A partial order on the symmetric groups defined by 3-cycles

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A PARTIAL ORDER ON THE SYMMETRIC GROUPS DEFINED BY 3-CYCLES

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ABSTRACT. We define a partial order on the symmetric group S_n of degree n by $x \leq y$ iff $y = a_1 \cdots a_k x$ with $i(y) = i(x) + 2k$ where $a_1, \cdots, a_k$ are 3-cycles of increasing or decreasing consecutive three letters and $i(*)$ is the number of inversions of the element $*$ of S_n, on the analogy of the weak Bruhat order. Whether an even permutation is comparable to the identity or not in this ordering is considered. It is shown that all of the even permutations of degree n which map 1 to n or $n-1$ are comparable to the identity.

Let G be a group generated by a subset $S \subset G$ not containing the identity of G. The pair (G, S) naturally defines an undirected simple regular graph (Cayley graph) with the vertex set G and an edge connecting x with y iff $y = ax$ for some $a \in S$, i.e. the edges meeting at a vertex x are $\{x, ax\}$ and $\{x, a^{-1}x\} = \{a^{-1}x, aa^{-1}x\}$ for $a \in S$. For instance, if G is a free group with basis S then the associated graph is a tree [Se,p26], and if G is a Coxeter group with the set S of simple reflections then the graph defines the weak Bruhat order [B]. In this note we investigate the graph associated with (G, S) where $G = A_n$ is the alternating group and $S = \{a_1, \cdots, a_{n-2}\}$ with the 3-cycles $a_j = (j, j+1, j+2)$ of consecutive three letters for $1 \leq j \leq n-2$.

Since the 3-cycle a_j is a product of two adjacent transpositions, if we write an element π of A_n as a product of $a_1, \cdots, a_{n-2}, a_1^{-1}, \cdots, a_{n-2}^{-1}$ then the number of the 3-cycles in the product is greater than or equal to $i(\pi)/2$, the half of the number of inversions $i(\pi)$ of π. Natural questions arises ; which elements of A_n can be represented by a product of exactly $i(\pi)/2$ 3-cycles $a_1, \cdots, a_{n-2}, a_1^{-1}, \cdots, a_{n-2}^{-1}$ and what is the canonical (reduced) expression of a general element of A_n by the product of these 3-cycles ? The problem is explained in terms of words

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on $[n] = \{1, \cdots, n\}$ (= arrangements of the n letters $1, \cdots, n$ without repetition) as follows. Let $x = x_1 \cdots x_n$ be a word on $[n]$. For each $2 \leq j \leq n-1$ the sequence of three numbers $x_{j-1}x_jx_{j+1}$ in the word x is equal to one of the six triples *abc*, *acb*, *bac*, *cab*, *bca*, *cba* where a, b, c are integers satisfying $1 \leq a < b < c \leq n$. In order to decrease the number of inversions by applying one of the 3-cycles $a_{j-1}^{\pm 1}$, the operations on the word x are allowed only in the cases *cab*, *bca*, *cba* above, i.e. the leftmost letter of the triple is greater than the rightmost letter:

$$cab \to abc, \qquad bca \to abc, \qquad cba \to acb \text{ or } bac \tag{1}$$

Let $S_n(l) = \{x \in S_n; i(x) = l\}$ be the set of words on $[n]$ with the number of inversions $i(x)$ equal to l. We define a partial order on S_n analogous to the weak Bruhat order.

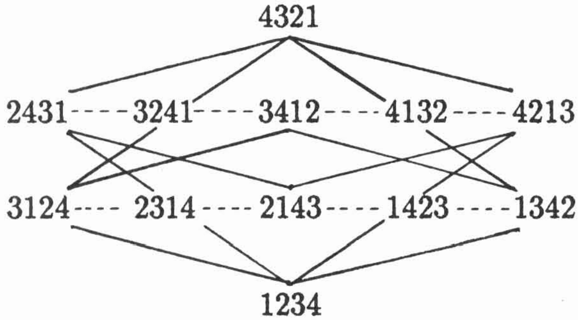
Definition 1. *$x \leq y$ for $x \in S_n(l)$ and $y \in S_n(m)$ if $m - l = 2k \geq 0$ is even and if x is obtained from y by applying k times the operations of type (1) above. An even word x on $[n]$ is called 'general' (resp. 'special') if x is comparable (resp. not comparable) to the identity $12 \cdots n$ by this partial order.*

An even permutation π corresponding to a special word, which is the main object of this note, means that it cannot be expressed by the product of $i(\pi)/2$ 3-cycles $a_j^{\pm 1}$. Definition 1 is formally generalized to Coxeter groups generated by simple reflections $s_1, \cdots, s_n$ and 3-cycles a_j to $s_j s_{j+1}$ for $1 \leq j \leq n-1$. We consider S_n as the poset (=partially ordered set) with the partial order of Definition 1, which is a union of $S_n^{(e)}$ and $S_n^{(o)}$ of even and odd words. We shall prove in Section 1

Theorem 2. *All of the even words $x = n, x_2 \cdots x_n$ and $y = y_1, n, y_3, \cdots, y_n$ on $[n]$ with the maximal letter n in the leftmost position and the second position from the left, are general, i.e. comparable to the identity $12 \cdots n$.*

The same conclusion holds for even words $x = x_1 \cdots x_{n-1}, 1$ and $y = y_1 \cdots y_{n-2}, 1, y_n$ on $[n]$ with the minimal letter 1 in the rightmost position and the second position from the right, by applying the involution $*$ on the poset S_n defined by $(x_1 \cdots x_n)^* = n+1-x_n, n+1-x_{n-1}, \cdots, n+1-x_1$ (cf. Section 2). On the other hand there are even words on $[n]$ with the maximal letter n in the third position from the left not comparable to the identity, e.g. 2143 and 246135. Theorem is easy to check for $n = 3, 4$ and proved by induction on n and by considering canonical expressions of odd words.

The associated Hasse diagram (y is drawn above x if $x < y$) of $S_4^{(e)}$ of even words on [4] is given by



where the dotted horizontal edges mean that the two vertices joined by them are transformed each other by a single 3-cycle $a_j^{\pm}$. We denote by Γ_n the Cayley graph of S_n, e.g. adjoined the dotted edges to the Hasse diagram of $S_4^{(e)}$ above. The circuits in the Cayley graph Γ_n correspond to the relations among the 3-cycles a_j for $1 \leq j \leq n-2$. The subgraph of Γ_n consisting of odd words as vertices is obtained by interchanging two letters (e.g. 1 and 2) in the subgraph of Γ_n consisting of even words as vertices so that they are isomorphic each other as a graph (not as a poset). The above graph of $S_4^{(e)}$ shows that 2143 is the only special word on [4]. In general, if x is an odd word on $[k]$ and y is an odd word on $\{k+1, \cdots, n\}$ then the juxtaposition xy is a special word on $[n]$. In order to separate these uninterested types of special words we define

Definition 3. *(i) A word $x = x_1 \cdots x_n$ on $[n]$ is decomposable if $x_1 x_2 \cdots x_k$ is a word on $[k]$ for some $1 \leq k \leq n-1$, and indecomposable if there are no such k's.*

(ii) An even word x is 'nearly decomposable' if x is indecomposable and minimal in the partial order of Definition 1, or if x is indecomposable and all of the words covered by x are decomposable.

For instance, 415263 and 3257146 are nearly decomposable words because the former is indecomposable and minimal, and the latter is indecomposable and the words covered by 3257146 are 3215746 and 3251476, both of whcih are decomposable. Let $S_n^{(d,s)}$ (resp. $S_n^{(i,s)}$) be the set of decomposable (resp. indecomposable) and special words on $[n]$ and let $S_n^{(s)} = S_n^{(d,s)} \cup S_n^{(i,s)}$. The decomposable part $S_n^{(d,s)}$

is described by various subsets of S_k for $k < n$ (Lemma 9) so we focus on the indecomposable part $S_n^{(i,s)}$. The definitions above leads to three facts ; (i) the set of indecomposable words is a dual order ideal of the poset S_n, i.e. if x is indecomposable and $x \leq y$ then y is indecomposable (cf. Lemma 7), (ii) since the set of general words is the dual order ideal of S_n generated by the identity $12\cdots n$, the set of special words is an order ideal of S_n, i.e. if x is special and $x \geq y$ then y is special, (iii) for any indecomposable even word x there is a chain in the poset S_n connecting x with a nearly decomposable word. These three observations imply that the set $S_n^{(i,s)}$ of ibdecomposable special words on $[n]$ is the union of $V_x \cap S_n^{(s)}$ where x's run over the set of nearly decomposable special words and $V_x = \{\ y \in S_n;\ x \leq y\}$ is the dual order ideal generated by x. Once we find the set of nearly decomposable special words, the set $V_x \cap S_n^{(s)}$ of the special words dominating x is obtained by an easy, but lengthy calculation by hand (cf. Section 4).

