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Ryukyu Math. J., 13(2000), 23-46

GENERIC G-LINEAR MAPS

TAKASHI MAEDA

ABSTRACT. A short proof is given for the existence of an $\mathrm{SL}(W) \times \mathrm{SL}(V)$-equivariant DG-algebra structure on the Eagon-Northcott complex associated with a linear map $V \to W^*$ over rational numbers, based on the Littlewood-Richardson rule. A generic G-linear map is defined and it is proved that the linear maps defined from the tensor product of three symmetric tensor representations of $\mathrm{GL}(d, \mathbb{Q})$ is generic.

This note consists of two parts. The first part is concerned with DG-algebra structures on finite free complex over a polynomial ring. It was constructed in [Sr] DG-algebra structures on the minimal free resolutions of cyclic modules R/I over a noether local ring R when I is (i) a power of an ideal generated by a regular sequence (over $\mathbb{Z}$) and (ii) the ideal of maximal minors of a generic $n \times m$ matrix (over rational numbers $\mathbb{Q}$). A simple proof for the existence of a DG-algebra structure in the case (i) was given in [M], assuming that R contains $\mathbb{Q}$. In this note we investigate a sufficient condition for the existence of a G-equivariant DG-algebra structure on complex with an action of a group G (Lemma 3) and deduce from it a short proof for the existence in the case (ii) above using the Littlewood-Richardson rule.

In the second part we consider the following problem on the tensor product of three representations of a group G all of whose finite-dimensional reprentations are completely reducible. We denote by V_λ, $V_\mu, \cdots$ irreducible representaions of G and by $V_{(\lambda)}$ the V_λ-isotypic component of a representaion V of G, i.e. $V_{(\lambda)}$ is the image of the canonical map from $V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, V)$ to V. A G-linear map $\varphi : V \to W$ restricts to each isotypic component : $\varphi_\lambda = \varphi|_{V_{(\lambda)}} : V_{(\lambda)} \to W_{(\lambda)}$.

Received November 30, 2000.

Definition. *A G-linear map $\varphi : V \to W$ is generic if, for each isotypic component $V_{(\lambda)}$ of V, the restriction φ_λ of φ to $V_{(\lambda)}$ is of maximal rank, that is, injective or surjective according to* $\dim V_{(\lambda)} \leq \dim W_{(\lambda)}$ *or* $\dim V_{(\lambda)} \geq \dim W_{(\lambda)}$.

For three representations V, U and W of G let $\varphi_\lambda : (V \otimes U)_{(\lambda)} \to V \otimes U$ and $\phi_\mu : U \otimes W \to (U \otimes W)_{(\mu)}$ be the canonical injection and the canonical projection, respectively, and let us consider the composite map

$$(V \otimes U)_{(\lambda)} \otimes W \xrightarrow{\varphi_\lambda \otimes 1} (V \otimes U) \otimes W \xrightarrow{1 \otimes \phi_\mu} V \otimes (U \otimes W)_{(\mu)} \qquad (1)$$

In general, (1) is not generic in the sense of Definition; a simple example for a finite group is in Section 4. However, for $G = \mathrm{GL}(d, \mathbb{Q})$, the canonical homomorphisms $S_{n+m} \otimes S_p \to S_n \otimes S_m \otimes S_p \to S_n \otimes S_{m+p}$ and $\wedge^n \otimes S_m \to \wedge^{n-1} \otimes S_1 \otimes S_m \to \wedge^{n-1} \otimes S_{m+1}$ are exapmles of generic maps where S_n (resp. $\wedge^n$) is the n-th symmetric (resp. exterior) representation of $\mathrm{GL}(d, \mathbb{Q})$. In view of this we consider

Problem. Is the composite map (1) generic for $\mathrm{GL}(d, \mathbb{Q})$ and for a connected reductive group over a field of characteristic zero ?

We may assume that V, U and W are irreducible in the Problem. In this note we show that the above problem is affirmative when V, U and W are symmetric tensor representations of $\mathrm{GL}(d, \mathbb{Q})$.

Theorem 1. *For five non-negative integers n, m, p, s, t such that $0 \leq s \leq min(n, m)$ and $0 \leq t \leq min(m, p)$ the composite map $\varphi_{s,t}$*

$$S_{n+m-s,s} \otimes S_p \xrightarrow{\varphi_s \otimes 1} S_n \otimes S_m \otimes S_p \xrightarrow{1 \otimes \phi_t} S_n \otimes S_{m+p-t,t} \qquad (2)$$

is generic in the sense of Definition where $S_{n+m-s,s}$ is the irreducible repesentation of $GL(d)$ associatd with the partition $(n + m - s, s)$ of $n + m$.

Theorem means that the restriction of $\varphi_{s,t}$ to each common irreducible submodules of $S_{n+m-s,s} \otimes S_p$ and $S_n \otimes S_{m+p-t,t}$, which is with multiplicity one by Pieri's formula, is a nonzero scalar multiple. Although this scalar is explicitly expressed in the case $d = 3$ (Lemma 5) it is not clear whether this scalar is nonzero or not. The outline of the proof of Theorem 1 is as follows. We reduce to the case $d = 3$

(the case $d = 2$ is included in the case $d = 3$) and prove the Theorem by induction on $p - t \geq 0$. The case $p - t = 0$ is proved by describing the composite map (2) explicitly (Corollary 6). If $p - t > 0$ then $m + p - t > t$ and we note in Lemma 7 that the diagram

$$\begin{array}{ccc} S_{n+m-s,s} \otimes S_p & \xrightarrow{\varphi_{s,t}} & S_n \otimes S_{m+p-t,t} \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ S_{n+m-s,s} \otimes S_{p-1} \otimes S_1 & \xrightarrow{\varphi'_{s,t} \otimes 1} & S_n \otimes S_{m+(p-1)-t,t} \otimes S_1 \end{array} \tag{3}$$

is commutative where ϕ_1 and ϕ_2 are induced from the polarizations $S_p \to S_{p-1} \otimes S_1$ and $S_{m+p-t,t} \to S_{m+p-1-t,t} \otimes S_1$. To prove Theorem 1 we have to show that if T and T' are common irreducible constituents of $S_{n+m-s,s} \otimes S_p$ and $S_n \otimes S_{m+p-t,t}$ respectively, then the restriction $\varphi_{s,t}|_T$ to T is nonzero. We prove in Lemma 9 that if $\varphi_{s,t} : S_{n+m-s,s} \otimes S_p \to S_n \otimes S_{m+p-t,t}$ is generic then so is $\varphi_{s,t} \otimes 1 : S_{n+m-s,s} \otimes S_p \otimes S_1 \to S_n \otimes S_{m+p-t,t} \otimes S_1$. Next we find an isomorphic irreducible submodules U and U' of $S_{n+m-s,s} \otimes S_{p-1}$ and $S_n \otimes S_{m+p-1-t,t}$ respectively, such that $U \otimes S_1$ and $U' \otimes S_1$ contain an irreducible constituent isomorphic to $T \cong T'$. Finally we show in Lemma 10 that the composite map

$$T \xrightarrow{\phi_1|_T} S_{n+m-s,s} \otimes S_{p-1} \otimes S_1 \xrightarrow{\pi} U \otimes S_1 \tag{4}$$

is nonzero where π is the projection. Since $\varphi'_{s,t} : S_{n+m-s,s} \otimes S_{p-1} \to S_n \otimes S_{m+(p-1)-t,t}$ is generic by the inductive hypothesis the restriction $\varphi'_{s,t}|_U$ to U is an isomorphism so the composite map

$$T \xrightarrow{\phi_1|_T} S_{n+m-s,s} \otimes S_{p-1} \otimes S_1 \xrightarrow{\varphi'_{s,t} \otimes 1} S_n \otimes S_{m+p-1-t,t} \otimes S_1$$

is nonzero since (4) is nonzero. Thus the commutativity of the diagram (3) implies that the restriction $\varphi_{s,t}|_T$ to T is nonzero.

The paper is organized as follows. In Section one we review briefly algebra structures on a complex and prove the existence of $\mathrm{SL}(V) \times \mathrm{SL}(W)$-equivariant DG-algebra structure on Eagon-Northcott complex. In Section two we collect identities in invariant theory of $\mathrm{GL}(3)$ [GY,p246] which are used in Section three. We give a proof of Theorem 1 in Section three and a counterexample of the Problem for the symmetric group of degree three in Section four. The base field is rational numbers $\mathbb{Q}$. The irreducible representation of $\mathrm{GL}(d)$ associated with a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$ is denoted by $S_\lambda = S_{\lambda_1, \cdots, \lambda_d}$.

1. DG-algebra structures

Let $V = \mathbb{Q}^n$ be the vector space of dimension n over $\mathbb{Q}$ and let $R = S_*V = \oplus_{d\geq 0} R_d$ be the symmetric algebra of V over $\mathbb{Q}$. For a reductive subgroup G of $\mathrm{GL}(V)$ let I be a G-stable ideal of R with a graded R-free resolution of R/I

$$\mathbb{Y} : \cdots \to Y_2 \overset{d_2}{\to} Y_1 \overset{d_1}{\to} Y_0 = R \to R/I \to 0$$

which is we assume G-equivariant. Then the tensor products $\mathbb{Y}\otimes\mathbb{Y}$ and $\mathbb{Y}\otimes\mathbb{Y}\otimes\mathbb{Y}$ are graded R-free complex augmented by R/I with the natural G-actions which are induced from that on $\mathbb{Y}$. The complex $\mathbb{Y}$ has a DG-algebra structure means that there is a graded homomorphism of complex $\mu : \mathbb{Y}\otimes\mathbb{Y} \to \mathbb{Y}$ by which $\mathbb{Y}$ becomes a commutative associative DG (differential graded) algebra, i.e.

