (-a)^{d-2}b_{i,k-1} + (-a)^{d-3}b_{i,k-2} + \cdots$$
$$+ (-a)^{d-k}b_{i,1} + (-a)^{d-k}\beta b'_{ik}$$
$$= (-a)^{d-k}\{(-a)^{k-1}b_{ik} + (-a)^{k-2}b_{i,k-1} + \cdots - ab_{i,2} + b_{i,1} + \beta\}$$
$$(rj) \quad 0 = (-a)^{d-1}c_j + (-a)^{d-k}\beta c'_1 = (-a)^{d-k}\{a^{k-1}c_j + (-1)^{k-1}\beta c'_1\}$$
$$(11)' \quad 0 = a'a + c'_{k-1}a' = a'(a + c'_{k-1})$$
$$(i1)' \quad 0 = a'b_{i1} + c'_{k-1}b'_{i1}$$
$$(r1)' \quad 0 = a'c_1 + c'_{k-1}c'_1$$
$$(ij)' \quad b'_{i,j-1} = a'b_{ij} + c'_{k-1}b'_{ij}$$
$$(rj)' \quad c'_{j-1} = a'c_j + c'_{k-1}c'_j$$

We shall show that $a + c'_{k-1} = 0$. Suppose $a + c'_{k-1} \neq 0$. Then $a' = 0$ by $(11)'$, which implies $a = 0$ by (11) so $c'_{k-1} \neq 0$ by the assumption. Then $c'_1 = 0$ by $(r1)'$, from which $c'_2 = \cdots = c'_{k-1} = 0$ successively by $(rj)'$, a contradiction to $c'_{k-1} \neq 0$. Hence $a + c'_{k-1} = 0$. (i) Case $a \neq 0$. The linear terms

$$\begin{array}{lll} (ik) & b_{i1} & 2 \leq i \leq r-1 \\ (11)' & c'_{k-1} = & -a \\ (ij)' & b'_{ij} & 1 \leq j \leq \lambda_i - 1 \\ (rj)' & c'_j & 1 \leq j \leq k-2 \end{array}$$

are expressed by the remaining $|\lambda| - d + 1$ variables

$$\begin{array}{lll} a, & b_{ij}\ (2 \leq i \leq r-1,\ 2 \leq j \leq \lambda_i), & c_j\ (1 \leq j \leq k-1) \\ a' & b'_{ij}\ (2 \leq i \leq r-1,\ j = \lambda_i) & \end{array}$$

with one relation $f := a^k + (-1)^{k-1}\beta a' = 0$ from (11). (ii) Case $a = 0$. Then $f^{d-1}(A) = 0$, and $a + c'_{k-1} = 0$ implies $c'_{k-1} = 0$ so $f(B) = a'A$. If $a' = 0$ then $f(B) = 0$ and W is not a cyclic $\mathbb{C}[f]$-module, i.e. not contained in S_T. Hence $a' \neq 0$. Then $f(B) = a'A$ implies that $b_{i1} = 0$ from $(i1)'$ and $\{c'_{k-1}, b'_{ij}, c'_j\}$ are expressed by the remaining $|\lambda| - d + 1$ variables as in the case (i). In this case $c_1 = 0$ by $(r1)'$ so $\beta = c_1 - ac_2 + \cdots = 0$ hence $f = a^k + (-1)^{k-1}\beta a' = 0$. $\square$

Remark 22. Let T' be the LR-tableau of content $(d-1, 1)$ deleting the leftmost cell of the first row of T, replacing the letter i in the remaining cells in the first row of T by $i - 1$ and adding the cell filled with the letter 1 in the cell $(r-1, \lambda_r)$. If T is the example in Introduction before Proposition C then

$T =$

							7
					6		
		1	4	5			
1	2	3					

We see $\dim X_{T'} = |\lambda| - d + r - s - 2$, $T' \triangleright \bar{T}$ and $T_{X'}$ is nonsingular along $S_{\bar{T}}$. For, if $W \in S_{T'}$ then $f^{d-1}(W) = 0$. Hence $f^{d-1}(A) = 0$ means $a = b_{1j} = 0$ for $2 \leq j \leq s$. Similarly, $f^{d-1}(B) = 0$ means $a' = 0$, and $a + c'_{k-1} = a = 0$ implies $c'_{k-1} = 0$, so $f(B) = a'A + c'_{k-1}B = 0$, from which we see $b'_{ij} = c_j = 0$ except b_{ij} for $2 \le i \le r-1$ and $j = \lambda_i$. Hence the nonzero parameters in the entries of A and B are

$$\begin{array}{ll} b_{ij} & 2 \le i \le s,\ 2 \le j \le \lambda_i;\ s < i \le r-1,\ 1 \le j \le \lambda_i \\ c_j & 1 \le j \le k-1 \\ b'_{ij} & 2 \le i \le r-1,\ j = \lambda_i \end{array}$$

the number of which is equal to $(|\lambda| - d - s) + (r-2) = |\lambda| - d + r - s - 2 = \dim X(\lambda, (d-1,1), \bar{\mu})$. Thus $X_{T'}$ is nonsingular along $S_{\bar{T}}$. $\square$

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