In Section 3 we consider minimal words. As remarked above, a word $x = x_1 \cdots x_n$ on $[n]$ is minimal in the partial order of Definition 1 iff the letters in odd-numbered and even-numbered positions in x are in ascending order ; $x_1 < x_3 < x_5 < \cdots < x_{2\lfloor n/2 \rfloor \pm 1}$ and $x_2 < x_4 < x_6 < \cdots < x_{2\lfloor n/2 \rfloor}$, that is, $x = x_1 \cdots x_n$ is a two-ordered sequence appearing in shellsort [K,p86]. By regarding the two-ordered sequences as lattice paths we shall show in Lemma 10 that the number of indecomposable minimal words on $[n]$ is equal to the $r-1$-th Catalan number $\frac{1}{r}\binom{2r-2}{r-1}$ for even $n = 2r$ while all of the minimal words are decomposable for odd n. In Section 4 are given the sets of nearly decomposable special words on $[n]$ for $n \leq 8$.

The rest of the paper is organized as follows. Theroem 2 above is proved in Section 1 and the set $S_n^{(d,s)}$ of decomposable special words is described explicitly in Section 2. Some results about minimal words are given in Section 3. Nearly decomposable special words, which we shall call primitive words, are considered in Section 4. The author would like to thank Professor Takeuchi for providing a computer program as well as for many useful discussions.

Notations

S_n : the set of words on $[n] = \{1, \cdots, n\}$ endowed with the partial order of Definition 1, denoted by $x \to y$ if $x > y$. A word $x = x_1 \cdots x_n$ on $[n]$ is identified with the bijection π of $[n]$ if $\pi(i) = x_i$. A permutation π acts on the set of words on $[n]$ through the positions rather than the letters, i.e. $\pi x = x_{\pi(1)} \cdots x_{\pi(n)}$ for $x = x_1 \cdots x_n$. In order to make S_n acts from *left* on the set of words the product of elements of S_n is read from left to right, e.g. $(12)(23) = (12)(23)123 = (12)132 = 312 = (132)123 = (132)$.

Γ_n : the Cayley graph associated to $(S_n, \{a_1, \cdots, a_{n-2}\})$ with the 3-cycles $a_j = (j, j+1, j+2)$ for $1 \leq j \leq n-2$,

$i(x)$: the number of inversions of a word x, i.e. the number of pairs (i,j) for $1 \leq i < j \leq n$ with $x_i > x_j$,

For a subset X of S_n denote by $X(l) = \{x \in X; i(x) = l\}$ and $X^{(*)}$ the set of the words on $[n]$ contained in X with the property expressed by $*$. Here $*$ is e, o, s, d, i, n, p means even, odd, special, decomposable, indecomposable, nearly decomposable, primitive, respectively. Similarly, $X^{(m)}$ (resp. $X^{(M)}$) is the set of the minimal (resp. maximal) elements of S_n in the partial order of Definition 1 contained in X. For instance, $X^{(i,m)}$ means the set of indecomposable and minimal words on $[n]$ contained in X.

V_x, Λ_x : the dual order and order ideals of the poset S_n generated by a word $x \in S_n$, i.e. $V_x = \{y \in S_n; x \leq y\}$ and $\Lambda_x = \{y \in S_n; x \geq y\}$.

1. Proof of Theorem 2

In this section we prove Theorem 2 in Introduction. It is easy to check Theorem 2 from the definition $312, 231 \to 123$ for $n = 3$, and from the graph of $S_4^{(e)}$ in Introduction for $n = 4$. First we consider words $x = n, x_2 \cdots x_n$ on $[n]$.

Proposition 4. *Let $x = nx_2 \cdots x_n$ be a word on $[n]$ with the maximal letter n in the leftmost position. If x is even (resp. odd) then x is comparable to the identity $12 \cdots n$ (resp. $12 \cdots, n-2, n, n-1$).*

Proof. The assertions on odd words are true from $321 \to 132$ for $n = 3$, and from $4231, 4312 \to 4123 \to 1243$ for $n = 4$. Proposition will be proved by induction on n for even and odd words simultaneously. It may be assumed that the subword $x_2 \cdots x_n$ on $[n-1]$ is minimal so $n-1 \in \{x_{n-1}, x_n\}$. Three cases are to be considered : (i) $x_n = n-1$, (ii) $x_{n-1} = n-1$ and $x_n = n-2$, (iii) $x_{n-1} = n-1$ and $x_n \neq n-2$.

(i) Case $x_n = n-1$. If the subword $nx_2 \cdots x_{n-1}$ of x deleting the rightmost letter $x_n = n-1$, is even then it is comparable to $12 \cdots, n-2, n$ by the inductive hypothesis, so

$$x = nx_2, \cdots, x_{n-1}, n-1 \ \to \ 12, \cdots n-2, n, n-1$$

If $nx_2 \cdots x_{n-1}$ is odd then it is comparable to $12 \cdots n-3, n, n-2$ by the inductive hypothesis, hence

$$\begin{aligned} x = nx_2, \cdots, x_{n-1}, n-1 &\rightarrow 12, \cdots, n-3, n, n-2, n-1 \\ &\rightarrow 12, \cdots, n-3, n-2, n-1, n \end{aligned}$$

(ii) Case $x_{n-1} = n-1$ and $x_n = n-2$. If $nx_2 \cdots x_{n-1}$ is odd then

$$\begin{aligned} x = nx_2, \cdots, x_{n-2}, n-1, n-2 &\rightarrow 12, \cdots, n-3, n, n-1, n-2 \\ &\rightarrow 12, \cdots, n-3, n-2, n, n-1 \end{aligned}$$

If $nx_2 \cdots x_{n-1}$ is even then

$$\begin{aligned} x = nx_2, \cdots, x_{n-2}, n-1, n-2 &\rightarrow 12, \cdots, n-1, n, n-2 \\ &\rightarrow 12, \cdots, n-2, n-1, n \end{aligned}$$

(iii) Case $x_{n-1} = n-1$ and $x_n \neq n-2$. Then $x_{n-3} = n-2$ because $x_1 \cdots, x_{n-3}, x_{n-2}, n-1, x_n$ is minimal with $x_{n-2} < x_n \leq n-3$.

(a) Case $nx_2 \cdots x_{n-1}$ odd.

$$\begin{aligned} x =\ & nx_2 \cdots, x_{n-2}, n-1, x_n \\ \rightarrow\ & 12, \cdots, x_n - 1, x_n + 1, \cdots, n, n-1, x_n \\ \rightarrow\ & \begin{cases} 12, \cdots, x_n - 1, x_n + 1, \cdots, n-1, x_n, n & \text{if } x \text{ even} \\ 12, \cdots, x_n - 1, x_n + 1, \cdots, x_n, n, n-1 & \text{if } x \text{ odd} \end{cases} \end{aligned}$$

from which, Proposition holds.

(b) Case $nx_2 \cdots x_{n-1}$ even. If x is even then

$$\begin{aligned} x = nx_2 \cdots x_{n-1} x_n &\rightarrow 12, \cdots, x_n - 1, x_n + 1, \cdots, n, x_n \\ &\rightarrow 12, \cdots, n \end{aligned}$$

Suppose x is odd. That the word $n, x_2 \cdots x_{n-1} = n, x_2 \cdots x_{n-2}, n-1$ is even means that $nx_2 \cdots, x_{n-2}$ is odd, hence

$$\begin{aligned} x =\ & nx_2, \cdots, x_{n-2}, n-1, x_n \\ \rightarrow\ & 12, \cdots, x_n - 1, x_n + 1, \cdots, n, n-2, n-1, x_n \\ \rightarrow\ & 12, \cdots, x_n - 1, x_n + 1, \cdots, n, x_n, n-2, n-1 \\ \rightarrow\ & 12, \cdots, x_n - 1, x_n + 1, \cdots, x_n, n-2, n, n-1 \\ \rightarrow\ & 12, \cdots, n-2, n, n-1 \quad \square \end{aligned}$$

Next consider words $x = x_1, n, x_3 \cdots x_n$. If $x_1 = 1$ then Theorem 2 follows from the previous Proposition 4 so we assume $x_1 \neq 1$.