(commutativity) $\mu(x\otimes y) = (-1)^{\deg x\cdot\deg y}\cdot\mu(y\otimes x)$
(associativity) $\mu(\mu(x\otimes y)\otimes z) = \mu(x\otimes\mu(y\otimes z))$
(Leibniz rule) $d(\mu(x\otimes y)) = dx\otimes y + (-1)^{\deg x}\cdot x\otimes d(y)$

for all homogeneous elements $x, y, z \in \mathbb{Y}$. A DG-algebra structure is said to be G-equivariant if the map μ is so. Koszul complex is a typical example with a DG-algebra structure while there are finite minimal free resolutions of cyclic modules which do not admit a DG-algebra structures [A2,p21]. We first show a Lemma, which follows from the complete reducibility of the reductive group G.

Lemma 2. *There is a G-equivariant map of complex $\mu : \mathbb{Y}\otimes\mathbb{Y}\to\mathbb{Y}$ inducing the identity on $(\mathbb{Y}\otimes\mathbb{Y})_0 = \mathbb{Y}_0 = R$.*

Proof. We construct $\mu_{r,s} : Y_r\otimes Y_s \to Y_{r+s}$ by the induction on the total degree $r+s$. Let $W\otimes R(-m)$ be a direct factor of $Y_r\otimes Y_s$ with an irreducible G-module W and let us consider the diagram

$$\begin{array}{ccccc}
W & \xrightarrow{\delta_{r+s}} & (\mathbb{Y}\otimes\mathbb{Y})^{(m)}_{r+s-1} & \xrightarrow{\delta_{r+s-1}} & (\mathbb{Y}\otimes\mathbb{Y})^{(m)}_{r+s-2} \\
 & & \downarrow{\scriptstyle \mu_{r+s-1}} & & \downarrow{\scriptstyle \mu_{r+s-2}} \\
Y^{(m)}_{r+s} & \xrightarrow{d_{r+s}} & Y^{(m)}_{r+s-1} & \xrightarrow{d_{r+s-1}} & Y^{(m)}_{r+s-2}
\end{array}$$

where $Y^{(m)}_{r+s}$ is the degree m component of the graded R-module Y_{r+s} etc. Assume μ_i are constructed for all $i \leq r+s-1$. Then

$$d_{r+s-1}\circ\mu_{r+s-1}\circ\delta_{r+s} = \mu_{r+s-2}\circ(\delta_{r+s-1}\circ\delta_{r+s}) = 0$$

so the image of $\mu_{r+s-1} \circ \delta_{r+s}$ is contained in Ker d_{r+s-1} = Image d_{r+s}. Since $d_{r+s} : Y^{(m)}_{r+s} \to$ Image d_{r+s} is G-split there is a G-homomorphism $\mu_{r,s} : W \to Y^{(m)}_{r+s}$ with $d_{r+s} \circ \mu_{r,s}$ equal to $\mu_{r+s-1} \circ \delta_{r+s}$. $\square$

For the G-equivariant map of complex μ above we set $\xi = \mu \circ (\mu \otimes 1) - \mu \circ (1 \otimes \mu) : \mathbb{Y} \otimes \mathbb{Y} \otimes \mathbb{Y} \to \mathbb{Y}$, which is a G-equivariant map of complex. For $r, s, t \geq 0$ let

$$Y_r \otimes Y_s \otimes Y_t = \oplus\, W_i \otimes R(-m_i) \qquad 1 \leq i \leq k(r,s,t) \tag{5}$$

be the direct sum decomposotion with irreducible G-modules W_i and let us consider the diagram

$$\begin{array}{ccccc} & & W & \xrightarrow{\delta_{r+s}} & (\mathbb{Y} \otimes \mathbb{Y} \otimes \mathbb{Y})^{(m)}_{r+s+t-1} \\ & & {\scriptstyle \xi_{r,s,t}}\downarrow & & \downarrow{\scriptstyle \xi_{r+s+t-1}} \\ Y^{(m)}_{r+s+t+1} & \xrightarrow[d_{r+s+t+1}]{} & Y^{(m)}_{r+s+t} & \xrightarrow[d_{r+s+t}]{} & Y^{(m)}_{r+s+t-1} \end{array}$$

The following Lemma gives a sufficient condition for the map ξ to be zero, which implies that the homomorphism of complex $\mu : \mathbb{Y} \otimes \mathbb{Y} \to \mathbb{Y}$ defines a multiplication of a G-equivariant DG-algebra structure on the complex $\mathbb{Y}$.

Lemma 3. *If, for all $r, s, t \geq 0$, the homogeneous component $Y^{(m_i)}_{r+s+t+1}$ of degree m_i in the $r+s+t+1$-th component $Y_{r+s+t+1}$ of $\mathbb{Y}$, contains no G-submodules isomorphic to irreducible G-module W_i for any $1 \leq i \leq k(r,s,t)$ in (5) then $\xi : \mathbb{Y} \otimes \mathbb{Y} \otimes \mathbb{Y} \to \mathbb{Y}$ is a zero map.*

Proof. We shall show the lemma by the induction on the total degree $r+s+t$. In the diagram above we see $d_{r+s+t} \circ \xi_{r,s,t} = \xi_{r+s+t-1} \circ \delta_{r+s+t} = 0$ by the inductive hypothesis $\xi_{r+s+t-1} = 0$ so the Image $\xi_{r,s,t}$ is contained in Ker d_{r+s+t} = Image $d_{r+s+t+1}$. The hypothesis of Lemma implies Image $\xi_{r,s,t} = 0$ since ξ and d are G-equivariant. $\square$

Now we show that the Eagon-Northcott complex satisfies the hypothesis of Lemma 2 so that it has a DG-algebra structure. Let $W = \mathbb{Q}^m$ (resp. $V = \mathbb{Q}^n$) be the vector space of dimension m (resp. n) over $\mathbb{Q}$ with the dual vector spce W^* of W and let $R = S_*(W \otimes V) = \oplus_{d \geq 0} S_d$ be the polynomial ring of mn variables over $\mathbb{Q}$. Assume $n > m$ and let $\phi : V = \mathrm{id} \otimes V \subset W^* \otimes W \otimes V$ be the canonical homomorphism.

We note that all maps below together with ϕ are $\mathrm{SL}(W) \times \mathrm{SL}(V)$-equivariant. By the isomorphisms $\wedge^m W^* = \mathbb{Q}$ we see $\wedge^m \phi$ induces a map

$$\wedge^m V \to \wedge^m W^* \otimes S_m(W \otimes V) = S_m$$

hence a map of free R-modules $\wedge^m V \otimes R(-m) \to R$ with the image equal to the ideal I generated by the maximal minors of the generic $n \times m$-matrix. The graded minimal R-free resolution of R/I is given by the Eagon-Northcott complex $\mathbb{E}$ of length $n - m + 1$ [S,p182] :

$$\mathbb{E} \ : 0 \to Y_{n-m+1} \stackrel{d_{n-m+1}}{\to} Y_{n-m} \to \cdots \to Y_2 \stackrel{d_2}{\to} Y_1 \stackrel{d_1}{\to} Y_0 = R$$
$$\text{where} \quad Y_{k+1} = F_k \otimes R(-m-k), \qquad F_k = S_k W \otimes \wedge^{m+k} V \qquad (6)$$

The differential $d_{k+1} : Y_{k+1} = F_k \otimes R(-m-k) \to Y_k = F_{k-1} \otimes R(-m-k+1)$ is induced from the canonical map

$$D_k \otimes \wedge^{m+k} \subset D_{k-1} \otimes W \otimes V \otimes \wedge^{m+k-1} = D_{k-1} \otimes \wedge^{m+k-1} \otimes S_1$$

where we set $D_k = S_k W$ and $\wedge^{m+k} = \wedge^{m+k} V$. We see from (6) that

$$\begin{aligned} Y_r \otimes Y_s \otimes Y_t &= F_{r-1} \otimes F_{s-1} \otimes F_{t-1} \otimes R(-3m - r - s - t + 3) \\ Y_{r+s+t+1}^{(3m+r+s+t-3)} &= F_{r+s+t} \otimes R(-m - r - s - t)^{(3m+r+s+t-3)} \\ &= F_{r+s+t} \otimes S_{2m-3}. \end{aligned}$$

For the condition on Lemma 2 we have to show that there are no common $\mathrm{SL}(W) \times \mathrm{SL}(V)$-irreducible components of $F_{r-1} \otimes F_{s-1} \otimes F_{t-1}$ and $F_{r+s+t} \otimes S_{2m-3}$, where $F_{r-1} \otimes F_{s-1} \otimes F_{t-1}$ is equal to

$$D_{r-1} \otimes D_{s-1} \otimes D_{t-1} \otimes \wedge^{m+r-1} \otimes \wedge^{m+s-1} \otimes \wedge^{m+t-1}$$

and $F_{r+s+t} \otimes S_{2m-3}$ is isomphic to

$$D_{r+s+t} \otimes \wedge^{m+r+s+t} \otimes \bigoplus_{\lambda \vdash 2m-3} (S_\lambda W \otimes S_\lambda V)$$

by Cauchy's formula [Mc,p63] where $S_\lambda W$ is the irreducible representation of $\mathrm{GL}(W)$ associated with the partition λ of $2m - 3$. Hence, as $\mathrm{GL}(W)$-modules, $F_{r-1} \otimes F_{s-1} \otimes F_{t-1}$ (resp. $F_{r+s+t} \otimes S_{2m-3}$) is a direct sum of

$$D_{r-1} \otimes D_{s-1} \otimes D_{t-1} \qquad (7)$$
$$(\text{resp.} \quad D_{r+s+t} \otimes S_\lambda W \quad \text{for } \lambda \vdash 2m-3)$$

We denote the $\mathrm{GL}(W) = \mathrm{GL}(m)$-irreducible components of (7) by the associated partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$. Then we see from the Littlewood-Richardson rule [Mc,p142] that

$$\lambda_1 - \lambda_m \leq \lambda_1 \leq (r-1) + (s-1) + (t-1) \tag{8}$$

for all partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ which appear in $D_{r-1} \otimes D_{s-1} \otimes D_{t-1}$ while

$$\mu_1 - \mu_m \geq (r+s+t) - \mu_m \geq r+s+t-2 \tag{9}$$

for all partitions $\mu = (\mu_1 \geq \cdots \geq \mu_m \geq 0)$ which appear in $D_{r+s+t} \otimes S_\lambda W$ with $\lambda \vdash 2m-3$, because, if $\mu_m \geq 3$ then $\mu_i \geq \mu_m \geq 3$ for all i so $3(m-1) \leq \mu_2 + \cdots + \mu_m \leq |\lambda| = 2m-3$, a contradiction. On the other hand two partitions λ and μ determine the same representation of $\mathrm{SL}(m)$ if and only if $\lambda_i - \lambda_{i+1} = \mu_i - \mu_{i+1}$ for all $1 \leq i \leq m-1$, hence, in particular, $\lambda_1 - \lambda_m = \mu_1 - \mu_m$. Therefore (8) and (9) imply that the two modules (7) have no common $\mathrm{SL}(W)$-components and the hypothiesis of Lemma 2 is satisfied. Thus we conclude that the Eagon-Northcott complex has an $\mathrm{SL}(W) \times \mathrm{SL}(V)$-equivariant DG-algebra structure.

Remark. An $\mathrm{SL}(W) \times \mathrm{SL}(V)$-equivariant multiplication $\mu : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ is defined as follows [cf.Sr,p184].

$$\mu_{r+1,s+1} : Y_{r+1} \otimes Y_{s+1} \to Y_{r+s+2} \otimes R(-m+1)$$

$$\begin{aligned} \mu_{r+1,s+1} : F_r \otimes F_s &\overset{1\otimes\theta}{\to} D_r \otimes D_s \otimes \wedge^{m-1} \otimes \wedge^{m+r+s+1} \overset{1\otimes\wedge^{m-1}\phi}{\longrightarrow} \\ &D_r \otimes D_s \otimes W \otimes S_{m-1} \otimes \wedge^{m+r+s+1} \to F_{r+s+1} \otimes S_{m-1} \end{aligned}$$

Here θ and $\wedge^{m-1}\phi$ are canonical homorphisms (where $\wedge^k = \wedge^k V$)

$$\theta : \wedge^{m+r} \otimes \wedge^{m+s} \to \wedge^{m-1} \otimes \wedge^{r+1} \otimes \wedge^{m+s} \to \wedge^{m-1} \otimes \wedge^{m+r+s+1}$$

$$\wedge^{m-1}\phi : \wedge^{m-1} \to \wedge^{m-1} W^* \otimes S_{m-1}(W \otimes V) = W \otimes S_{m-1}$$

Any homomorphism of complex $\mu : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$, which is $\mathrm{SL}(W) \times \mathrm{SL}(V)$-equivariant, automatically satifies the associative law by our Lemma 3.

2. GL(3)-linear maps

Let $\{x_1, x_2, x_3\}$ be a basis of the three-dimensional vector space W with the dual basis $\{u_1, u_2, u_3\}$ of the dual space W^*. We denote by $S_n W = S_n^{(x)}$ (resp. $S_n W^* = S_{n,n}^{(u)}$) the representation space of GL(3) with the basis consisting of the monomoials of $\{x_1, x_2, x_3\}$ (resp. $\{u_1, u_2, u_3\}$) of degree n, and denote a vector of $S_n^{(x)}$ (resp. $S_{n,n}^{(u)}$) by

$$a_x^n = \sum_{i+j+k=n} \binom{n}{ijk} a_{ijk} x_1^i x_2^j x_3^k, \quad u_\alpha^n = \sum_{i+j+k=n} \binom{n}{ijk} \alpha_{ijk} u_1^i u_2^j u_3^k$$

for scalars a_{ijk}, α_{pqr}. The basis of the tensor product $S_n^{(x)} \otimes S_{n,n}^{(u)}$ consists of the monomials of $\{x_i\}$ and $\{u_i\}$ of bidegree (n, n) so a vector of $S_n^{(x)} \otimes S_{n,n}^{(u)}$ is written by $a_x^n u_\alpha^n$. We write $a_{ijk} = a_i a_j a_k$ and $\alpha_{ijk} = \alpha_i \alpha_j \alpha_k$ symbolically, and the symbolic factors

$$a_\alpha = \sum_{i=1}^{3} a_i \alpha_i, \qquad (axy) = \begin{vmatrix} a_1 & x_1 & y_1 \\ a_2 & x_2 & y_2 \\ a_3 & x_3 & y_3 \end{vmatrix}$$

where $\{y_1, y_2, y_3\}$ is a basis of W which are transformed in the same way as $\{x_1, x_2, x_3\}$, and the differential operators

$$(y\frac{\partial}{\partial x}) = \sum_{i=1}^{3} y_i \frac{\partial}{\partial x_i}, \qquad (\frac{\partial^2}{\partial x \partial u}) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i \partial u_i}$$

$$(\partial_x \partial_y u) = \begin{vmatrix} \partial/\partial x_1 & \partial/\partial y_1 & u_1 \\ \partial/\partial x_2 & \partial/\partial y_2 & u_2 \\ \partial/\partial x_3 & \partial/\partial y_3 & u_3 \end{vmatrix}, \qquad (xy\partial_u) = \begin{vmatrix} x_1 & y_1 & \partial/\partial u_1 \\ x_2 & y_2 & \partial/\partial u_2 \\ x_3 & y_3 & \partial/\partial u_3 \end{vmatrix}$$

which are SL(3)-invariant by definition. The differential operator $(\partial^2/\partial x \partial u)$ defines the contraction map $S_n^{(x)} \otimes S_{m,m}^{(u)} \to S_{n-1}^{(x)} \otimes S_{m-1,m-1}^{(u)}$, so a vector $a_x^n u_\alpha^m$ of $S_n^{(x)} \otimes S_{m,m}^{(u)}$ is contained in the irreducible subspace $S_{n+m,m}^{(x,u)}$ if and only if

$$(\frac{\partial^2}{\partial x \partial u}) a_x^n u_\alpha^m = nm \cdot a_\alpha a_x^{n-1} u_\alpha^{m-1} = 0 \tag{10}$$

We write $a_\alpha = 0$ symbolically if (10) holds.

Lemma 4. *(i) For $0 \le s \le min(n,m)$ the image of a vector $a_x^n b_y^m$ of $S_n^{(x)} \otimes S_m^{(y)}$ under the projection $S_n^{(x)} \otimes S_m^{(y)} \to S_{n+m-s,s}^{(x,u)}$ is given by*

$$(u\partial_x\partial_y)^s\{a_x^n b_y^m\}_{(y=x)} = (abu)^s a_x^{n-s} b_x^{m-s}. \tag{11}$$

The equality in (11) (and in what follows) means modulo a nonzero scalar multiple when the actual scalar is not important.

(ii) For $0 \le s \le min(n,m)$ the image of a vector $a_x^{n+m-2s} u_\alpha^s$ of $S_{n+m-s,s}^{(x,u)}$ under the injection $S_{n+m-s,s}^{(x,u)} \to S_n^{(x)} \otimes S_m^{(y)}$ is given by

$$(xy\partial_u)^s (y\partial_x)^{m-s}\{a_x^{n+m-2s} u_\alpha^s\} = a_x^{n-s} a_y^{m-s} (xy\alpha)^s.$$

(iii) For a vector $a_x^{n-m} u_\alpha^m$ (resp. b_y^p) of $S_{n,m}^{(x,u)}$ (resp. $S_p^{(y)}$) let

$$\begin{aligned} P &= (\frac{\partial^2}{\partial u \partial y})^t_{(y=x)} (u\partial_x\partial_y)^s \{a_x^{n-m} u_\alpha^m \cdot b_y^p\} \\ &= b_\alpha^t u_\alpha^{m-t} (uab)^s a_x^{n-m-s} b_x^{p-s-t} \end{aligned}$$

Then the image of a vector $a_x^{n-m} u_\alpha^m b_y^p$ of $S_{n,m}^{(x,u)} \otimes S_p^{(y)}$ under the projection

$$\pi : S_{n,m}^{(x,u)} \otimes S_p^{(y)} \to S_{n+(p-s-t),m+s,t}^{(x,u)} \tag{12}$$

where $0 \le s \le n-m$ and $0 \le t \le m$, is given by

$$P + \sum_{k \ge 1} \lambda_k u_x^k \cdot b_\alpha^{t+k} u_\alpha^{m-t-k} (uab)^s a_x^{n-m-s} b_x^{p-s-t-k}.$$

Here $\lambda_k \in \mathbb{Q}$ are determined so as to be annihilated by $(\partial^2/\partial x \partial u)$.