Proposition 5. *(I) Let $x = n-1, n, x_3 \cdots, x_n$ be a word on $[n]$ with $n-1$ and n in the first and the second position from left, respectively. If x is even (resp. odd) then x is comparable to the identity $12\cdots n$ (resp. both $12, \cdots, n-3, n-1, n-2, n$ and $12\cdots, n-4, n-1, n-3, n, n-2$).*

(II) Let $x = x_1, n, x_3 \cdots, x_n$ be a word on $[n]$ with the leftmost letter x_1 equal to neither 1 nor $n-1$, and the maximal letter n in the second position from left. If x is even (resp. odd) then x is comparable to the identity $12\cdots n$ (resp. $12, \cdots, n-2, n, n-1$).

Proof. The assertions on odd words are true for $n = 4$ from $3412 \to 3214 \to 1324$, $3412 \to 3142$ and $2413 \to 1243$. Proposition will be proved by induction on n. It may be assumed that the subword $x_3 \cdots x_n$ of x is minimal from the beginning, so $1 \in \{x_3, x_4\}$. If $x_3 = 1$ then $x = x_1, n, 1, x_4 \cdots x_n$ covers the word $1, x_1, n, x_4 \cdots x_n$ and the inductive hypothesis is applied to the subword $x_1, n, x_4 \cdots x_n$. Hence we assume $x_4 = 1$ from now on.

(I) Since $n > x_3 > 1$ in the word $x = n-1, n, x_3, 1, \cdots x_n$ it can be transformed n to the rightmost position : $x \to n-1, y_2 \cdots y_{n-1}, n$. If the subword $n-1, y_2 \cdots y_{n-1}$ on $[n-1]$ is even (resp. odd) then it is comparable to $12\cdots, n-1$ (resp. $12\cdots, n-3, n-1, n-2$) by the previous Proposition, which implies

$$x \to \begin{cases} 12\cdots, n-1, n & \text{if } x \text{ is even} \\ 12\cdots, n-3, n-1, n-2, n & \text{if } x \text{ is odd} \end{cases}$$

Next we shall show that if x is odd then it is comparable to $12\cdots, n-1, n-3, n, n-2$. That the subword $x_3 \cdots x_n$ of x is minimal on $[n-2]$ means that $n-2 \in \{x_{n-1}, x_n\}$.

(i) Case $x_n = n-2$. The word $x = n-1, n, x_3 \cdots, x_{n-1}, n-2$ is odd so is the subword $n-1, n, x_3 \cdots x_{n-1}$ on $\{1, \cdots, n-3, n-1, n\}$, which is comparable to $12\cdots n-1, n-3, n$ by the inductive hypothesis. Thus

$$x = n-1, n, x_3 \cdots x_{n-1}, n-2 \to 12\cdots, n-1, n-3, n, n-2.$$

(ii) Case $x_{n-1} = n-2$. Since $n > x_3 > 1$ the letter n in the subword $n-1, n, x_3, 1, \cdots x_{n-2}$ of x is transformed to the rightmost position, hence

$$\begin{aligned} x &= n-1, n, x_3, 1, \cdots x_{n-2}, n-2, x_n \\ &\to n-1, y_2, \cdots, y_{n-3}, n, n-2, x_n \\ &\to n-1, y_2, \cdots, y_{n-3}, x_n, n, n-2 \qquad (*) \end{aligned}$$

The word x is odd so is the subword $n-1, y_2, \cdots, y_{n-3}, x_n$ on $\{1, \cdots, n-3, n-1\}$, which is comparable to $12\cdots, n-1, n-3$ by the previous Proposition. Hence $(*) \to 12\cdots, n-1, n-3, n, n-2$.

(II) That the subword $x_3 \cdots x_n$ is minimal means that $n-1 \in \{x_{n-1}, x_n\}$. First consider the case $x_n = n-1$.

(i) Case $x_n = n-1$ and $x_1 = n-2$. Applying the results of (I) to the subword $y = n-2, n, x_3 \cdots x_{n-1}$ on $\{1, \cdots, n-2, n\}$ of x we see

$$y \to \begin{cases} 12\cdots, n-2, n & \text{if } y \text{ is even} \\ 12\cdots, n-2, n-4, n, n-3 & \text{if } y \text{ is odd} \end{cases}$$

Hence, if y is even then $x = y, n-1 \to 12\cdots, n-2, n, n-1$, and if y is odd then

$$\begin{aligned} x = y, n-1 &\to 12\cdots, n-2, n-4, n, n-3, n-1 \\ &\to 12\cdots, n-2, n-4, n-3, n-1, n \\ &\to 12\cdots, n-4, n-3, n-2, n-1, n \end{aligned}$$

(ii) Case $x_n = n-1$ and $x_1 \neq n-2$. Applying the inductive hypothesis of (II) to the subword $y = x_1, n, x_3 \cdots x_{n-1}$ on $\{1, \cdots, n-2, n\}$ of x we see

$$y \to \begin{cases} 12\cdots, n-2, n & \text{if } y \text{ is even} \\ 12\cdots, n-3, n, n-2 & \text{if } y \text{ is odd} \end{cases}$$

Hence, if y is even then $x = y, n-1 \to 12\cdots, n-2, n, n-1$, and if y is odd then

$$\begin{aligned} x = y, n-1 &\to 12\cdots, n-3, n, n-2, n-1 \\ &\to 12\cdots, n-3, n-2, n-1, n \end{aligned}$$

Next consider the case $x_{n-1} = n-1$.

(iii) Case $x_{n-1} = n-1$ and $x_n = n-2$. Since $x_1 \neq n-1$, applying the inductive hypothesis of (II) to the subword $y = x_1, n, x_3 \cdots x_{n-2}, n-1$ on $\{1, \cdots, n-3, n-1, n\}$ of x we see

$$y \to \begin{cases} 12\cdots, n-3, n-1, n & \text{if } y \text{ is even} \\ 12\cdots, n-3, n, n-1 & \text{if } y \text{ is odd} \end{cases}$$

Hence, if y is even then

$$\begin{aligned} x = y, n-2 &\to 12\cdots, n-3, n-1, n, n-2 \\ &\to 12\cdots, n-3, n-2, n-1, n \end{aligned}$$

and if y is odd then

$$\begin{aligned} x = y, n-2 &\to 12\cdots, n-3, n, n-1, n-2 \\ &\to 12\cdots, n-3, n-2, n, n-1 \end{aligned}$$

(iv) Case $x_{n-1} = n-1$ and $x_n \neq n-2$. If the subword $x_1, n, x_3 \cdots x_{n-2}, n-1$ on $\{1, \cdots, x_n - 1, x_n + 1, \cdots, n\}$ of x is odd then, since $x_1 \neq n-1$, it is comparable to $12\cdots, x_n - 1, x_n + 1, \cdots, n, n-1$ by the inductive hypotyesis of (II). Hence

$$\begin{aligned} x &= x_1, n, x_3, \cdots x_{n-2}, n-1, x_n \\ &\to 12\cdots, x_n - 1, x_n + 1, \cdots, n, n-1, x_n \\ &\to \begin{cases} 12\cdots, x_n - 1, x_n + 1, \cdots, n-1, x_n, n & \text{if } x \text{ is even} \\ 12\cdots, x_n - 1, x_n + 1, \cdots, x_n, n, n-1 & \text{if } x \text{ is odd} \end{cases} \end{aligned}$$

from which, Proposotion holds. Next suppose $x_1, n, x_3, \cdots, x_{n-2}, n-1$ is even, i.e. $x_1, n, x_3, \cdots, x_{n-2}$ is an odd word on $\{1\cdots, x_n - 1, x_n + 1, \cdots, n-2, n\}$. If x is even then the even word $x_1, n, x_3, \cdots, x_{n-2}, n-1$ is comparable to $12\cdots, x_n - 1, x_n + 1, \cdots, n$ by the inductive hypothesis of (II) we see

$$\begin{aligned} x &= x_1, n, x_3, \cdots, x_{n-2}, n-1, x_n \\ &\to 12\cdots, x_n - 1, x_n + 1, \cdots, n, x_n \\ &\to 12\cdots n \end{aligned}$$

If x is odd then

(a) Case $x_1 = n-2$. Applying the result of (I) to the odd word $n-2, n, x_3, \cdots x_{n-2}$ on $\{1, 2, \cdots, x_n - 1, x_n + 1, \cdots, n-2, n\}$ we see

$$\begin{aligned} x &= n-2, n, x_3, \cdots, x_{n-2}, n-1, x_n \\ &\to 12\cdots, x_n - 1, x_n + 1, \cdots, n-4, n-2, n-3, n, n-1, x_n \\ &\to 12\cdots, x_n - 1, x_n + 1, \cdots, n-4, n-2, n-3, x_n, n, n-1 \\ &\to 12\cdots, x_n - 1, x_n + 1, \cdots, n-4, n-3, x_n, n-2, n, n-1 \\ &\to 12\cdots, n-2, n, n-1 \end{aligned}$$

(b) Case $x_1 \neq n-2$. Applying the inductive hypotyesis of (II) to the odd word $x_1, n, x_3, \cdots, x_{n-2}$ on $\{1, 2, \cdots, x_n - 1, x_n + 1, \cdots, n-2, n\}$

we see

$$\begin{aligned} x = x_1, n, x_3, \cdots, x_{n-2}, n-1, x_n \\ \to 12\cdots, x_n - 1, x_n + 1, \cdots, n, n-2, n-1, x_n \\ \to 12\cdots, x_n - 1, x_n + 1, \cdots, n, x_n, n-2, n-1 \\ \to 12\cdots, x_n - 1, x_n + 1, \cdots, x_n, n-2, n, n-1 \\ \to 12\cdots, n-2, n, n-1 \quad \square \end{aligned}$$

2. Decomposable special words

In this section we define an involution on the poset S_n and then determine the set $S_n^{(d,s)}$ of decomposable special words.