(iv) For a vector $a_x^M u_\alpha^N$ of $S_{n+(p-s-t),m+s,t}^{(x,u)}$ with $M = n + (p-s-t) - (m+t)$ and $N = (m+s) - t$ let

$$\begin{aligned} Q &= u_y^t (xy\partial_u)^s (y\partial_x)^{p-s-t} \{a_x^M u_\alpha^N\} \\ &= u_y^t u_\alpha^{N-s} (xy\alpha)^s a_x^{M-(p-s-t)} a_y^{p-s-t} \end{aligned}$$

Then the image of a vector $a_x^M u_\alpha^N$ of $S_{n+(p-s-t),m+s,t}^{(x,u)}$ under the injection

$$S_{n+(p-s-t),m+s,t}^{(x,u)} \to S_{n,m}^{(x,u)} \otimes S_p^{(y)} \tag{13}$$

where $0 \leq s \leq n-m$ *and* $0 \leq t \leq m$, *is given by*

$$Q + \sum_{k \geq 1} \mu_k u_x^k \cdot u_y^{t-k} u_\alpha^{N-s} (xy\alpha)^s a_x^{M-(p-s-t)-k} a_y^{p-s-t+k}$$

Here $\mu_k \in \mathbb{Q}$ *are determined so as to be annnihilated by* $(\partial^2/\partial x \partial u)$.

Proof. (i) follows since (11) is annihilated by $(\partial^2/\partial x \partial u)$. (ii) By (i) we have to show

$$(u\partial_x \partial_y)^i_{(y=x)} a_x^{n-s} a_y^{m-s} (xy\alpha)^s = \begin{cases} \lambda \cdot a_x^{n+m-2s} u_\alpha^s & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}$$

for a nonzero $\lambda \in \mathbb{Q}$. This follows from Lemma 5(i) since $a_\alpha = 0$ by the irreducibility of $a_x^{n+m-2s} u_\alpha^s$. (iii) The projection (12) factors through

$$S^{(x,u)}_{n,m} \otimes S^{(y)}_p \xrightarrow{\varphi} S^{(x)}_{n+(p-s-t)-(m+s)} \otimes S^{(u)}_{m+s-t,m+s-t} \to S^{(x,u)}_{n+(p-s-t),m+s-t}$$

and φ takes the vector $a_x^{n-m} u_\alpha^m b_y^p$ to P, which is contained in the subspace $\sum_{k \geq 0} S^{(x,u)}_{n+(p-s-t)-k,m+s,t+k}$ of $S^{(x)}_{n+(p-s-t)-(m+s)} \otimes S^{(u)}_{m+s-t,m+s-t}$. Hence the image $\pi(a_x^{n-m} u_\alpha^m b_y^p)$ in $S_{n+(p-s-t),m+s-t}$ is obtained by adding to P a vector which is zero mod u_x so as to be annihilated by $(\partial^2/\partial x \partial u)$. (iv) The injection (13) factors through

$$S^{(x,u)}_{n+(p-s-t),m+s,t} \xrightarrow{\varphi} S^{(x)}_{n-m} \otimes S^{(u)}_{m,m} \otimes S^{(y)}_p \to S^{(x,u)}_{n,m} \otimes S^{(y)}_p$$

and φ takes the vector $a_x^M u_\alpha^N$ to Q, which is contained in the $S_{n+(p-s-t),m+s,t}$-isotypic component of $S^{(x)}_{n-m} \otimes S^{(u)}_{m,m} \otimes S^{(y)}_p$. Hnece the image of $a_x^M u_\alpha^N$ in $S^{(x,u)}_{n,m} \otimes S^{(y)}_p$ is obtained by adding to Q a vector which is zero mod u_x so as to be annihilated by $(\partial^2/\partial x \partial u)$. □

Lemma 5. *(i)* $(u\partial_x \partial_y) a_x^n b_y^m (xy\alpha)^s$ *is equal to*

$$\begin{aligned} &nm a_x^{n-1} b_y^{m-1} (xy\alpha)^{s-1} + s(n+m+s+1) u_\alpha a_x^n y^m (xy\alpha)^{s-1} \\ &\quad - s(n a_\alpha u_x b_y + m a_x u_y b_\alpha) a_x^{n-1} b_y^{m-1} (\alpha xy)^{s-1} \end{aligned} \tag{14}$$

(ii) For $0 \leq k \leq min(n,m)$

$$(x\partial_y)^k a_y^n b_y^m = k! \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} a_x^i a_y^{n-i} b_x^{k-i} b_y^{m-k+i} \tag{15}$$

(iii) For $0 \le s \le min(m,n,p,q)$

$$(u\partial_x\partial_y)^s a_x^m b_x^n a_y^p b_y^q = (uab)^s \sum_{t=0}^{s} c_t \cdot a_x^{m-t} b_x^{n-s+t} a_y^{p-s+t} b_y^{q-t} \tag{16}$$

$$\textit{where} \quad c_t = (-1)^t \binom{s}{t} \frac{m!}{(m-s+t)!} \frac{q!}{(q-s+t)!} \frac{n!}{(n-t)!} \frac{p!}{(p-t)!}$$

Proof. (i) $(u\partial_x\partial_y)a_x^n b_y^m (xy\alpha)^s$ is equal to

$$\begin{aligned} & a_x^n\{m(u\partial_x b)b_y^{m-1}(xy\alpha)^s + b_y^m s(\widehat{xu\partial_x}\alpha)(xy\alpha)^{s-1}\} \\ = & -mb_y^{m-1}(ub\partial_x)a_x^n(xy\alpha)^s - sb_y^m(u\partial_x\widehat{x\alpha})a_x^n(xy\alpha)^{s-1} \end{aligned} \tag{17}$$

where $(ub\partial_x)a_x^n(xy\alpha)^s$ is equal to

$$\begin{aligned} & n(uba)a_x^{n-1}(xy\alpha)^s + a_x^n s(ub\widehat{y\alpha})(xy\alpha)^{s-1} \\ = & -n(abu)a_x^{n-1}(xy\alpha)^s + sa_x^n(u_y b_\alpha - u_\alpha b_y)(xy\alpha)^{s-1} \end{aligned} \tag{18}$$

$$\text{and} \quad (u\partial_x\widehat{x\alpha})a_x^n(xy\alpha)^{s-1} = \{(\partial_x)_\alpha u_x - u_\alpha(\partial_x)_x\}a_x^n(xy\alpha)^{s-1} \tag{19}$$

Here $(\partial_x)_\alpha u_x a_x^n (xy\alpha)^{s-1} = \sum_{i=1}^3 \alpha_i \partial_{x_i} u_x a_x^n (xy\alpha)^{s-1}$ is equal to

$$\begin{aligned} & \alpha_1\{u_1 a_x(xy\alpha) + u_x n a_1(xy\alpha)\}a_x^{n-1}(xy\alpha)^{s-2} + \cdots \\ = & (u_\alpha a_x + n u_x a_\alpha)a_x^{n-1}(xy\alpha)^{s-1} \end{aligned} \tag{20}$$

and $(\partial_x)_x a_x^n (xy\alpha)^{s-1} = \sum_{i=1}^3 \partial_{x_i} x_i a_x^n (xy\alpha)^{s-1}$ is equal to

$$\begin{aligned} & a_x^n(xy\alpha)^{s-1} + x_1\{na_1(xy\alpha) + a_x(s-1)(y\alpha)_{23}\}a_x^{n-1}(xy\alpha)^{s-2} + \cdots \\ = & (3+n+s-1)a_x^n(xy\alpha)^{s-1} \end{aligned} \tag{21}$$

Substituting (20) and (21) into (19) and then (18) and (19) into (17) we see that $(\partial_x\partial_y)a_x^n b_y^m (xy\alpha)^s$ is

$$\begin{aligned} & -mb_y^{m-1}\{-n(abu)a_x^{n-1}(xy\alpha)^{s-1} + s(u_y b_\alpha - u_\alpha b_y)a_x^n(xy\alpha)^{s-1}\} \\ & -sb_y^m\{(u_\alpha a_x + n u_x a_\alpha)a_x^{n-1}(xy\alpha)^{s-1} - (3+n+s-1)u_\alpha a_x^n(xy\alpha)^{s-1}\} \end{aligned}$$

which is equal to (14). (ii) When $k=1$ we see

$$
\begin{aligned}
(x\partial_y)a_y^n b_y^m &= x_1(na_1b_y + a_y mb_1)a_y^{n-1}b_y^{m-1} + \cdots \\
&= na_x a_y^{n-1}b_y^m + ma_y^m b_x b_y^{m-1}
\end{aligned}
$$

(15) follows from this and the induction on k. (iii) When $s=1$ we see

$$
\begin{aligned}
(u\partial_x\partial_y)a_x^m b_x^n a_y^p b_y^q &= a_x^m b_x^n (p(u\partial_x a)b_y + a_y(u\partial_x b))a_y^{p-1}b_y^{q-1} \\
&= (uab)(-npa_x^m b_x^{n-1}a_y^{p-1}b_y^q + mqa_x^{m-1}b_x^n a_y^p b_y^{q-1})
\end{aligned}
$$

(16) follows from this and the induction on s. $\square$

3. Proof of Theorem 1

First we reduce the proof of Theorem 1 to the case $d=3$. The case $d=2$ follows from the proof in the case $d=3$. If $\{x_1, \cdots, x_d\}$ is a standard basis of the $d(\geq 3)$-dimensional vector space V then the monomials of degree n in $\{x_1, \cdots, x_d\}$ is a basis of $S_n = Sym^n V$ on which the action of GL(d) is indueced from the action on $\{x_1, \cdots, x_d\}$. Let $W = \langle x_1, x_2, x_3\rangle$ be the three-dimensional subspace of V generated by $\{x_1, x_2, x_3\}$ and GL(3) = GL$(W)\times\{id_{d-3}\}$ be the canonical subgroup of GL(d) = GL(V). Then the irreducible components of $S_nV\otimes S_mV\otimes S_pV$ are of the forms $S_{i,j,k}V$ by Littlewood-Richardson rule so the decomposition into irreducible components of GL(d)-module $S_nV\otimes S_mV\otimes S_pV$ is exactly the same as that of GL(3)-module $S_nW\otimes S_mW\otimes S_pW$. It follows from this that if $\varphi_{s,t} = (1\otimes\phi_t)\circ(\varphi_s\otimes 1)$ is generic in the case $d=3$ then so is in general d.