If we place vertical lines | at both ends of a word $x = x_1 \cdots x_n$ on $[n]$ and between x_i and x_{i+1} whenever $\max(x_1, \cdots, x_i) = i$ then the indecomposable subwords of x are the segments between the lines. We define an involution $*$ on the poset S_n by $x^* = n+1-x_n, n+1-x_{n-1}, \cdots, n+1-x_1$ for $x = x_1 \cdots x_n$. In the Caylet graph of $S_4^{(e)}$ in Introduction, $*$ induces the reflection about the vertical axis of symmetry. The fact that $n+1-x_i > n+1-x_{i+2}$ if $x_i > x_{i+2}$, means that if x covers y then x^* covers y^* so $*$ is an automorphism of the poset S_n. It follows from the identities $i(x_n \cdots x_1) = \binom{n}{2} - i(x) = i(n+1-x_1, \cdots n+1-x_n)$ that the number of inversions is preserved by $*$, $i(x) = i(x^*)$.

Lemma 6. *A word x is (i) minimal (resp. (ii) indecomposable, (iii) special, (iv) nearly decomposable) iff x^* is a word with the same property.*

Proof. (i) If x is not minimal with $x_i > x_{i+2}$ then $n+1-x_{i+2} > n+1-x_i$ so the word x^* is not minimal. (ii) If x is decomposable with a subword $x_1 \cdots x_k$ on $[k]$ then $n+1-x_k, \cdots n+1-x_1$ is a word on $\{n+1-k, n+2-k, \cdots, n\}$ so the word x^* is decomposable. (iii) If x is not special with a chain $x \to x_{(1)} \to \cdots \to x_{(r)} = 12\cdots n$ connectiong x with the identity then $x^* \to x^*_{(1)} \to \cdots \to (12\cdots n)^* = 12\cdots n$ is a chain connecting x^* with the identity so x^* is not special. (iv) follows from the results of (i) and (ii). $\square$

Let C_n be the set of compositions (=ordered partitions) of n, e.g. $C_3 = \{(3), (2,1), (1,2), (1,1,1)\}$. A partial order on C_n is defined by the refinement ; $(a_1,\cdots,a_r) \le (b_1,\cdots,b_s)$ in C_n iff there are $i_1 = 0 < i_2 < \cdots < i_s < i_{s+1} = r$ such that $(a_{i_k+1}, a_{i_k+2}, \cdots, a_{i_{k+1}})$ is a composition of b_k for each $1 \le k \le s$. We define the map $f : S_n \to C_n$ by $f(x) = (a_1,\cdots,a_r)$ where the word $x = x^{(1)}\cdots x^{(r)}$ on $[n]$ is the juxtaposition of indecomposable subwords $x^{(i)}$ of length a_i for $1 \le i \le r$. Then $\cup_{a \ge b} f^{-1}(b) = S_{a_1} \times \cdots \times S_{a_r}$ for $a = (a_1,\cdots,a_r) \in C_n$ and $f^{-1}((n)) \subset S_n$ is the set of indecomposable words on $[n]$.

Lemma 7. *The map $f : S_n \to C_n$ is order-preserving, i.e. $f(x) \le f(y)$ in C_n if $x \le y$ in the poset S_n.*

Proof. $231 > 123 < 312$ and $213 < 321 > 132$ in S_n is transformed by f to $(3) > (1,1,1) < (3)$ and $(21) < (3) > (1,2)$, respectively. □

Hence the dual order ideal $V_x = \{y \in S_n; x \le y\}$ for a word x is contained in the union of $f^{-1}(b)$ with $b \in C_n$ satisfying $f(x) \le b$, in particular, if x is an indecomposable word then V_x is contained in the set $f^{-1}((n))$ of indecomposables. Let us denote the number of even (resp. odd) indecomposable words on $[n]$ by $\gamma_n^{(e)}$ (resp. $\gamma_n^{(o)}$) and $\gamma_n = \gamma_n^{(e)} + \gamma_n^{(o)}$. The construction of the indecomposable words is contained in the proof of the following Lemma.

Lemma 8. *(i) γ_n is expressed by a linear combination of $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$.*

$$\gamma_n = \gamma_1 \cdot 1 \cdot (n-2)! + \gamma_2 \cdot 2 \cdot (n-3)! + \gamma_3 \cdot 3 \cdot (n-4)! + \cdots + \gamma_{n-2} \cdot (n-2) \cdot 1! + \gamma_{n-1} \cdot (n-1) \cdot 0!$$

(ii) $\gamma_n^{(e)} - \gamma_n^{(o)} = (-1)^{n-1}$

n	1	2	3	4	5	6	7	8	9
$n!$	1	2	6	24	120	720	5040	40320	362980
γ_n	1	1	3	13	71	461	3447	29093	273343

Proof. (i) The indecomposable words on $[n]$ are constructed from words on $[n-1]$ as follows. Let $x = yz$ be a word on $[n-1]$ with an indecomposable subword $y = y_1 \cdots y_i$ of x of length $1 \le i \le n-1$. Then the i words

$$ny_1 \cdots y_i z, \quad y_1 n y_2 \cdots y_i z, \quad \cdots \quad , \ y_1 \cdots \ y_{i-1} n y_i z$$

inserting the letter n in the word $x = yz$, are indecomposable words on $[n]$ and all of the indecomposable words on $[n]$ are obtained in this way from words on $[n-1]$. (i) follows from this. (ii) Let $H_k = S_k \times S_{n-k} \subset S_n$ for $1 \leq k \leq n-1$. Then the set of indecomposables $f^{-1}((n))$ is the complement of $H_1 \cup \cdots \cup H_{n-1}$ in S_n so $\gamma_n = |f^{-1}((n))|$ is equal to

$$n! - \sum_{i=1}^{n-1} |H_i| + \sum_{i<j} |H_i \cap H_j| - \cdots + (-1)^{n-1} |H_1 \cap \cdots \cap H_{n-1}|$$

by the Principle of Inclusion-Exclusion[S,vol 1,p64]. The numbers of even and odd permutations in $H_{i_1} \cap \cdots \cap H_{i_k}$ are same for all $1 \leq i_1 < \cdots < i_k \leq n-1$ except $H_1 \cap \cdots \cap H_{n-1} = \{e\}$. (ii) follows from this. $\square$

(i) also follows from the identity $n! = \gamma_1 \cdot (n-1)! + \gamma_2 \cdot (n-2)! + \cdots + \gamma_n \cdot 0!$ and the generating function $\sum_{n\geq 1} \gamma_n x^n$ is equal to $1 - (\sum_{n\geq 0} n! x^n)^{-1}$ [C; S,vol 1,p49]. Alternatively, indecomposable words on $[n]$ are divided into the following three types.

(i) $x = AnB1C$ where A, B, C are subwords of x (or empty).

(ii) $x = A1BnC$ and there are $a \in A$ and $c \in C$ with $a > c$,

(iii) $x = A1BnC$ and $a < c$ for all $a \in A$ and all $c \in C$ but there are $a \in A, b, b' \in B$ and $c \in C$ such that b is in the left of b' ; $x = \cdots a \cdots 1 \cdots b \cdots b' \cdots n \cdots c \cdots$ with $a > b'$ and $b > c$.

If a special word x on $[n]$ is decomposed by $x = yz$ with an indecomposable z on $\{n-k+1, \cdots, n\}$ then the three cases occur : (i) y and z are odd words, (ii) y is a special word on $[n-k]$, (iii) z is a special word on $\{n-k+1, \cdots, n\}$. We see from this

Lemma 9. *The set $S_n^{(d,s)}$ of the decomposable special words on $[n]$ is the disjoint union $B_1 \coprod B_2 \coprod \cdots \coprod B_{n-1}$ with $B_k = \{\ yz \in S_n^{(d,s)}\ ;\ z \in S_k^{(i)}\}$, which, in turn , is the disjoint union of B'_k, B_k" and B_k"' where*

$$B'_k = \{\ yz\ ;\ y \in S_{n-k}^{(o)},\ z \in S_k^{(i,o)}\}, \quad 2 \leq k \leq n-2$$

$$B_k" = \{\ yz\ ;\ y \in S_{n-k}^{(s)},\ \textit{non-special}\ z \in S_k^{(i,e)}\}, \quad k \neq 2,\ n-k \geq 4$$

$$B_k"' = \{\ yz\ ;\ y \in S_{n-k}^{(e)},\ z \in S_k^{(i,s)}\}, \quad k \geq 6$$

For $1 \leq k \leq 6$ we see

$$B_1 = B_1" = S^{(s)}_{n-1} \cdot S^{(i,e)}_1 \cong S^{(s)}_{n-1}$$

$$B_2 = B_2' = S^{(o)}_{n-2} \cdot S^{(i,o)}_2 \cong S^{(o)}_{n-2}$$

$$B_3 = B_3' \coprod B_3"$$

$$= S^{(o)}_{n-3} \cdot S^{(i,o)}_3 \coprod S^{(s)}_{n-3} \cdot (S^{(i,e)}_3 \backslash S^{(s)}_3) \cong S^{(o)}_{n-3} \coprod 2 \cdot S^{(s)}_{n-3}$$

$$B_4 = B_4' \coprod B_4"$$

$$= S^{(o)}_{n-4} \cdot S^{(i,o)}_4 \coprod S^{(s)}_{n-4} \cdot (S^{(i,e)}_4 \backslash S^{(s)}_4) \cong 7 \cdot S^{(o)}_{n-4} \coprod 6 \cdot S^{(s)}_{n-4}$$

$$B_5 = B_5' \coprod B_5"$$

$$= S^{(o)}_{n-5} \cdot S^{(i,o)}_5 \coprod S^{(s)}_{n-5} \cdot (S^{(i,e)}_5 \backslash S^{(s)}_5) \cong 35 \cdot S^{(o)}_{n-5} \coprod 36 \cdot S^{(s)}_{n-5}$$

$$B_6 = B_6' \coprod B_6" \coprod B_6"'$$

$$= S^{(o)}_{n-6} \cdot S^{(i,o)}_6 \coprod S^{(s)}_{n-6} \cdot (S^{(i,e)}_6 \backslash S^{(s)}_6) \coprod S^{(e)}_{n-6} \coprod S^{(i,s)}_6$$

$$\cong 231 \cdot S^{(o)}_{n-6} \coprod 226 \cdot S^{(s)}_{n-6} \coprod 4 \cdot S^{(e)}_{n-6}$$

n	$\lvert B_1\rvert$	$\lvert B_2\rvert$	$\lvert B_3\rvert$	$\lvert B_4\rvert$	$\lvert B_5\rvert$	$\lvert B_6\rvert$	$\lvert B_7\rvert$	$\lvert S_n^{(d,s)}\rvert$
4	0	1	0	–	–	–	–	1
5	0	3	2	0	–	–	–	5
6	5	12	3	7	0	–	–	27
7	31	60	14	21	35	4	–	165
8	177	360	70	90	105	232	12	649

Thus the decomposable part $S_n^{(d,s)}$ of the special words on $[n]$ are described from those of degree k smaller than n. Therefore the study of the set $S_n^{(s)} = S_n^{(d,s)} \cup S_n^{(i,s)}$ is reduced to that of the indecomposable part $S_n^{(i,s)}$ of $S_n^{(s)}$.

3. Minimal words

A word $x = x_1 \cdots x_n$ on $[n]$ is minimal in the partial order of Definition 1 iff the triples $x_{j-1}x_jx_{j+1}$ for all $1 \leq j \leq n-2$ are one of abc, bca, cab for some $1 \leq a < b < c \leq n$, i.e. the minimal a is in the left of the maximal c. This is equivalent to the condition that the subsequences of x in the odd-numbered and the even-numbered positions form increasing sequences.

$$x_1 < x_3 < x_5 < \cdots < x_{2\lfloor n/2 \rfloor \pm 1}, \quad x_2 < x_4 < x_6 < \cdots < x_{2\lfloor n/2 \rfloor} \qquad (3)$$

Thus the number of the set $S_n^{(m)}$ of the minimal words on $[n]$ is equal to the number of ways of choosing $\lfloor n/2 \rfloor$ elements $x_2, x_4, \cdots, x_{2\lfloor n/2 \rfloor}$ from $[n] = \{1, \cdots, n\}$, i.e. $|S_n^{(m)}| = \binom{n}{\lfloor n/2 \rfloor}$. The two inequalities (3) mean that a word $x = x_1 \cdots x_n$ on $[n]$ is minimal iff $\max(x_1, \cdots, x_i) = x_{i-1}$ or x_i for all $2 \leq i \leq n$, and the minimality of x implies that the subwords $x_i x_{i+1} \cdots x_j$ are minimal for all $1 \leq i < j \leq n$. A minimal words $x = x_1 \cdots x_{2r}$ on $[2r]$ corresponds bijectively to lattice pathes from $(0,0)$ to (r,r) with unit steps to the right and up where the i-th step is horizontal or vertical according to the integer i in an even-numbered position or in an odd numbered-position [K,p86]. We shall show

Lemma 10. *A minimal word $x = x_1 \cdots x_{2r}$ on $[2r]$ is indecomposable iff the corresponding lattice path is in the range $\{(i,j)\ ;\ 0 \leq j < i \leq r\} \cup \{(0,0), (r,r)\}$. Hence the number of indecomposable minimal words on $[2r]$ is equal to the $r-1$-th Catalan number $\frac{1}{r}\binom{2r-2}{r-1} = \binom{2r-2}{r-1} - \binom{2r-2}{r}$ [cf.St,vol2,p223]. If n is odd then all of the minimal words on $[n]$ are decomposable.*

Proof. The two inequalities (3) and x is indecomposable imply that $x_2 = 1$ and $x_4 = 2$. If x_3 is less than x_6 then $x_1x_2x_3x_4 = x_1, 1, x_3, 2$ is a subword of x on $[4]$ so x is indecomposable implies $x_3 > x_6$. Similarly, if x_5 is less than x_8 then $x_1 \cdots x_6$ is a subword of x on $[6]$ so $x_5 > x_8$. In general, x is indecomposable implies $x_{2k-3} > x_{2k}$ for all $3 \leq k \leq \lfloor n/2 \rfloor$. We see from this that x_{2k-1} is greater than $x_1, x_2, \cdots, x_{2k-2}$ for all $2 \leq k \leq \lfloor n/2 \rfloor$. Hence, if $x_{2k-1} \leq 2k-1$ then $x_{2k-1} = 2k-1$ and $x_1 \cdots x_{2k-1}$ is a subword of x on $[2k-1]$. Therfore, if $x = x_1 \cdots x_n$ is indecomposable and minimal then $x_{2k-1} \geq 2k$ for all $1 \leq k \leq \lfloor n/2 \rfloor$. Hence, if $n = 2r-1$ is odd then $x_{2r-1} \geq 2r$, a contradiction. Thus $n = 2r$ is even with $x_{2r-1} = 2r$ and $x_{2r-3} = 2r-1$. In conclusion, there is an indecomposable minimal word on $[n]$ iff $n = 2r$ is even, and indecomposable minimal words on $[2r]$ corresponds bijectively to two-array sequences with the following inequalities

$$\begin{array}{ccccccccccccccc} & & 1 & < & 2 & < & x_6 & < & x_8 & < & x_{10} & < \cdots < & x_{2r-2} & < & x_{2r} \\ & & \wedge & & \wedge & & \wedge & & \wedge & & & & \wedge & & \wedge \\ x_1 & < & x_3 & < & x_5 & < & x_7 & < & \cdots & < & x_{2r-5} & < & 2r-1 & < & 2r \end{array}$$

These two-array, in turn, corresponds bijectively to the lattice path stated in Lemma. □

The $r-1$-th Catalan numbers $|S_{2r}^{(i,m)}| = \frac{1}{r}\binom{2r-2}{r-1}$ for small $2r$ are given by

$2r$	2	4	6	8	10	12	14	16	18	20	22
$\lvert S_{2r}^{(i,m)}\rvert$	1	1	2	5	14	42	132	429	1430	4862	16796

and indecomposable minimal words for small $n = 2r$ are given as follows.