Let $(n, m, p\,; s, t\,; i, j, k, l)$ be nine integers such that

$$
\begin{aligned}
&0 \leq s \leq \min(n,m), \quad 0 \leq t \leq \min(m,p) && (22)\\
&0 \leq i \leq n+m-2s, \quad 0 \leq j \leq s, \quad i+j \leq p && (23)\\
&0 \leq k \leq m+p-2t, \quad 0 \leq l \leq t, \quad k+l \leq n &&
\end{aligned}
$$

and let

$$
T = S_{(n+m-s)+(p-i-j),s+i,j} \qquad T' = S_{(m+p-k)+(n-k-l),t+k,l}
$$

be an irreducible component of $S_{n+m-s,s} \otimes S_p$ and $S_n \otimes S_{m+p-t,t}$ respectively. If we set

$$M = (n+m+p) - 2(s+i) - j, \qquad N = (s+i) - j$$
$$M' = (n+m+p) - 2(t+k) - l, \qquad N' = (t+k) - l$$

then there are isomorphisms $T \cong S_{M,N}$ and $T' \cong S_{M',N'}$ of SL(3)-modules. Suppose T and T' are isomorphic GL(3)-modules. Then $l = j$ and $t + k = s + i$, i.e.

$$l = j, \qquad k = s + i - t \tag{24}$$

so the ranges of i and j are

$$\begin{gathered} \max(0, t-s) \leq i \leq \min(n+m-2s, m+p-s-t) \\ 0 \leq j \leq \min(s,t), \qquad i+j \leq \min(p, n-s+t) \end{gathered} \tag{25}$$

Let us consider the compostion of GL(3)-maps.

$$T \xrightarrow{\varphi_1} S^{(x,u)}_{n+m-s,s} \otimes S^{(z)}_p \xrightarrow{\varphi_2} S^{(x)}_n \otimes S^{(y)}_m \otimes S^{(z)}_p \xrightarrow{\varphi_3} S^{(x)}_n \otimes S^{(z,u)}_{m+p-t,t} \tag{26}$$

where $\varphi_1, \varphi_2, \varphi_3$ are uniquely determined up to scalar multiples. From Lemma 4(iv) a vector $a_x^M u_\alpha^N$ of $S_{(n+m+p)-(s+i+j),s+i,j}$ is transformed under φ_1 to

$$\sum_{k \geq 0} \mu_k u_x^k \cdot u_z^{j-k} u_\alpha^{N-i} (xz\alpha)^i a_x^{M-(p-i-j)-k} a_z^{p-i-j+k} \tag{27}$$

where $\mu_0 = 1$ and $\mu_k \in \mathbb{Q}$ are determined so as to be annihilated by $(\partial^2/\partial x \partial u)$. Since $N - i = s - j$ we see $(xy\partial_u)^s\{u_x^k u_z^{j-k} u_\alpha^{N-i}\} = 0$ for any $k > 0$ so (27) is mapped by $\varphi_2 = (y\partial_x)^{m-s}(xy\partial_u)^s$ to

$$\begin{aligned} &(y\partial_x)^{m-s}\{(xyz)^j (xy\alpha)^{N-i} (xz\alpha)^i a_x^{M-(p-i-j)} a_z^{p-i-j}\} \\ &= (y\partial_x)^{m-s}\{(xz\alpha)^i a_x^{n+m-2s-i}\}(xyz)^j (xy\alpha)^{s-j} a_z^{p-i-j} \end{aligned}$$

which in turn is transformed under $\varphi_3 = (u\partial_y\partial_z)^t_{(y=z)}$ to

$$u_x^j (u\partial_y\partial_z)^{t-j}_{(y=z)} (y\partial_x)^{m-s}\{a_x^{n+m-2s-i}(xz\alpha)^i\}(xy\alpha)^{s-j} a_z^{p-i-j} \tag{28}$$

From Lemma 5(ii) we see

$$
\begin{aligned}
&(y\partial_x)^{m-s}\{a_x^{n+m-2s-i}(xz\alpha)^i\} \\
&= (m-s)!\sum_k \lambda_k \cdot a_x^{(n+m-2s-i)-(m-s)+k} a_y^{(m-s)-k}(xz\alpha)^{i-k}(yz\alpha)^k
\end{aligned}
$$

$$
\text{with} \qquad \lambda_k = \binom{n+m-2s-i}{m-s-k}\binom{i}{k} \tag{29}
$$

Here k's in the summation run over the range

$$
\max(0, s+i-n) \leq k \leq mini(m-s, i) \tag{30}
$$

Hence (28) divided by $(m-s)!$ is equal to

$$
\begin{aligned}
&u_x^j(u\partial_y\partial_z)^{t-j}_{(y=z)}\{\sum_k \lambda_k a_x^{n-s-i+k} a_y^{m-s-k}(xz\alpha)^{i-k}(yz\alpha)^k\}(xy\alpha)^{s-j}a_z^{p-i-j} \\
&= u_x^j a_x^{n-s-i+k}\sum_k \lambda_k (u\partial_y\partial_z)^{t-j}_{(y=z)}\{a_y^{m-s-k}(xz\alpha)^{i-k}(yz\alpha)^k(xy\alpha)^{s-j}a_z^{p-i-j}\} \\
&= u_x^j a_x^{n-s-i+k}\sum_k \lambda_k 2^k k! u_\alpha^k (u\partial_y\partial_z)^{t-j-k}_{(y=z)}\{a_y^{m-s-k}(xz\alpha)^{i-k}(xy\alpha)^{s-j}a_z^{p-i-j}\}
\end{aligned}
\tag{31}
$$

since $(u\partial_y\partial_z)^{t-j}(yz\alpha)^k = 2^k k! u_\alpha^k (u\partial_y\partial_z)^{t-j-k}$. From Lemma 4(iii) we see

$$
\begin{aligned}
&(u\partial_y\partial_z)^{t-j-k}\{a_y^{m-s-k}(xz\alpha)^{i-k}(xy\alpha)^{s-j}a_z^{p-i-j}\} \\
&= (-1)^{(i-k)+(s-j)}(u\partial_y\partial_z)^{t-j-k}\{a_y^{m-s-k}(x\alpha z)^{i-k}(x\alpha y)^{s-j}a_z^{p-i-j}\} \\
&= (-1)^{i-k+s-j}(ua\widehat{x\alpha})^{t-j-k}\sum_l \mu_{k,l} \\
&\quad \times a_y^{(m-s-k)-(t-j-k)+l}(x\alpha z)^{(i-k)-(t-j-k)+l}(x\alpha y)^{(s-j)-l}a_z^{p-i-j-l} \\
&= (-1)^{t-j-k}(ua\widehat{x\alpha})^{t-j-k}\sum_l \mu_{k,l} \\
&\quad \times a_y^{m-s-t+j+l}(xz\alpha)^{i-t+j+l}(x\alpha y)^{(s-j)-l}a_z^{p-i-j-l}
\end{aligned}
\tag{32}
$$

Here $(ua\widehat{x\alpha})^{t-j-k} = (u_x a_\alpha - u_\alpha a_x)^{t-j-k} = (-1)^{t+j+k}(u_\alpha a_x)^{t-j-k}$ and $\mu_{k,l} \in \mathbb{Q}$ is equal to

$$
\begin{aligned}
\mu_{k,l} = (-1)^l \binom{t-j-k}{l} &\frac{(m-s-k)!}{(m-s-t+j+l)!}\frac{(i-k)!}{(i+j-t+l)!} \\
&\times \frac{(s-j)!}{(s-j-l)!}\frac{(p-i-j)!}{(p-i-j-l)!}
\end{aligned}
\tag{33}
$$

and l's in the summation in (32) run over

$$\max(0, s+t-m-j, t-i-j) \le l \le \min(t-j-k, s-j, p-i-j) \quad (34)$$

We set $y = z$ in (32), which becomes equal to

$$(u_\alpha a_x)^{t-j-k} a_z^{m+p-s-t-i} (xz\alpha)^{s-t+i} \sum_l \mu_{k,l}$$

Substituting this into (31) we see that the image of $a_x^M u_\alpha^N$ under the composition (26) is equal to

$$\begin{aligned} & u_x^j a_x^{n-s-i+k} \sum_k \lambda_k 2^k k! u_\alpha^k (u_\alpha a_x)^{t-j-k} a_z^{m+p-s-t-i} (xz\alpha)^{s-t+i} \sum_l \mu_{k,l} \\ = & (\sum_k \lambda_k 2^k k! \sum_l \mu_{k,l}) u_x^j u_\alpha^{t-j} a_x^{n-s+t-i-j} a_z^{m+p-s-t-i} (xz\alpha)^{s-t+i} \end{aligned}$$

Therefore we have proved

Lemma 5. *The restriction of $\varphi_{s,t}$ to $T = S_{(n+m-s)+(p-i-j),s+i,j}$ is an isomorphism if and only if the scalar*

$$\sum_k \lambda_k 2^k k! \sum_l \mu_{k,l} \quad (35)$$

is nonzero where λ_k and $\mu_{k,l}$ are the rational numbers (29) and (33) with the ranges of k and l given by (30) and (34), respectively.