$n=2$ 21 $(i=1)$; $n=4$ 3142 $(i=3)$

$n=6$ 415263 $(i=6)$ 315264 $(i=5)$

$n=8$ 51627384 $(i=10)$ 41627385 $(i=9)$ 41527386 $(i=8)$

31627485 $(i=8)$ 31527486 $(i=7)$

Next let us consider the number of inversions. The number of inversions of $x = x_1 \cdots x_n$ is written by $i(x) = \sum_{k=1}^{n} i_k(x)$ where $i_k(x)$ is the number of the pairs (k,j) with $k < j \leq n$ and $x_k > x_j$. Then we see

$$\max(x_k - k, 0) \leq i_k(x) \leq \min(x_k - 1, n - k)$$

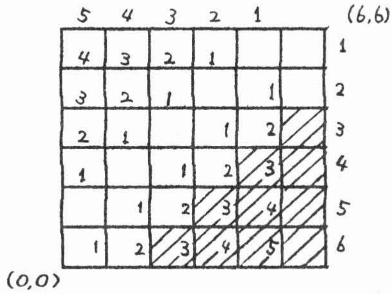
for all $1 \leq k \leq n$ and $x \in S_n$. The numbers of inversions of minimal words on $[n]$ are described as follows.

Lemma 11. *If $x = x_1 \cdots x_n$ is minimal then (i) $i_k(x) = \max(x_k - k, 0)$ for all $1 \leq k \leq n$ so $i(x) = \sum_{k=1}^{n} \max(x_k - k, 0)$, and (ii) $x_1, x_2, \cdots, x_{k-1} < x_k$ for all k such that $x_k \geq k$.*

Proof. For a minimal word x on $[n]$ suppose $i_k(x) > 0$. Then there is an $l > k$ with $x_k > x_l$. Since x is minimal we see $l \equiv k+1 \bmod 2$ so $x_1, \cdots, x_{k-1} < x_k$, which implies $i_k(x) = x_k - k$. □

Remark. The converse of (i) does not hold, e.g. $x = 231$ is not minimal, but $i_1(x) = i_2(x) = 1$ and $i_3(x) = 0$.

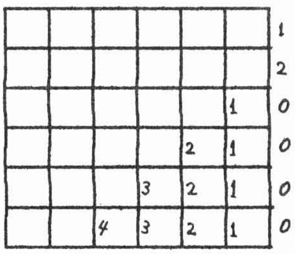
The number of inversions of a minimal word x on $[2r]$ is the sum of the weights in the corresponding lattice path. Here the weights in the unit edges are given below in the case $2r = 12$.



In particular, the number of inversions of an indecomposable minimal word x on $[2r]$ is calculated by Lemma 11 : $i(x) = \sum_{k=1}^{r}(x_{2k-1} - (2k-1))$, which is, by the diagram above, equal to $y_1 + \cdots + y_{r-2} + 3$, where y_j is the weight of the vertical unit edge from $(i, j-1)$ to (i, j) for some i. Since the path corresponding to an indecomposable x is in the shaded region the weight y_j for $1 \le j \le r-2$ satisfy

$$\max(2, y_{j-1} - 1) \le y_j \le r + 1 - j$$

If we set $\lambda_j = r - y_j - j + 1$ for $1 \le j \le r-2$ then the weight diagram above changes to



with

$$\begin{aligned}\lambda_1 + \cdots + \lambda_{r-2} &= r(r-2) - \sum_{j=1}^{r-2} y_j - \frac{(r-3)(r-2)}{2} \\ &= \frac{(r-2)}{2}\{2r - (r-3)\} - i(x) + 3 \\ &= \frac{r(r+1)}{2} - i(x)\end{aligned}$$

and λ_j for $1 \leq j \leq r-2$ satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-2} \geq 0, \qquad \lambda_j \leq r-j-1$$

Therefore we have shown

Lemma 12. *The indecomposable minimal words on $[2r]$ with the number of inversions equal to l, corresponds bijectively to the partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{r-2})$ of $\frac{r(r+1)}{2} - l$ with the height $ht(\lambda) \leq r-2$ and contained in the staircase diagram $\delta = (r-2, r-3, \cdots, 2, 1)$.*

We see from the weight diagram above that among the indecomposable minimal words on $[2r]$ the minimal (resp. maximal) value of the number of inversions is given by

$$2(r-1)+3 = 2r-1 \quad (\text{resp.} \ \sum_{k=1}^{r} k = r(r+1)/2\ )$$

when $x_{2k} = 2k-2$ and $x_{2k-1} = 2k+1$, i.e. $x = 31527496\cdots 2r-1, 2r-4, 2r, 2r-2$ (resp. $x_{2k} = k$ and $x_{2k+1} = r+k$, i.e. $x = r+1, 1, r+2, 2, r+3, 3, \cdots, 2r-1, r-1, 2r, r$). For a mimimal word $x = x_1 \cdots x_n$ on $[n]$ let m be the integer such that $x_{m+1} \cdots x_n$ is an indecomposable subword of x. Then the subword $x_1 \cdots x_m$ is minimal on $[m]$ and x is written by juxtaposing a mimimal words on $[m]$ and an indecomposable minimal word on $[n-m+1, \cdots, n]$. We see from this that the set $S_n^{(m)}$ of miminal word on $[n]$ is decomposed by

$$S_{n-1}^{(m)} S_1^{(m)} \coprod S_{n-2}^{(m)} S_2^{(i,m)} \coprod S_{n-3}^{(m)} S_3^{(i,m)} \coprod \cdots \coprod S_1^{(m)} S_{n-1}^{(i,m)} \coprod S_n^{(i,m)}$$

We saw $S_n^{(i,m)} = \phi$ for all odd n so $S_n^{(m)}$ is equal to

$$S_{2m-1}^{(m)} S_1 \coprod S_{2m-2}^{(m)} S_2^{(i,m)} \coprod S_{2m-4}^{(m)} S_4^{(i,m)} \coprod S_{2m-6}^{(m)} S_6^{(i,m)} \coprod \\ \cdots\cdots \coprod S_2^{(m)} S_{2m-2}^{(i,m)} \coprod S_{2m}^{(i,m)}$$

The set $S_n^{(m)}(l)$ is decomposed by $S_n^{(m)}(l) = \coprod_y S_n^{(m)}(l, y)$ where y runs over indecomposable minimal word on $[m]$ for $m \leq n$ of length $i(y) \leq l$ and $S_n^{(m)}(l, y)$ is the set

$$\{\ yz\ ;\ z \text{ is a minimal word on } \{m+1, \cdots, n\} \text{ with } i(z) = l - i(y)\ \}$$

Hence the number of the set $S_n^{(m)}(l)$ is written by

$$|S_n^{(m)}(l)| = \sum_y |S_n^{(m)}(l,y)|$$

with $|S_n^{(m)}(l,y)| = |S_{n-m}^{(m)}(l-i(y))|$. From the number of indecomposable minimal words given above we see $|S_n^{(m)}(l)|$ is equal to

$$\begin{aligned}&|S_{n-1}^{(m)}(l)| + |S_{n-2}^{(m)}(l-1)| + |S_{n-4}^{(m)}(l-3)| + |S_{n-6}^{(m)}(l-5)|\\ &+ |S_{n-6}^{(m)}(l-6)| + |S_{n-8}^{(m)}(l-7)| + 2\cdot|S_{n-8}^{(m)}(l-8)| + |S_{n-8}^{(m)}(l-9)|\\ &+ |S_{n-8}^{(m)}(l-10)| + |S_{n-10}^{(m)}(l-9)| + 3\cdot|S_{n-10}^{(m)}(l-10)|\\ &+ 3\cdot|S_{n-10}^{(m)}(l-11)| + \cdots\end{aligned}$$

From the binomial identity $\sum_{i=0}^{n}\binom{k+i}{k} = \binom{n+k+1}{k+1}$ we see

$$|S_n^{(m)}(1)| = \binom{n-1}{1}, \qquad |S_n^{(m)}(2)| = \binom{n-2}{2}$$

$$|S_n^{(m)}(3)| = \binom{n-3}{3} + \binom{n-3}{1}, \quad |S_n^{(m)}(4)| = \binom{n-4}{4} + 2\binom{n-4}{2}.$$

$$|S_n^{(m)}(5)| = \binom{n-5}{5} + 3\binom{n-5}{3} + \binom{n-5}{1}$$

$$|S_n^{(m)}(6)| = \binom{n-6}{6} + 4\binom{n-6}{4} + 3\binom{n-6}{2} + \binom{n-6}{1} + \binom{n-6}{0}$$

$$|S_n^{(m)}(7)| = \binom{n-7}{7} + 5\binom{n-7}{5} + 6\binom{n-7}{3} + 2\binom{n-7}{2} + 3\binom{n-7}{1}$$

By the induction on l we see that $|S_n^{(m)}(l)|$ is a polynomial of n of degree l expressed in the form

$$\begin{aligned}&\binom{n-l}{l} + (l-2)\binom{n-l}{l-2} + \binom{l-3}{2}\binom{n-l}{l-4}\\ &+ a_{l-5}\binom{n-l}{l-5} + a_{l-6}\binom{n-l}{l-6} + \cdots + a_1\binom{n-l}{1} + a_0\binom{n-l}{0}\end{aligned}$$

for some nonnegative integers $a_0, a_1, \cdots, a_{l-5}$.

4. Primitive words

Primitive words in the title means nearly decomposable and special words and denote the set consisting of these words on $[n]$ by $S_n^{(p)}$. In this section we shall determine $S_n^{(p)}$ explicitly for $n = 5, 6, 7, 8$. We first make a remark on nearly decomposable words.

Lemma 13. *Let $x = x_1 \cdots x_n$ be a nealy decomposable word. If $x_{i-1} > x_i > x_{i+1}$ for some i then $x_i = i$.*

Proof. $x \to x_1 \cdots x_{i-2} x_i x_{i+1} x_{i-1} x_{i+2} \cdots x_n$ is decomposable implies that $x_1 \cdots x_{i-2} x_i$ is a subword on $[i-1]$ or $x_1 \cdots x_{i-2} x_i x_{i+1}$ is a subword on $[i]$ so that $x_i \leq i$. Similarly, $x \to x_1 \cdots x_{i-2} x_{i+1} x_{i-1} x_i x_{i+2} \cdots x_n$ is decomposable means that $x_1 \cdots x_{i-2} x_{i+1}$ is a subword on $[i-1]$ or $x_1 \cdots x_{i-2} x_{i+1} x_{i-1}$ is a subword on $[i]$ so that $x_i \geq i$. □

Now we start from special words on [5].

Proposition 14. $S_5^{(s)} = S_5^{(s,d)} = \{$ 21354, 13254, 21435, 21543, 32154 $\}$ *and* $S_5^{(p)} = S_5^{(s,i)} = \phi$.

Proof. It is easily seen that the five words in Proposition are special. Since the minimal even words on [5] are {21354, 13254, 21435} we shall show; (i) the special words covering these minimal words are 32154 and 21543, (ii) 32154 and 21543 are not covered by the special words. (i) The words covering 21354 are {32154, 23514, 25134, 21543}, where 23514 → 23145 → 12345 and 25134 → 12534 → 12345. Similarly, the words covering 13254 are {32154, 15324, 13542}, where 15324 → 12534 → 12345 and 13542 → 13425 → 12345. The same result holds for 21435 because 21435 = 13254*. (ii) The words covering 21543 are {52143, 25413}, where 52143 → 15243 → 12453 → 12345 and 25413 → 24153 → 12453 → 12345. The same result holds for 32154 because 32154 = 21543*. □

Next we consider special words on [6].

Proposition 15. $S_6^{(i,s)} = S_6^{(p)} = \{$ 246135, 245163, 416235, 415263 $\}$.

Proof. It is easily seen that the four words in Proposition are indecomposable and special. Let $x = abcdef \in S_6^{(p)}$ be a primitive word on [6]. We first show that x is one of the four words in Proposition 15, according to the position of the letter 6 in the word x. We have to consider the three cases $6 \in \{c, d, e\}$ by Theorem 3.

(I) $c = 6$. $x = ab6def \to abde6f \in S_6^{(d,s)}$ implies (i) $f = 5$ and $x = ab6de5$ or (ii) $(e,f) = (5,4)$ and $x = ab6d54$. (i) If $d > e$ then $x \to abe6d5 \to abed56 \in S_6^{(s)}$ implies $abed \in S_4^{(s)} = \{2143\}$, so $x = 216354$ is decomposable. If $b > d$ then $x \to adb6e5 \to adbe56 \in S_6^{(s)}$ implies $adbe = 2143$, from which we obtain a primitive word $x = 246135$. If $b < d < e$ then $b = 1$ and $a = 3$ or 4 so $x = a16de5$ is equal to 316245 odd or 416235, which is a primitive word. (ii) $x \to abd564 \to abd456$ so $abd \in S_3^{(s)} = \phi$.

(II) $d = 6$. $x = abc6ef \to abcef6 \in S_6^{(d,s)}$ implies $abcef \in S_5^{(s)}$. Proposition 14 shows that $\{a,b\} = \{1,2\}$ or $\{a,b,c\} = \{1,2,3\}$ so x is decomposable.

(III) $e = 6$ and $x = abcd6f$. If $d > f$ then $x \to abcfd6 \in S_6^{(d,s)}$ implies $abcfd \in S_5^{(s)}$, a contradiction as in the case (II). Hence we assume $d < f$ in what follows. Since $x = abcd6f$ with $d < f$ is indecomposable we see $5 \in \{a,b,c\}$. (i) $a = 5$. $x = 5bcd6f \to bc5d6h \in S_6^{(d,s)}$ implies $b = 1$ or $(b,c) = (2,1)$. If $b = 1$ then $x = 51cd6f \to 1c5d6f$ so $c5d6f$ is special. If $(b,c) = (2,1)$ then $x = 521d6f \to 152d6f$ so $52d6f$ is special. Both cases cannot occur by Proposition 14. (ii) $b = 5$. $x = a5cd6f \to acd56f \in S_6^{(d,s)}$ implies $(a,c) = (2,1)$ or $f = 4$. If $(a,c) = (2,1)$ then $x = 251d6f = 251463 \to 125463 \to 125346 \to 123456$. If $f = 4$ then $x = a5cd64$, which is equal to $253164 \to 231564 \to\to 123456$ or $x = 351264 \to 312564 \to\to 123456$. (iii) $c = 5$ and $x = ab5d6f$. If $b > d$ then $x \to adb56f$ implies $(a,d) = (2,1)$ or $f = 4$. If $(a,d) = (2,1)$ then $x = 2b516f = 245163$ is primitive. If $f = 4$ then $x = ab5d64$, which is equal to 235164 odd or $325164 \to 312564 \to\to 123456$. If $b < d$ then $b < d < f$ implies $b = 1$ and $x = a15d6f$, which is equal to 315264 odd or 415263 is primitive. Thus the set of primitive words on [6] consists of the four words in Proposition 15.

Next, we have to show that any words covering these four words are not special. The words covering the primitive words 246315, 245136, 415263 are

$$\begin{array}{lllll} 246315 \leftarrow & 462135, & 624135, & 246351, & 246513 \\ 245136 \leftarrow & 452163, & 524163, & 246513, & 245631 \\ 415263 \leftarrow & 541263, & 452163, & 416523, & 415632 \end{array}$$

respectvely. Thses twelve words are non-special, e.g. $462135 \to 421635 \to\to 123456$, $624135 \to 641235 \to 164235 \to 126435 \to 123645 \to$

123456, 246351 $\to$ 234651 $\to$ 234516 $\to\to$ 123456, 246513 $\to$ 241653 $\to\to$ 124536 $\to$ 123456. The same conclusion holds for the primitive word 416235 because 416235 = 245163*. Thus any indecomposable special words on [6] is primitive and Proposition is proved. □

Proposition 16. *$S_7^{(i,s)} = S_7^{(p)}$ and it consists of the following twelve words.*

2471365,	3257146,	5217346,	4162735,	4152763,	3516274
4172365,	3517246,	2461735,	2451763,	3256174,	5216374

Proof. Let $x = abcdefg$ be a primitive word on [7]. We first show that x is one of the twelve words in Proposition according to the position of the letter 7 in the word x, i.e. $7 \in \{c, d, e, f\}$.

(I) $c = 7$. $x = ab7defg \to abde7fg \in S_7^{(d,s)}$ implies $\{f, g\} = \{5, 6\}$ or $\{a, b, d\} = \{1, 2, 3\}$. If $(f, g) = (5, 6)$ then $x = ab7de56 \to abde756 \to abde567$ implies $abde \in S_4^{(s)} = \{2143\}$ so $x = 2173456$ is decomposable. Suppose $(f, g) = (6, 5)$. (i) If $b > d$ then $x = ab7de65 \to adb7e65 \in S_7^{(d,s)}$ implies $(a, d) = (2, 1)$ or $e = 4$. If $(a, d) = (2, 1)$ then

$$x = 2b71e65 = 2471365 \in S_7^{(p)}.$$

If $e = 4$ then $x = ab7d465$. Since $b > d$, x is equal to 2371465, 3172465, which are odd, or 3271465 $\to$ 3127465 $\to\to$ 1234675 $\to$ 1234567. (ii) If $d > e$ then $x = ab7de65 \to abe7d65 \in S_7^{(d,s)}$ implies $d = 4$ so $x \to abe7465 \to abe4675 \to abe4567 \in S_7^{(s)}$, which is impossible. (iii) If $b < d < e$ then $(b, d) = (1, 2)$ so

$$x = a172e65 = 4172365 \in S_7^{(p)}.$$

(II) $d = 7$. $x = abc7efg \to abcef7g \in S_7^{(d,s)}$ implies that $g = 6$ or $(f, g) = (6, 5)$. If $(f, g) = (6, 5)$ then $x = abc7e65 \to abce675 \to abce567 \in S_7^{(s)}$ so $abce = 2143$ and $x = 2147365$ is decomposable. Suppose $g = 6$ and $x = abc7ef6$. (i) If $e > f$ then $x \to abcf7e6 \to abcfe67$ so $abcfe \in S_5^{(s)}$. Then Proposition 14 shows $\{a, b\} = \{1, 2\}$ or $\{a, b, c\} = \{1, 2, 3\}$ and $x = abc\cdots$ is decomposable. (ii) If $c > e$ then $x = abc7ef6 \to abec7f6 \to abecf67$ so $abecf \in S_5^{(s)}$ with $e < c$. Since $a \neq 1$ and $ab \neq 21$ we see $abecf = 32154 \in S_5^{(s)}$ by Proposition 14 and

$$x = 3257146 \in S_7^{(p)}.$$

(iii) If $c < e < f$ then $1 \in \{b, c\}$. If $b = 1$ then $c = 2$ and $x = a127ef6 \to 12a7ef6$ so $a7ef6 \in S_5^{(s)}$, whcih is impossible by Proposition 14. If $c = 1$ then $x = ab17ef6 \to 1ab7ef6$. If $a = 2$ then $b7ef6 \in S_5^{(s)}$, which is impossible by Proposition 14, so $2 \in \{b, e\}$. If $b = 2$ then $ab7ef6 = a27ef6$ with $e < f$ is equal to 327456 (then $x = 3217456$ is decomposable) or 427356 (odd), or 527346, from which we obtain

$$x = 5217346 \in S_7^{(p)}.$$

If $e = 2$ then $ab7ef6 = ab72f6 \to a2b7f6 \to a2bf67 \in S_6^{(s)}$ implies $a2bf = 3254$, from which

$$x = 3517246 \in S_7^{(p)}.$$

(III) $e = 7$. $x = abcd7fg \to abcdfg7 \in S_7^{(d,s)}$ implies that $abcdfg \in S_6^{(s)}$. If this is decomposable then $g = 6$ and $abcdf \in S_5^{(s)}$, which is impossible by Proposition 14 because x is indecomposable. Hence $abcdfg \in S_6^{(i,s)}$. Then Proposition 15 shows that x is equal to

$$2461735, \quad 4162735, \quad 2451763, \quad 4152763.$$

These four words are primitive.

(IV) $f = 7$ and $x = abcde7g$. If $e > g$ then $x \to abcdge7 \in S_7^{(d,s)}$ so $abcdge \in S_6^{(s)}$. If this is decomposable then $e = 6$ and $abcdg \in S_5^{(s)}$, which is impossible by Proposition 14 because x is indecomposable. Hence $abcdge \in S_6^{(i,s)}$. Since $g < e$ we see from Proposition 15 that $abcdge = 246135$ or 416235, so $x = 2461573 \to 2415673 \to\to 1245367 \to 1234567$ or $x = 4162573 \to 4125673 \to\to 1245367 \to 1234567$. Thus we assume $e < g$ in what follows, which imlies $6 \in \{a, b, c, d\}$ because x is indecompoasble.

(i) $a = 6$. $x = 6bcde7g \to bc6de7g \in S_7^{(d,s)}$ so $b = 1$ or $(b, c) = (2, 1)$. If $(b, c) = (2, 1)$ then $x = 621de7g \to 162de7g \in S_7^{(d,s)}$ so $62de7g \in S_6^{(i,s)}$, which is impossible by Proposition 15. If $b = 1$ then $x = 61cde7g \to 1c6de7g$ so $c6de7g \in S_6^{(s)}$, which is decomposable by Proposition 15. Then $c = 2$ and $6de7g \in S_5^{(s)}$, which is impossible by Proposition 14.

(ii) $b = 6$. $x = a6cde7g \to acd6e7g \in S_7^{(d,s)}$ so $(a, c) = (2, 1)$ or $(e, g) = (4, 5)$ since $e < g$. If $(a, c) = (2, 1)$ then $x = 261de7g$ with

$e < g$, which is equal to $2615374 \to 1265374 \to 1236574 \to 1236457 \to 1234567$ or 2614375 odd or $2614375 \to 1263475 \to 1234675 \to 1234567$. If $(e, g) = (4, 5)$ then $x = a6cd475 \to acd6475$. Since 6475 is odd we see $acd = 213$ or 321 so $x = 2613475 \to 1263475 \to 1234675 \to 1234567$ or $x = 3621475 \to 2361475 \to 2314675 \to 1234567$.

(iii) $c = 6$. $x = ab6de7g \to abde67g \in S_7^{(d,s)}$ implies $g = 5$ (since $e < g$). Then $abde \in S_4^{(s)} = \{2143\}$, which implies x is decomposable.

(iv) $d = 6$ and $x = abc6e7g$. If $c > e$ then $x \to abec67g \in S_7^{(d,s)}$ so $g = 5$ or $(c, g) = (5, 4)$. If $g = 5$ then $abec \in S_4^{(s)} = \{2143\}$, which imples x is decomposable. If $(c, g) = (5, 4)$ then $x = ab56e74 \to abe5674$. Since 5674 is odd we see $abe = 321$ and

$$x = 3256174 \in S_7^{(p)}$$

If $c < e < g$ then $1 \in \{b, c\}$. If $b = 1$ then $x = a1c6e7g$ with $c < e < g$, which is equal to $x = 5126374$ odd or $x = 4126375 \to 1246375 \to 1234675 \to 1234567$. If $c = 1$ then $x = ab16e7g \to 1ab6e7g \in S_7^{(s)}$ so $ab6e7g \in S_6^{(s)}$. If this is decomposable then $a = 2$ and $b6e7g \in S_5^{(s)}$, which is impossible by Proposition 14. Hence $ab6e7g \in S_6^{(i,s)}$ with $e < g$, which is equal to 356274 or 526374 by Proposition 15, hence

$$x = 3516274, \qquad 5216374,$$

both of which are primitive. Thus the set $S_7^{(p)}$ of primitive words on [7] consists of the twelve words in Proposition 16.

Next we shall show that any word covering these twelve words are general. The words covering 2471365 are $\{4721365, 7241365, 2473615, 2476135, 2471653\}$, each of which is general because

$$\begin{aligned}
&4721365 \to 4217365 \to 1427365 \to 1423675 \to\to 1234567\\
&7241365 \to 7124365 \to 1274365 \to 1237465 \to 1234675 \to 1234567\\
&2473615 \to 2347615 \to 2346175 \to 2314675 \to\to 1234567\\
&2476135 \to 2417635 \to\to 1246375 \to 1234675 \to 1234567\\
&2471653 \to 2416753 \to\to 1245673 \to 1245367 \to 1234567
\end{aligned}$$

and the remaining eleven words are shown to be general similarly. □

For $n = 8$ we write down the primitive words without proof.

Proposition 17. *$S_8^{(p)}$ consists of the following 50 words.*

24813567,	23518467,	24613857,	24713586,	51627384
41823567,	23618457,	41623857,	41723586,	61427385
32581476,	24518367,	24516837,	24617583,	41527386
35182476,	25618347,	41526837,	41627583,	31627485
52183476,	26138457,	52163874,	41267385,	25617384
31468257,	61238457,	52173846,	24167385,	24517386
41268357,	31628457,	32571846,	23467185,	23617485
23468157,	41528367,	35172846,	31467285,	24613785
24168357,	51628347,	35162874,	61237485,	41623785
41526783,	61428357,	32561874,	26137485,	24516783

and the set of indecomposable special words consists of 150 words.

Lastly we propose a problem; what is the range of the number of inversions of primitive words, i.e. the set $P(n) = \{i(x)\ ;\ x \in S_n^{(p)}\}$? We see from the results in this section that $P(6) = \{8\}$, $P(7) = \{10\}$ and $P(8) = \{8, 10\}$.

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