As a consequence we see

Corollary 6. *The restriction of $\varphi_{s,t}$ to T is an isomorphism in the case $t = p$.*

Proof. We see $\lambda_k > 0$ for any k by (29), and l's satisfying the range (24) consists of one element $l = p-i-j$. In fact , $i-t+j+l \ge 0$ and $p-i-j-l \ge 0$ in (32) implies $l = p-i-j$. $\square$

Now we prove Theorem 1 by induction on $p-t \ge 0$. The case $p-t=0$ is Corollary 6 so we assume $p-t>0$ in what follows.

Lemma 7. *If $p-t>0$ then the square*

$$\begin{array}{ccc} S^{(x,u)}_{n+m-s,s}\otimes S^{(z)}_p & \xrightarrow{\varphi_{s,t}} & S^{(x)}_n\otimes S^{(z,u)}_{m+p-t,t} \\ \Big\downarrow {\scriptstyle 1\otimes(w\partial_z)} & & \Big\downarrow {\scriptstyle 1\otimes(w\partial_z)} \\ S^{(x,u)}_{n+m-s,s}\otimes S^{(z)}_{p-1}\otimes S^{(w)}_1 & \xrightarrow{\varphi'_{s,t}\otimes 1} & S^{(x)}_n\otimes S^{(z,u)}_{m+(p-1)-t,t}\otimes S^{(w)}_1 \end{array} \tag{36}$$

is commutative up to scalar multiples.

Proof. We note that $p-t>0$ implies $m+(p-1)-t\geq m\geq t$ by (22) so $S_{m+(p-1)-t,t}$ is contained in $S_m\otimes S_{p-1}$. The square (36) is decomposed to

$$\begin{array}{ccc} S^{(x,u)}_{n+m-s,s}\otimes S^{(z)}_p & \xrightarrow{\varphi_s\otimes 1} & S^{(x)}_n\otimes S^{(y)}_m\otimes S^{(z)}_p \\ {\scriptstyle 1\otimes(w\partial_z)}\Big\downarrow & & {\scriptstyle 1\otimes 1\otimes(w\partial_z)}\Big\downarrow \\ S^{(x,u)}_{n+m-s,s}\otimes S^{(z)}_{p-1}\otimes S^{(w)}_1 & \xrightarrow{\varphi'_s\otimes 1\otimes 1} & S^{(x)}_n\otimes S^{(y)}_m\otimes S^{(z)}_{p-1}\otimes S^{(w)}_1 \end{array} \tag{37}$$

$$\begin{array}{ccc} S^{(x)}_n\otimes S^{(y)}_m\otimes S^{(z)}_p & \xrightarrow{1\otimes\phi_t} & S^{(x)}_n\otimes S^{(z,w)}_{m+p-k,k} \\ {\scriptstyle 1\otimes 1\otimes(w\partial_z)}\Big\downarrow & & {\scriptstyle 1\otimes(w\partial_z)}\Big\downarrow \\ S^{(x)}_n\otimes S^{(y)}_m\otimes S^{(z)}_{p-1}\otimes S^{(w)}_1 & \xrightarrow{1\otimes\phi'_t\otimes 1} & S^{(x)}_n\otimes S^{(z)}_{m+p-1-t,t}\otimes S^{(w)}_1 \end{array} \tag{38}$$

where $\varphi_s=\varphi'_s=(xy\partial_u)^s(y\partial_x)^{m-s}$ and $\phi_t=\phi'_t=(u\partial_y\partial_z)^t_{(y=z)}$ by Lemma 4(i) and (ii). In the square (37) a vector $a_x^{n+m-2s}u_\alpha^s c_z^p$ of $S^{(x,u)}_{n+m-s,s}\otimes S^{(z)}_p$ is transformed, up to scalar multiples, as

$$\begin{array}{ccc} a_x^{n+m-2s}u_\alpha^s c_z^p & \longrightarrow & (xy\alpha)^s a_x^{n-s}a_y^{m-s}c_z^p \\ \Big\downarrow & & \Big\downarrow \\ a_x^{n+m-2s}u_\alpha^s c_z^{p-1}c_w & \longrightarrow & (xy\alpha)^s a_x^{n-s}a_y^{m-s}c_z^{p-1}c_w \end{array}$$

In the square (38) a vector $a_x^n b_x - mc_z^p$ of $S^{(x)}_n\otimes S^{(y)}_m\otimes S^{(z)}_p$ is transformed to

$$\begin{array}{ccc} a_x^n b_y^m c_z^p & \longrightarrow & a_x^n(bcu)^t b_y^{m-t}c_z^{p-t} \\ \Big\downarrow & & \Big\downarrow \\ a_x^n b_y^m c_z^{p-1}c_w & \longrightarrow & a_x^n(bcu)^t b_y^{m-t}c_z^{p-t-1}c_w \end{array} \quad \square$$

Let $\varphi: V\to W$ is a G-linear map with $K=\operatorname{Ker}\varphi$ and $C=\operatorname{Cok}\varphi$.

Lemma 8. *Suppose $K \otimes S$ and $C \otimes S$ has no common irreducible components for a G-module S and $\varphi : V \to W$ is generic (see the Definition in Introduction). Then $\varphi \otimes 1 : V \otimes S \to W \otimes S$ is genric.*

Proof. Write $V = K \oplus L$ and $W = L' \oplus C$ with an isomorphism $\varphi : L \to L'$. Let $(V \otimes S)_{(\lambda)}$ be an isotypic component of $V \otimes S$. If $(K \otimes S)_{(\lambda)}$ is nonzero then $(C \otimes S)_{(\lambda)} = 0$ by the hypothesis, so $(\varphi \otimes 1)_{(\lambda)} : (V \otimes S)_{(\lambda)} \to (W \otimes S)_{(\lambda)} = (L' \otimes S)_{(\lambda)}$ is surjective. If $(K \otimes S)_{(\lambda)} = 0$ then $(\varphi \otimes 1)_{(\lambda)} : (V \otimes S)_{(\lambda)} = (L \otimes S)_{(\lambda)} \to (W \otimes S)_{(\lambda)}$ is injective. $\square$

We apply Lemma 8 to $\varphi_{s,t} : S_{n+m-s,s} \otimes S_p \to S_n \otimes S_{m+p-t,t}$ and $S = S_1$.

Lemma 9. *If $\varphi_{s,t}$ above is generic then so is $\varphi_{s,t} \otimes 1 : S_{n+m-s,s} \otimes S_p \otimes S_1 \to S_n \otimes S_{m+p-t,t} \otimes S_1$.*

Proof. We shall show that for any irreducible component

$$U = S_{(n+m-s)+(p-i-j),s+i,j} \quad (\text{resp. } U' = S_{(m+p-t)+(n-k-l),t+k,l})$$

of Ker φ (resp. Cok φ), $U \otimes S_1$ and $U' \otimes S_1$ has no isomorphic irreducible components. For, suppose $U \otimes S_1$ and $U' \otimes S_1$ has a common irreducible component Z. Then Z is obtained by adjoining one box to the e_1-th row of U and to the e_2-th row of U' for distinct integers e_1 and e_2 with $1 \leq e_1, e_2 \leq 3$.

(i) $(e_1, e_2) = (1, 2)$. Then $Z = S_{(n+m-s)+(p-i-j)+1,s+i,j} = S_{(m+p-t)+(n-k-l),t+k+1,l}$. If $i > 0$ then $V = S_{n+m-s,s} \otimes S_p$ contains

$$S_{(n+m-s)+(p-i-j)+1,s+i-1,j} = S_{(m+p-t)+(n-k-l),t+k,l} = U'.$$

since $\varphi_{s,t}$ is genric. A contradiction to $U' \subset$ Cok φ. Similarly, if $n - k - l > 0$ then $W = S_n \otimes S_{m+p-t,t}$ contains

$$S_{(m+p-t)+(n-k-l)-1,t+k+1,l} = S_{(n+m-s)+(p-i-j),s+i,j} = U.$$

A contradiction to $U \subset$ Ker φ. If $i = n - k - l = 0$ then $Z = S_{(n+m-s)+(p-j)+1,s,j} = S_{m+p-t,t+k+1,l}$. Then $t \geq l$ implies $s = t + k + 1 = t + (n - l) + 1 > n$, which contradicts (22).

(ii) $(e_1, e_2) = (1, 3)$. Then $Z = S_{(n+m-s)+(p-i-j)+1,s+i,j} = S_{(m+p-t)+(n-k-l),t+k,l+1}$. Hence $j = l + 1 \geq 1$ and $V = S_n \otimes S_{m+p-t,t}$ contains $S_{(n+m-s)+(p-i-j)+1,s+i,j-1} = S_{(m+p-t)+(n-k-l),t+k,l} = U'$.

(iii) $(e_1, e_2) = (2, 3)$. Then $Z = S_{(n+m-s)+(p-i-j),s+i+1,j} = S_{(m+p-t)+(n-k-l),t+k,l+1}$. Hence $j = l + 1 \geq 1$ and $V = S_n \otimes S_{m+p-t,t}$ contains $S_{(n+m-s)+(p-i-j),s+i+1,j-1} = S_{(m+p-t)+(n-k-l),t+k,l} = U'$.

The cases $(e_1, e_2) = (2, 1), (3, 1), (3, 2)$ are similarly proved. $\square$

By induction on $p-t$ we assume $\varphi'_{s,t} : S_{n+m-s,s} \otimes S_{p-1} \to S_n \otimes S_{m+(p-1)-t,t}$ is generic. Then $\varphi'_{s,t} \otimes 1$ is generic by Lemma 9. To prove Theorem 1 we have to show that if $T = S_{(n+m-s)+(p-i-j),s+i,j}$ (resp. $T' = S_{m+p-t+(n+k+l),t+k,l}$) is an irreducible component of $S_{n+m-s,s} \otimes S_p$ (resp. $S_n \otimes S_{m+p-t,t}$) and if T is isomorphic to T' as GL(3)-module then $\varphi_{s,t}|_T : T \to T'$ is an isomorphism, i.e. nonzero. Let

$$\begin{aligned} U_1 &= S_{n+m-s+(p-i-j)-1,s+i,j} && \text{if } p-i-j>0 \\ U_2 &= S_{n+m-s+(p-i-j),s+i-1,j} && \text{if } i>0 \\ U_3 &= S_{n+m-s+(p-i-j),s+i,j-1} && \text{if } j>0 \end{aligned} \tag{39}$$

be irreducible submodules of $S_{n+m-s,s} \otimes S_{p-1}$, and let

$$\begin{aligned} U'_1 &= S_{m+p-t+(n-k-l)-1,t+k,l} && \text{if } m+p-t>t+k \\ U'_2 &= S_{m+p-t+(n-k-l),t+k-1,l} && \text{if } k>0 \\ U'_3 &= S_{m+p-t+(n-k-l),t+k,l-1} && \text{if } l>0 \end{aligned}$$

be irreducible submodules of $S_n \otimes S_{m+(p-1)-t,t}$. Then the T-isotypic (resp. T'-isotypic) component of $S_{n+m-s,s} \otimes S_{p-1} \otimes S_1$ (resp. $S_n \otimes S_{m+(p-1)-t,t} \otimes S_1$) is contained in $(U_1 \oplus U_2 \oplus U_3) \otimes S_1$ (resp. $(U'_1 \oplus U'_2 \oplus U'_3) \otimes S_1$) and the multiplicity is at most three. We denote by

$$T_r \subset U_r \otimes S_1, \quad T'_r \subset U'_r \otimes S_1 \qquad r = 1,2,3$$

the submodule isomorphic to $T \cong T'$. The square (36) is commutative by Lemma 7 so $\varphi_{s,t}|_T : T \to T'$ factors through

$$T \subset \oplus_{r=1}^3 T_r \subset \oplus_{r=1}^3 U_r \otimes S_1 \xrightarrow{\varphi'_{s,t}\otimes 1} \oplus_{r=1}^3 U'_r \otimes S_1$$

where the left-most inclusion is induced from $1 \otimes (w\partial_z)$. On the other hand $\varphi'_{s,t}$ is generic by the inductive hypothesis, hence, if both U_r and U'_r appear for some common $1 \le r \le 3$ then $\varphi'_{s,t} : U_r \cong U'_r$ is an ismorphism and we have the composite of the injections

$$T_r \subset U_r \otimes S_1 \cong U'_r \otimes S_1 \subset S_n \otimes S_{m+p-1-t,t} \otimes S_1$$

Therefore, in order to complete the proof of Theorem 1 we have only to show

$$T \subset \oplus_{r=1}^3 T_i \to T_r \tag{40}$$

where π_i is the projection, is nonzero. We prove this in Lemma 10 below. As to the existence of U_r and U'_r for some common $1 \leq r \leq 3$ we see

(I) If $p-i-j>0$ and $m+p-t>t+k$ then U_1 and U'_1 exist.
(II) If $i>0$ and $k=s-t+i>0$ then U_2 and U'_2 exist.
(III) If $j=l>0$ then U_3 and U'_3 exist.

All the cases when (i,j) satisfying none of (I), (II), (III) are reduced to $t=p$, the initial hypothesis of the induction :

(i) $p-i-j=k=j=0$. Then $i=t-s$ by (24), so $0=p-i-j= p-(t-s)=(p-t)+s$ hence $t=p$ since $p-t \geq 0$ and $s \geq 0$ by (22).
(ii) $p-i-j=i=j=0$. Then $p=0=t$.
(iii) $(m+p-t)-(t+k)=i=j=0$. Then $k=s-t$ by (24) so $m+p=2t+k=t+s$. Since $s \leq m$ and $t \leq p$ we see $t=p$.
(iv) $(m+p-t)-(t+k)=k=j=0$. Then $0=m+p-2t= (m-t)+(p-t)$ so $t=p$ since $t \leq \min(m,p)$ by (22).

Now we shall show that (40) is nonzero. We assume $p-i-j>0$, $i>0$ and $j>0$ and denote by

$$T \xrightarrow{\varphi} S^{(x,u)}_{n+m-s,s} \otimes S^{(z)}_p \xrightarrow{\phi} S^{(x,u)}_{n+m-s,s} \otimes S^{(z)}_{p-1} \otimes S^{(w)}_1$$

$$T \xrightarrow{\varphi_r} U_r \otimes S^{(w)}_1 \xrightarrow{\phi_r} S^{(x,u)}_{n+m-s,s} \otimes S^{(z)}_{p-1} \otimes S^{(w)}_1 \qquad r=1,2,3$$

where φ and φ_r are induced from $1 \otimes (w\partial_z)$ and U_r is the submodule of $S_{n+m-s,s} \otimes S_{p-1}$ defined in (39). The multiplicity of T in $S_{n+m-s,s} \otimes S_p$ is equal to three so the composite map $\phi \circ \varphi$ is expressed by a linear combination of $\phi_r \circ \varphi_r$ for $r=1,2,3$:

$$\phi \circ \varphi = \lambda_1 \phi_1 \circ \varphi_1 + \lambda_2 \phi_2 \circ \varphi_2 + \lambda_3 \phi_3 \circ \varphi_3$$

All of $\varphi, \varphi_i, \phi, \phi_i$ are determined up to nonzero scalar multiples so it is well-determined whether each of $\lambda_1, \lambda_2, \lambda_3$ is zero or not.

Lemma 10. *All of $\lambda_1, \lambda_2, \lambda_3$ are nonzero.*

Proof. By using Lemma 4(iv) we calculate the image of a vector $a_x^M u_\alpha^N$ of $T = S^{(x,u)}_{(n+m-s)+(p+1-i-j),s+i,j}$ by mod u_x and modulo scalars. We note that $p-i-j>0$ (resp. $i>0$) implies $M>0$ (resp. $N>0$). For simplicity we denote by $L=p-i-j$.

(0) $a_x^M u_\alpha^N$ is transformed under φ mod u_x to

$$u_z^j (xz\partial_u)^i (z\partial_x)^L \{a_x^M u_\alpha^N\} \equiv u_z^j u_\alpha^{N-i} (xz\alpha)^i a_x^{M-L} a_z^L$$

which is mapped by $\phi = (w\partial_z)$ to

$$\begin{aligned} &u_z^{j-1} u_\alpha^{N-i} (xz\alpha)^{i-1} a_x^{M-L} a_z^{L-1} \\ &\times \{j \cdot u_w (xz\alpha) a_z + i \cdot u_z (xw\alpha) a_z + L \cdot u_z (xz\alpha) a_w\} \end{aligned}$$

(i) $\varphi_1(a_x^M u_\alpha^N) = (w\partial_x) a_x^M u_\alpha^N \equiv u_\alpha^N a_x^{M-1} a_w$ is transformed under $\phi_1 \equiv u_z^j (xz\partial_u)^i (z\partial_x)^{L-1}$ mod u_x to

$$u_z^j u_\alpha^{N-i} (xz\alpha)^i a_x^{(M-1)-(L-1)} a_z^{L-1} a_w$$

(ii) $\varphi_2(a_x^M u_\alpha^N) = (xw\partial_u) a_x^M u_\alpha^N \equiv u_\alpha^{N-1} (xw\alpha) a_x^M$ is transformed under $\phi_2 \equiv u_z^j (xz\partial_u)^{i-1} (z\partial_x)^L$ mod u_x to

$$u_z^j u_\alpha^{N-i} (xz\alpha)^{i-1} a_x^{M-L} a_z^{L-1} \{(M-L+1)(xw\alpha) a_z + L(zw\alpha) a_x\}$$

(iii) $\varphi_3(a_x^M u_\alpha^N) = u_w u_\alpha^N a_x^M + \cdots$, which have to be annihilated by $(\partial^2/\partial x \partial u)$ so that

$$\varphi_3(a_x^M u_\alpha^N) = u_w u_\alpha^N a_x^M + \mu_1 u_x \cdot u_\alpha^N a_x^{M-1} a_w \tag{41}$$
$$\text{with} \qquad \mu_1 = -M/(3 + (M-1) + N) \tag{42}$$

Here we used the irreducibility condition $a_\alpha = 0$ in (10). (41) is transformed by $\phi_3 = u_z^{j-1} (xz\partial_u)^i (z\partial_x)^L$ mod u_x to

$$\begin{aligned} &u_z^{j-1} u_\alpha^{N-i} (xz\alpha)^{i-1} a_x^{M-L} a_z^{L-1} \\ &\times M a_z \{i(xzw) u_\alpha + (N-i+1) u_w (xz\alpha)\} + \mu_1 L (N-i+1) a_w u_z (xz\alpha) \end{aligned}$$

Dividing by $u_z^{j-1} u_\alpha^{N-i} (xz\alpha)^{i-1} a_x^{M-L} a_z^{L-1}$ we set

$$\begin{aligned} f &= j \cdot u_y (xz\alpha) a_z + i \cdot u_z (xw\alpha) a_z + L \cdot u_x (xz\alpha) a_w \\ f_1 &= u_z (xz\alpha) a_w \\ f_2 &= (M-L+1) \cdot u_z (xw\alpha) a_z + L \cdot u_z (zw\alpha) a_x \\ f_3 &= (N-i+1) \cdot u_w (xz\alpha) a_z + i \cdot u_\alpha (xzw) a_z - \mu_2 L \cdot u_z (xzy) a_w \end{aligned}$$

with $\mu_2 = (N - i + 1)/(M + N + 2)$. Substituting the relations

$$\begin{aligned} u_z(zw\alpha)a_x &= u_z(xw\alpha)a_z + u_w(zx\alpha)a_y + u_z(zwx)a_\alpha \\ &= u_z(xw\alpha)a_z - \phi_1 \quad \text{since } a_\alpha = 0, \\ \text{and} \quad u_\alpha(xzw)a_z &= u_x(\alpha zw)a_z + u_z(x\alpha w)a_z + u_w(xz\alpha)a_z \end{aligned}$$

into f_2 and f_3 we obtain

$$\begin{aligned} f_2 &= (M+1)\cdot u_z(xw\alpha)a_z - L\cdot\phi_1 \\ f_3 &= (N+1)\cdot u_w(xz\alpha)a_z - i\cdot u_z(xw\alpha)a_z - \mu_2 L\cdot u_z(xzy)a_w \end{aligned}$$

Hence we see

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -L & M+1 & 0 \\ -\mu_2 L & -i & N+1 \end{pmatrix} \begin{pmatrix} f_1 \\ u_z(xw\alpha)a_z \\ u_w(xz\alpha)a_z \end{pmatrix}$$

If we set $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ then

$$(\lambda_1\ \lambda_2\ \lambda_3) \begin{pmatrix} 1 & 0 & 0 \\ -L & M+1 & 0 \\ -\mu_2 L & -i & N+1 \end{pmatrix} = (L\ i\ j)$$

from which

$$\begin{aligned} \lambda_1 &= L(1 + \frac{i}{M+1} + \mu_2 \frac{j}{N+1}) \\ \lambda_2 &= \frac{i}{M+1}(1 + \frac{j}{N+1}), \qquad \lambda_3 = \frac{j}{N+1} \end{aligned}$$

We note $M, N > 0$ and $\mu_2 > 0$ since $N - i = s - j \geq 0$. Hence, if $L = p - i - j$, i, $j > 0$ then λ_1, λ_2, λ_3 is positive. $\square$

We see from the above proof that if one of L, i, j is nonzero then the corresponding λ_r is positive. Thus the composite map $T \to T_r$ in (40) is nonzero if $U_r \otimes S_1$ has a component isomorphic to T. This means that the restriction $\varphi_{s,t}|_T$ is nonzero for any common component T of $S_{n+m-s,s} \otimes S_p$ and $S_n \otimes S_{m+p-t,t}$ and the proof of Theorem 1 is now complete.

4. An example

Let $G = S_3$ be the symmetric group of degree three, which has three irreducible representations : the trivial representation ϵ, the alternating repesentation χ, and the two dimensional representation $V = V_\rho$. Here $V = V_\rho$ is defined by

$$\sigma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x-y \end{pmatrix} \tag{43}$$

$$\tau\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -y \end{pmatrix}$$

for a basis $\{x, y\}$ of V, and the generators $\{\sigma, \tau\}$ of S_3 with the relations $\sigma^3 = \tau^2 = (\sigma\tau)^2 = 1$. We shall show in this section that the canonical map

$$V \otimes (V \otimes V)_{(\rho)} \to V \otimes V \otimes V \to (V \otimes V)_{(\rho)} \otimes V \tag{44}$$

is not a generic map. Since $V \otimes V \cong V_\epsilon + V_\chi + V_\rho$ we see

$$\{(V \otimes V) \otimes V\}_{(\rho)} \cong V_\epsilon \otimes V + V_\chi \otimes V + (V \otimes V)_{(\rho)} \tag{45}$$

$$\{V \otimes (V \otimes V)\}_{(\rho)} \cong V \otimes V_\epsilon + V \otimes V_\chi + (V \otimes V)_{(\rho)} \tag{46}$$

Let $\{a, b\}$, $\{x, y\}$ and $\{\xi, \eta\}$ be the three set of the basis of $V = V_\rho$ which are transformed by σ and τ in the same way as (43). If we set

$$^t(z_1\ z_2\ z_3\ z_4) = A \cdot {}^t\,(ax\ by\ ay\ bx), \quad A = \begin{pmatrix} 2 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \tag{47}$$

then we see from (43) that

$$\sigma(z_1\ \cdots\ z_4) = (z_1\ \cdots\ z_4) \cdot B, \quad \tau(z_1\ \cdots\ z_4) = (z_1\ \cdots\ z_4) \cdot C$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

so z_1 (resp. z_2) is a basis of V_ϵ (resp. V_χ) and $\{z_3, z_4\}$ is a basis of $(V \otimes V)_{(\rho)}$. Hence, in the decomposition (45)

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} z_1\xi \\ z_1\eta \end{pmatrix} = \begin{pmatrix} \{2(ax+by) - (ay+bx)\}\xi \\ \{2(ax+by) - (ay+bx)\}\eta \end{pmatrix}$$

is a basis of $\{(V \otimes V)_{(\epsilon)} \otimes V\}_{(\rho)}$,

$$\begin{pmatrix} f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} z_2\xi \\ z_2\eta \end{pmatrix} = \begin{pmatrix} (ay-bx)\xi \\ (ay-bx)\eta \end{pmatrix}$$

is a basis of $\{(V \otimes V)_{(\chi)} \otimes V\}_{(\rho)}$, and

$$\begin{pmatrix} f_5 \\ f_6 \end{pmatrix} = \begin{pmatrix} z_3\xi - (z_3\eta + z_4\xi) \\ z_4\eta - (z_3\eta + z_4\xi) \end{pmatrix} = \begin{pmatrix} (ax-by)\xi - \{a(x-y)-bx\}\eta \\ \{ay + b(x-y)\}\xi - (ax-by)\eta \end{pmatrix}$$

is a basis of $\{(V \otimes V)_{(\rho)} \otimes V\}_{(\rho)}$. Similarly, if we set

$$^t(w_1 \ \cdots \ w_4) = A \cdot {}^t\,(x\xi \ y\eta \ x\eta \ y\xi)$$

using A in (47) then w_1 (resp. w_2) is a basis of V_ϵ (resp. V_χ) and $\{z_3, z_4\}$ is a basis of $(V \otimes V)_{(\rho)}$. In the decomposition (45)

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} aw_1 \\ bw_1 \end{pmatrix} = \begin{pmatrix} a\{2(x\xi + y\eta) - (x\eta + y\xi)\} \\ b\{2(x\xi + y\eta) - (x\eta + y\xi)\} \end{pmatrix}$$

is a basis of $\{V \otimes (V \otimes V)_{(\epsilon)}\}_{(\rho)}$,

$$\begin{pmatrix} g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} aw_2 \\ bw_2 \end{pmatrix} = \begin{pmatrix} a(x\eta - y\xi) \\ b(x\eta - y\xi) \end{pmatrix}$$

is a basis of $\{V \otimes (V \otimes V)_{(\chi)}\}_{(\rho)}$, and

$$\begin{pmatrix} g_5 \\ g_6 \end{pmatrix} = \begin{pmatrix} aw_3 - (aw_4 + bw_3) \\ bw_4 - (aw_4 + bw_3) \end{pmatrix} = \begin{pmatrix} a(x\xi - y\eta) - b\{(x-y)\xi - x\eta\} \\ a\{y\xi + (x-y)\eta\} - b(x\xi - y\eta) \end{pmatrix}$$

is a basis of $\{V \otimes (V \otimes V)_{(\rho)}\}_{(\rho)}$. Putting all this together we obtain

$$^t(f_1 \ \cdots \ f_6) = X \cdot {}^t\,(ax\xi \ ax\eta \ \ ay\xi \ ay\eta \ \ bx\xi \ bx\eta \ \ by\xi \ by\eta)$$
$$^t(g_1 \ \cdots \ g_6) = Y \cdot {}^t\,(ax\xi \ ax\eta \ \ ay\xi \ ay\eta \ \ bx\xi \ bx\eta \ \ by\xi \ by\eta)$$

where X and Y are 6×8 matrices given by

$$X = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 2 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Then we see ${}^t(2f_1\ 2f_2\ 6f_3\ 6f_4\ 2f_5\ 2f_6) = Z \cdot {}^t(g_1\ \cdots\ g_6)$ with the 6×6 matrix Z equal to

$$Z = \begin{pmatrix} 1 & 0 & 1 & -2 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 1 & -2 & -3 & 0 & -2 & 4 \\ 2 & -1 & 0 & -3 & -4 & 2 \\ 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \end{pmatrix}$$

Thus the basis $\{f_5, f_6\}$ of $\{V \otimes (V \otimes V)_{(\rho)}\}_{(\rho)}$ is contained in the subspace $\langle g_1, \cdots, g_4\rangle = \{V \otimes (V \otimes V)_{(\epsilon)}\}_{(\rho)} + \{V \otimes (V \otimes V)_{(\chi)}\}_{(\rho)}$ and the restriction of (44) to the V_ρ-isotypic component is a zero map.

Remark. The projection $P_\lambda : V \to V_{(\lambda)} \subset V$ to the V_λ-isotypic component $V_{(\lambda)}$ is expressed using the character χ_λ of V_λ [S,p34] :

$$P_\lambda = \frac{\dim\ V_\lambda}{|G|} \cdot \sum_{g \in G} \bar{\chi}_\lambda(g)\rho(g)$$

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Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN