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A REALIZATION OF TWISTED GRASSMANN VARIETIES

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ABSTRACT. A twisted Grassmann variety (a form of Grassmann variety), which is the variety representing the functor of right ideals of prescribed rank in a central simple algebra over a field, is represented by a linear section of a Grassmann variety (Theorem A). The Severi-Brauer schemes of some R-orders in the matrix ring $M_4(R)$ of degree 4 over a regular local ring R are constructed (Theorem B). The variety of rank 4 right ideals of the associative k-algebra generated by x, y with the relations $x^4 = y^4 = 0$ and $yx = \sqrt{-1}xy$ is described (Theorem C).

Introduction

Let A be a central simple algebra of degree n over a field F of characteristic zero with the unit group A^* and the opposite algebra A^{op} of A. If we regard A as $A^* \times (A^{op})^*$-module by $(u, v) \cdot x = uxv$ for $u, v \in A^*$ and $x \in A$ then the scalar extension $A \otimes F_a$ over the algebraic closure F_a of F is isomorphic to $V \otimes W$ for the standard $A^* \otimes F_a = GL_n(F_a)$-modules V and W of dimension n over F_a. By Cauchy's formula [A-B-W,p246,F-H,p80] we see

$$\wedge^n (V \otimes W) = \oplus_\lambda (S_\lambda V \otimes S_{\tilde{\lambda}} W) \tag{1}$$

where λ runs over partitions of the integer n and $S_\lambda V$ (resp. $S_{\tilde{\lambda}} W$) is the irreducible representation of $GL(V)$ (resp. $GL(W)$) associated with λ (resp. $\tilde{\lambda}$ conjugate to λ). In particular $\wedge^n (V \otimes W)$ contains the tensor product of the n-th symmetric tensor representation $S_{(n)} V$ of $GL(V)$ and the one-dimensional representation $S_{(1^n)} W = \wedge^n W$ of $GL(W)$, which descends to an $A^* \times (A^{op})^*$-submodule $\mathcal{V}_{(n)}$ of $\wedge^n A$ over

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F. We will show that the Severi-Brauer variety X_A of A is represented by an $A^* \times (A^{op})^*$-equivariant cartesian square [cf.H,p113,Ex.9.23]

$$\begin{array}{ccc} X_A & \longrightarrow & G(n,A) \\ \downarrow & & \downarrow \\ \mathbb{P}\mathcal{V}_{(n)} & \longrightarrow & \mathbb{P}[\wedge^n A] \end{array}$$

where $G(n, A)$ is the Grassmann variety of n-dimensional subspaces of A and the left (resp. the right) vertical arrow is the twisted Veronese embedding of degree n (resp. the Plücker embedding). We generalize this construction of Severi-Brauer varieties to that of twisted Grassmann varieties as follows. Let $A = M_l(D)$ be a central simple algebra of degree n over F with a division algebra D of index $m = n/l$. For an integer r dividing m the r-th twisted Grassmann variety $X_{A,r}$ of A is defined by the F-variety representing the functor of right ideals of A of dimension nr over F and is a form of the Grassmann variety $G(r, n)$ of r-dimensional subspaces in an n-dimansional vector space over F [B,p98]. We show in Section 1

Theorem A. *$X_{A,r}$ is represented by an $A^* \times (D^{op})^*$-equivariant cartesian square (fibre product)*

$$\begin{array}{ccc} X_{A,r} & \longrightarrow & G(rm,L) \\ \downarrow & & \downarrow \\ \mathbb{P}\mathcal{V}_{(m^r)} & \longrightarrow & \mathbb{P}[\wedge^{rm} L]. \end{array} \tag{2}$$

Here L is the minimal left ideal of A of dimension nm over F, the left (resp. the right) vertical arrow is defined by the line bundle $\mathcal{L}$ with $\mathcal{L} \otimes F_a = \mathcal{O}_{G(r,n)}(m)$ (resp. the Plücker embedding) and the lower arrow is a linear section.

If K is a maximal subfield of D then the minimal left ideal L is a left vector space of dimension n over K and the upper arrow in (2) factors through the Weil restriction of the Grassmann variety $G(r, L)$ over K:

$$X_{A,r} \to R_{K/F}\, G(r, L) \to G(rm, L).$$

Here the left arrow is the canonical embedding corresponding to the isomorphisms $X_{A,r} \otimes K = G(r, n) = G(r, L)$ over K [V,p39], and the right arrow is induced by regarding an r-dimensional subspace over K in L as an rm-dimensional subspace over F [S,p325].

Let R be a regular local ring of dimension two with the maximal ideal $(f,g)R$. In Section 2 we consider the two R-orders

$$\Lambda_1 = \begin{pmatrix} R & R & R & R \\ (fg) & R & (f) & R \\ (g) & (g) & R & R \\ (fg) & (g) & (fg) & R \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} R & R & R & R \\ (f) & R & (f) & R \\ (g) & (g) & R & R \\ (fg) & (g) & (f) & R \end{pmatrix}$$

of the matrix ring $M_4(R)$ of degree four (where (f) means the principal ideal of R generated by f). Let V_1 and V_2 be the Severi-Brauer schemes of Λ_1 and Λ_2, respectively, i.e. the R-schemes representing the functors of left ideals which are rank 4 subbundles of Λ_1 and Λ_2, respectively. We will show V_1 and V_2 are embedded into $(\mathbb{P}^3_R)^4$ with the defining ideal generated by the minors of degree two in 4×4-matrices. As for closed fibres we show in Section 2

Theorem B. *(i) The closed fibre of V_1 consists of eight components Z_i $(1 \le i \le 8)$, four of which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and the others are isomorphic to the closed subvariety of $\mathbb{P}^3_x \times \mathbb{P}^1_y \times \mathbb{P}^1_z$ defined by*

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & x_3 \\ z_1 & z_2 \end{vmatrix} = 0. \tag{3}$$

V_1 has singularity only at one point $p = \cap_{i=1}^{8} Z_i$, where the completion of the local ring is isomorphic to $k[|x_1, \cdots, x_9|]$ modulo

$$\text{rank} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \le 1.$$

(ii) The closed fibre of V_2 consists of six components Y_i $(1 \le i \le 6)$; two of which are isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$, and the others are isomorphic to the variety defined by (3) in (i) above. V_2 has singularity only at one point $p = \cap_{i=1}^{6} Y_i$, where the completion of the local ring is isomorphic to $k[|x_i, y_i; 1 \le i \le 4|]$ modulo

$$\text{rank} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \le 1.$$

The closed fibres are reduced in both cases (i)-(ii) and the total space V_1 and V_2 are irreducible.

Let $A = (0,0)_{4,k}$ be the 16-dimensional associative algebra over a field k generated by x, y with the relations $x^4 = y^4 = 0$ and $yx = \sqrt{-1}xy$. We consider the closed subscheme W_0 in Grassmann variety $G(4, A) = G(4, 16)$ of rank 4 right ideals in A. We show in Section 3

Theorem C. *(i) W_0 is an irreducible variety of dimension three which is non-normal along a Weil divisor $S \cong \mathbb{P}^2$.*

(ii) There is a birational morphism $\nu : X \to W_0$ such that $X = \mathbb{P}[\mathcal{E}]$ is a $\mathbb{P}^1$-bundle over $\mathbb{F}_3 = \mathbb{P}[\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)]$ with the rank two vector bundle $\mathcal{E}$ on $\mathbb{F}_3$ defined by the non-splitting exact sequence (note $Ext^1(\mathcal{O}_{\mathbb{F}_3}(2f), \mathcal{O}_{\mathbb{F}_3}(s+3f)) = H^1(\mathcal{O}_{\mathbb{F}_3}(s+f)) \cong k$)

$$0 \to \mathcal{O}_{\mathbb{F}_3}(s+3f) \to \mathcal{E} \to \mathcal{O}_{\mathbb{F}_3}(2f) \to 0. \tag{4}$$

(iii) $\nu^{-1}(S)$ consists of two irreducible divisors $D_1 \cup D_2$ where

(iii-a) $D_1 = \mathbb{P}[\mathcal{E}|_s] \cong \mathbb{F}_0$ and the restricion $D_1 \to S$ of ν is a double cover ramified along a conic q on $S = \mathbb{P}^2$,

(iii-b) $D_2 \cong \mathbb{F}_3$ is the section defined by the surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{F}_3}(2f)$ in (4) above with the $\mathbb{P}^1$-bundle structure $\nu : D_2 = \mathbb{F}_3 \to C$ over a conic C on S.

(iv) For a point p of W_0

(iv-a) p is in S if and only if p corresponds to the rank 4 right ideals of A contained in the rank 9 ideal $xyA \subset A$.

(iv-b) p is in the conic C in (iii-b) if and only if p corresponds to non-principal rank four right ideals of A.

The results in Section 2 and Section 3 will be used in the construction of a fibre space associated with the cyclic algebra $(f, g)_{4,F}$ of degree 4 over a function field F.

The paper is organized as follows. In Section 1 we prove Theorem A. In Section 2 (resp. Section 3) we prove Theorem B (resp. Theorem C), respectively. In Section 4 we write down explicitly an ideal basis of the Severi-Brauer variety in $\mathbb{P}^9$ of a cyclic algebra $A = (f, g)_{3,F}$ of degree three over a field F (Proposition 8). The base field is assumed to have characteristic zero for the sake of completely reducibility.

1. Representation

Let $\nu = (\nu_1 \geq \cdots \geq \nu_d \geq 0)$ be a partition of a natural number $d = \Sigma\, \nu_i$ and let V_ν be the irreducible representation of the symmetric group of degree d associated with ν. We denote by S_ν the Schur functor corresponding to the partition ν, i.e. $S_\nu V$ is the irreducible representation of GL(V) asssociated with ν for a vector space V over a field F. The decomposition of $S_\nu(V \otimes W)$ into irreducible GL$(V) \times$ GL(W)-module is given by

$$S_\nu(V \otimes W) = \oplus_{\lambda,\mu}\, C_{\lambda\mu\nu}(S_\lambda V \otimes S_\mu W) \qquad \text{[F-H,p80]} \tag{1}$$

where λ and μ are partitions of d and the coefficients $C_{\lambda\mu\nu}$ are calculated by the decomposition of the tensor product representation of the symmetric group of degree d: $V_\lambda \otimes V_\mu = \oplus_\nu\, C_{\lambda\mu\nu} V_\nu$. In particular if $n =$ dim $V \geq$ dim $W = m$ and $\nu = (1^{rs})$, 1 repeats rs-times, for integers r, s then the decomposition (1) reduces to Cauchy's formula

$$\wedge^{rs}(V \otimes W) = \oplus_\lambda\, (S_\lambda V \otimes S_{\tilde{\lambda}} W) \tag{2}$$

where $\tilde{\lambda}$ is the conjugate to each partition λ of weight rs. If $\{v_1, \cdots, v_n\}$ (resp. $\{w_1, \cdots, w_m\}$) is an F-basis of V (resp. W) and if $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p > 0)$ is a partition of rs, then the maximal vector of the component $S_\lambda V \otimes S_{\tilde{\lambda}} W$ with respect to a Borel subgroup of $GL(V) \times GL(W)$ is given by

$$\begin{array}{l} (v_1 \otimes w_1) \wedge (v_1 \otimes w_2) \wedge \cdots \wedge (v_1 \otimes w_{\lambda_1}) \\ \wedge\ (v_2 \otimes w_1) \wedge (v_2 \otimes w_2) \wedge \cdots \wedge (v_2 \otimes w_{\lambda_2}) \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \wedge\ (v_p \otimes w_1) \wedge (v_p \otimes w_2) \wedge \cdots \wedge (v_p \otimes w_{\lambda_p}) \end{array}$$

[A-B-W,p248, cf. D-E-P,p147]. Therefore $\wedge^{rs}$ $(V \otimes W)$ contains the tensor product $S_{(s^r)}V \otimes S_{(r^s)}W$ of the representation $S_{(s^r)}V$ (resp. $S_{(r^s)}W$) of $GL(V)$ (resp. $GL(W)$).

Lemma 1. *The inclusion of $S_{(s^r)}V \otimes S_{(r^s)}W$ into $\wedge^{rs}(V \otimes W)$ induces a $GL_n(F_a) \times GL_m(F_a)$-equivariant commutative diagram*

$$\begin{array}{ccc} G(r, V) \times G(s, W) & \longrightarrow & G(rs, V \otimes W) \\ \downarrow & & \downarrow \\ \mathbb{P}[S_{(s^r)}V \otimes S_{(r^s)}W] & \longrightarrow & \mathbb{P}[\wedge^{rs}(V \otimes W)]. \end{array} \tag{3}$$

Here $G(r,V)$ is the Grassmann variety of r-dimensional subspacs (r-subspaces, for short) in the vector space V, the left (resp. the right) arrow is the embedding by the line bundle $\mathcal{O}_{G(r,V)}(s) \otimes \mathcal{O}_{G(s,W)}(r)$ followed by the Segre embedding (resp. the Plücker embedding).

Proof. For an r-subspace V_1 of V and an s-subspace W_1 of W the upper arrow in (3) maps the pair (V_1, W_1) to the subspace $V_1 \otimes W_1$ of $V \otimes W$ so the image is equal to $\wedge^{rs}(V_1 \otimes W_1)$ in $\wedge^{rs}(V \otimes W)$. Since $\dim V_1 = r$ and $\dim W_1 = s$ we see from (2) that $S_\lambda V_1 \otimes S_{\tilde{\lambda}} W_1 = 0$ except for $\lambda = (s^r)$, which in turn implies $\wedge^{rs}(V_1 \otimes W_1)$ is just equal to $S_{(s^r)}V_1 \otimes S_{(r^s)}W_1$. □

Let $A = M_l(D)$ be a central simple algebra of degree n over F with a division algebra D of index (resp. exponent) equal to $m = n/l$ (resp. e). Let r, s be integers with $e = rs$. Denote by $X_{A,r}$ the r-th twisted Grassmann variety of A, i.e. the F-variety representing the functor of right ideals of A of dimension nr over F. Since the base field extension $X_{A,r} \otimes F_a = G(r,n)$ to the algebraic closure F_a of F (where $n = \deg A$) [B,p102,Cor.2], the Hochshild-Serre spectral seqence $E_2^{pq} = H^p(F, H^q(G(r,n), \mathbb{G}_m)) \Rightarrow H^{p+q}(X_{A,r}, \mathbb{G}_m)$ provides the exact sequence

$$0 \to \text{Pic } X_{A,r} \to (\text{Pic } G(r,n))^{\mathcal{G}} \xrightarrow{f} \text{Br } F \to \text{Br } X_{A,r}$$

where $\mathcal{G}$ is the absolute Galois group of F. We see from [S-V,p511, Th.3.3] that the generator $\mathcal{O}(1)$ of Pic $G(r,n)$ is mapped to the class $[A^{\otimes r}]$ by f. Hence Pic $X_{A,r}$ is generated by a line bundle $\mathcal{L}$ such that $\mathcal{L} \otimes F_a = \mathcal{O}_{G(r,n)}(s)$ with $s = \exp A/r = e/r$. Similarly, Pic $X_{D,s}$ is generated by a line bundle $\mathcal{L}'$ with $\mathcal{L}' \otimes F_a = \mathcal{O}_{G(s,m)}(r)$.

If we regard the minimal left ideal $L = D^l = F^{mn}$ of A as the $A^* \times (D^{op})^*$-module via $(u,v) \cdot x = uxv$ for $u \in A^*$, $v \in D^*$ and $x \in L$ then the scalar extension $L \otimes F_a$ is isomorphic to $V \otimes W$ for the standard $A^* \otimes F_a = GL_n(F_a)$ (resp. $D^* \otimes F_a = GL_m(F_a)$)-modules V (resp. W) of dimension n (resp. m) over F_a. The following Lemma deduced from [Ti,p216,Th.7.2; cf.Lemma 7 in Section 4] guarantees the descent of the diagram (3) from F_a to the base field F.

Lemma 2. *Let B be a central simple algebra of degree n over F and let μ be a partition of an integer d. If the index of $B^{\otimes d}$ is equal to r then the homomorphisms*

$$GL_n(F_s) = GL(V) \to GL(S_\mu V) \xrightarrow{\delta} GL((S_\mu V)^{\oplus r})$$

over F_a, where δ is the diagonal, descends to the irreducible representation (ρ_μ, W_μ) over F of the unit group B^* of B :

$$\rho_\mu : B^* \to B_1^* \to GL(W_\mu).$$

Here B_1 is the central simple algebra over F Brauer equivalent to $B^{\otimes d}$ such that $B_1^* \otimes F_a = GL(S_\mu V)$. In particular if the exponent of B divides d $=$ weight μ then the absolutely irreducible representation $S_\mu V$ of $GL(V)$ descends to a representation of B^* over F.

Since weight$(1^{rs}) = e = \exp A$, Lemma 2 implies the decomposition (2) descends over F :

$$\wedge^{rs} L = \oplus_\lambda \, (\mathcal{V}_\lambda \otimes \mathcal{W}_{\tilde{\lambda}}) \qquad \text{weight } \lambda = rn$$

where $\mathcal{V}_\lambda$ (resp. $\mathcal{W}_{\tilde{\lambda}}$) is the irreducible A^* (resp. $(D^{op})^*$)-module with $\mathcal{V}_\lambda \otimes F_a = S_\lambda V$ (resp. $\mathcal{W}_{\tilde{\lambda}} \otimes F_a = S_{\tilde{\lambda}} W$). Therefore $\wedge^{rs} L$ contains the submodule $\mathcal{V}_{(s^r)} \otimes \mathcal{W}_{(r^s)}$ for which $\mathcal{V}_{(s^r)} \otimes \mathcal{W}_{(r^s)} \otimes F_a$ is isomorphic to the tensor product $S_{(s^r)} V \otimes S_{(r^s)} W$. Therefore the diagram (3) descends to an $A^* \times (D^{op})^*$-equivariant commutative diagram over F:

$$\begin{array}{ccc} X_{A,r} \times X_{D,s} & \longrightarrow & G(rs, L) \\ \downarrow & & \downarrow \\ \mathbb{P}[\mathcal{V}_{(s^r)} \otimes \mathcal{W}_{(r^s)}] & \longrightarrow & \mathbb{P}[\wedge^{rs} L]. \end{array} \tag{4}$$

In particular, if $s = m = \dim W$ then $G(m, W)$ is one-point and $S_{(r^m)} W$ is a one-dimensional $GL(W)$-module. Hence (4) gives the diagram (2) in Theorem A. To complete the proof of Theorem A we have to show

Lemma 3. *The commutative diagram*

$$\begin{array}{ccc} G(r, V) & \longrightarrow & G(rm, V \otimes W) \\ \downarrow & & \downarrow \\ \mathbb{P}[S_{(m^r)} V \otimes S_{(r^m)} W] & \longrightarrow & \mathbb{P}[\wedge^{rm} (V \otimes W)] \end{array}$$

is cartesian.

Proof. Let R be an rm-subspace of $V \otimes W$ such that $\wedge^{rm} R$ is contained in $S_{(m^r)} V \otimes S_{(r^m)} W$. Since $\wedge^{rm} R$ is fixed by the action of $GL(W)$

because of $S_{(r^m)}W = F$ the subspace R is an $GL(W)$-submodule of $V \otimes W = W^n$ ($n = \dim V$). Therefore R is equal to $V_1 \otimes W$ for an r-subspace V_1 of V. □

The $A^* \times (D^{op})^*$-isomorphism $A \cong L_1 \oplus \cdots \oplus L_l$ with $L_i = L$ ($1 \leq i \leq l$) induces the inclusions of $A^* \times (D^{op})^*$-subspaces

$$\mathcal{V}'_{(n^r)} \subset \mathcal{V}^{\otimes l}_{(m^r)} \subset (\wedge^{rm} L_1) \otimes \cdots \otimes (\wedge^{rm} L_l) \subset \wedge^{rn} A \tag{5}$$

with $\mathcal{V}'_{(n^r)} \otimes F_a = S_{(n^r)}V$ and $\mathcal{V}_{(m^r)} \otimes F_a = S_{(m^r)}V$. The next Corollary follows from considering the corresponding diagram over F_a.

Corollary 4. *The linear sections of $G(rn, A) \subset \mathbb{P}[\wedge^{rn} A]$ cut out by (5) induce the commutative diagram*

$$\begin{array}{ccccccc}
X_{A,r} & \xrightarrow{\Delta} & \Pi^l X_{A,r} & \longrightarrow & \Pi^l G(rm, L) & \longrightarrow & G(rn, A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}\mathcal{V}'_{(n^r)} & \longrightarrow & \mathbb{P}[\mathcal{V}^{\otimes l}_{(m^r)}] & \longrightarrow & \mathbb{P}[(\wedge^{rm} L)^{\otimes l}] & \longrightarrow & \mathbb{P}[\wedge^{rn} A]
\end{array}$$

where Δ is the diagonal and $\Pi^l *$ is the product of l factors of $*$.

2. Severi-Brauer schemes associated with orders in $M_4(R)$

Let R be a regular local ring of dimension two with the maximal ideal $(f, g)R$. In this section we consider the Severi-Brauer schemes of some R-orders in $M_4(R)$. First let us consider

$$\Lambda_1 = \begin{pmatrix} R & R & R & R \\ (fg) & R & (f) & R \\ (g) & (g) & R & R \\ (fg) & (g) & (fg) & R \end{pmatrix}.$$

Since Λ_1 contains four primitive idempotents we will realize the Severi-Brauer scheme of Λ_1 as a closed subscheme of $(\mathbb{P}^3_R)^4$ (cf. Cor.4). Let S be a local R-algebra and let L be a left ideal which is a rank 4 subbundle of $\Lambda_1 \otimes S$. For any non-zero element

$$\xi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 fg & b_2 & b_3 f & b_4 \\ c_1 g & c_2 g & c_3 & c_4 \\ d_1 fg & d_2 g & d_3 fg & d_4 \end{pmatrix} \tag{1}$$

of $\Lambda_1 \otimes S$ with $a_i, b_i, c_i, d_i \in S$ we see (where e_{ij} are matrix units)

$$\begin{pmatrix} e_{11}\xi \\ e_{12}\xi \\ e_{13}\xi \\ e_{14}\xi \end{pmatrix} = B_1 \cdot \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \end{pmatrix}, \quad \begin{pmatrix} fe_{21}\xi \\ e_{22}\xi \\ fe_{23}\xi \\ e_{24}\xi \end{pmatrix} = B_2 \cdot \begin{pmatrix} fe_{21} \\ e_{22} \\ fe_{23} \\ e_{24} \end{pmatrix}$$

$$\begin{pmatrix} fge_{31}\xi \\ ge_{32}\xi \\ e_{33}\xi \\ e_{34}\xi \end{pmatrix} = B_3 \cdot \begin{pmatrix} fge_{31} \\ ge_{32} \\ e_{33} \\ e_{34} \end{pmatrix}, \quad \begin{pmatrix} fge_{41}\xi \\ fge_{42}\xi \\ fe_{43}\xi \\ e_{44}\xi \end{pmatrix} = B_4 \cdot \begin{pmatrix} fge_{41} \\ fge_{42} \\ fe_{43} \\ e_{44} \end{pmatrix}$$

with $B_1, \cdots, B_4$ equal to

$$B_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ fgb_1 & b_2 & fb_3 & b_4 \\ gc_1 & gc_2 & c_3 & c_4 \\ fgd_1 & gd_2 & fgd_3 & d_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} a_1 & fga_2 & ga_3 & fga_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & fgc_2 & c_3 & fc_4 \\ d_1 & gd_2 & gd_3 & d_4 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} a_1 & a_2 & ga_3 & ga_4 \\ fgb_1 & b_2 & fgb_3 & gb_4 \\ c_1 & c_2 & c_3 & c_4 \\ fd_1 & d_2 & fgd_3 & d_4 \end{pmatrix}, \quad B_4 = \begin{pmatrix} a_1 & fa_2 & a_3 & fga_4 \\ gb_1 & b_2 & b_3 & gb_4 \\ gc_1 & fgc_2 & c_3 & fgc_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

This shows the rank 4 left ideal L decomposes into the direct sum $L_1 \oplus \cdots \oplus L_4$ with L_i contained in the i-th row of $M_4(R)$. Suppose L_i is identically zero. Then $B_i = 0$ so that $\xi = 0$, a contradiction. Therefore each L_i is R-free and hence a line subbundle of Λ_1. Hence all minors of deg 2 in the matrices $B_1, \cdots, B_4$ above are equal to zero. Let ξ_i be a generator of L_i which is uniquely deteremined up to the multiplication by units of S. Write the sum $\xi = \xi_1 + \cdots + \xi_4$ as in (1). Each L_i is a line subbundle of Λ_1 implies $(a_i), (b_i), (c_i), (d_i)$ are unimodular rows :

$$(a_1, \cdots, a_4)S = S, \qquad (b_1, \cdots, b_4)S = S$$
$$(c_1, \cdots, c_4)S = S, \qquad (d_1 \cdots, d_4)S = S.$$

Therefore L defines an S-valued point of V_1. Here V_1 is the closed subscheme of $(\mathbb{P}^3_R)^4 = \mathbb{P}^3_a \times \mathbb{P}^3_b \times \mathbb{P}^3_c \times \mathbb{P}^3_d$ whose defining ideal is generated by the minors of deg 2 in the above four matrices $B_1, \cdots, B_4$. This ideal is simply written by

$$I : fg = \{h \in R[a_i, b_i, c_i, d_i; 1 \le i \le 4] \mid fgh \in I\}$$

where I is the ideal of $R[a_i, b_i, c_i, d_i]$ generated by the minors of deg 2 in B_1. The construction shows V_1 is in fact the Severi-Brauer scheme of Λ_1, i.e. the R-scheme representing the functor of left ideals which are rank 4 subbundles of Λ_1. By the same argument the R-order

$$\Lambda_2 = \begin{pmatrix} R & R & R & R \\ (f) & R & (f) & R \\ (g) & (g) & R & R \\ (fg) & (g) & (f) & R \end{pmatrix}$$

defines the Severi-Brauer scheme V_2 which is represented by the closed subscheme of $(\mathbb{P}^3_R)^4$ with the defining ideal equal to $(I : fg)$ of $R[a_i, b_i, c_i, d_i]$ where I is generated by the minors of deg 2 in the matrix

$$B_1' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ fb_1 & b_2 & fb_3 & b_4 \\ gc_1 & gc_2 & c_3 & c_4 \\ fgd_1 & gd_2 & fd_3 & d_4 \end{pmatrix}.$$

If we put $f = 1$ in Λ_1 and Λ_2 we see the following Lemma which are special cases in [A1,p184,Th.1.4].

Lemma 5. *(i) The Severi-Brauer scheme of the R-order of $M_4(R)$ setting $f = 1$ in Λ_1 is isomorphic to the blow-up of $\mathbb{P}^3_R$ along*

$$\{g = a_1 = 0\} \supset \{g = a_1 = a_2 = 0\} \supset \{g = a_1 = a_2 = a_3 = 0\}.$$

The closed fibre consists of four components, each of which is isomorphic to the blow-up of $\mathbb{P}^3$ along a line l and a point lying on l.

(ii) The Severi-Brauer scheme of the R-order of $M_4(R)$ setting $f = 1$ in Λ_2 is isomorphic to the blow-up of $\mathbb{P}^3_R$ along $\{g = a_1 = a_2 = 0\}$. The closed fibre consists of two components, both of which are isomorphic to the blow-ups of $\mathbb{P}^3$ along a line.

Now we prove Theorem B. Let $V_i^o = \{f = g = 0\}$ be the closed fibre of V_i $(i = 1, 2)$. The complement $V_i - V_i^o$ are irreducible by Lemma 5. First we investigate the irreducible components $\{Z_j\}$ of the closed fibre V_i^o. Next we find an irreducible openset U of V_i such that $U \cap Z_j$ are dense in Z_j for all j, hence V_i is irreducible.

(Proof of Theorem B(i)) The group $G = \langle \sigma \rangle \cong \mathbb{Z}/(4)$ acts on V_1 by

$$\sigma : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \to \begin{pmatrix} b_2 \\ b_3 \\ b_4 \\ b_1 \end{pmatrix} \to \begin{pmatrix} c_3 \\ c_4 \\ c_1 \\ c_2 \end{pmatrix} \to \begin{pmatrix} d_4 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Let F be one of the irreducible components of V_1^o.

(α) Suppose $a_1 = a_2 = a_3 = 0$ on F. If $f = g = a_i = 0$ $(i = 1, 2, 3)$ and $a_4 = 1$ then the matrices in $B_1, \cdots, B_4$ become

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & b_2 & 0 & b_4 \\ 0 & 0 & c_3 & c_4 \end{pmatrix}, \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & 0 & c_3 & 0 \\ d_1 & 0 & 0 & d_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & b_2 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ 0 & d_2 & 0 & d_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & b_2 & b_3 & 0 \\ 0 & 0 & c_3 & 0 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

where we are deleting redundant rows. From this we see $b_2 = c_3 = 0$ on F so that

$$b_3(c_1, d_1, d_2, d_4), \quad c_1(b_4, d_2, d_4), \quad \begin{vmatrix} b_1 & b_4 \\ d_1 & d_4 \end{vmatrix}, \quad \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix}$$

vanish on F.

(1) If $b_3 = c_1 = 0$ on F then $F = F_1$ is equal to

$$F_1 = \{ \begin{pmatrix} 0 & 0 & 0 & a_4 \\ b_1 & 0 & 0 & b_4 \\ 0 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \mid \begin{vmatrix} b_1 & b_4 \\ d_1 & d_4 \end{vmatrix} = \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix} = 0 \}.$$

(2) If $b_3 = b_4 = d_2 = d_4 = 0$ on F then $F = F_2$ is equal to

$$F_2 = \{ \begin{pmatrix} 0 & 0 & 0 & a_4 \\ b_1 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & 0 & d_3 & 0 \end{pmatrix} \} \cong \mathbb{P}^1 \times \mathbb{P}^2.$$

(3) If $c_1 = d_1 = d_2 = d_4 = 0$ on F then $F = F_3$ is equal to

$$F_3 = \{ \begin{pmatrix} 0 & 0 & 0 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ 0 & c_2 & 0 & c_4 \\ 0 & 0 & d_3 & 0 \end{pmatrix} \} = \sigma^{-1}(F_2).$$

Considering the G-orbits of the above F_1, F_2, F_3 we obtain eight components Z_i $(1 \le i \le 8)$ where four of them are isomorphic to F_1 and the other four are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$.

(β) If a_1 is not identically zero on F then we see from $B_1, \cdots, B_4$ that the following determinants vanish on F.

$$b_2,\ b_3,\ b_4,\ c_3,\ c_4,\ d_2,\ d_4,\ \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},\ \begin{vmatrix} a_1 & a_3 \\ d_1 & d_3 \end{vmatrix}$$

that is, F coincides with $\sigma(F_1)$. By the action of $G = \langle \sigma \rangle$ we see the same holds if one of b_2, c_3, d_4 is not identically zero on F.

(γ) Suppose $a_1 = b_2 = c_3 = d_4 = 0$ in the four matrices $B_1, \cdots, B_4$:

$$\begin{pmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & c_4 \end{pmatrix}, \quad \begin{pmatrix} b_1 & 0 & b_3 & b_4 \\ c_1 & 0 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & a_2 & 0 & 0 \\ c_1 & c_2 & 0 & c_4 \\ 0 & d_2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & a_3 & 0 \\ 0 & 0 & b_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{pmatrix}.$$

We see all the components defined by the minors of deg 2 in the above four matrices are contained in one of $Z_1, \cdots, Z_8$.

We see from $(\alpha), (\beta), (\gamma)$ that the eight components Z_i are all the irreducible components of the closed fibre V_1^o of V_1, consisting of two G-orbits, and V_1^o is a reduced schemes.

Next consider the openset U of V_1 where $a_4 b_1 c_2 d_3$ is not zero. Setting $a_4 = b_1 = c_2 = d_3 = 1$ in $B_1, \cdots, B_4$ we see

$$\begin{aligned} g &= a_2 c_4 = b_3 d_1, & c_3 &= a_3 c_4 = b_3 c_1 \\ a_1 &= a_2 c_1 = a_3 d_1, & d_4 &= b_4 d_1 = c_4 d_2 \\ b_2 &= a_2 b_4 = b_3 d_2. \end{aligned}$$

Hence, at the point

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

the completion of the local ring of V_1 is isomorphic to

$$\operatorname{rank} \begin{pmatrix} a_2 & b_3 & a_3 \\ d_1 & c_4 & c_1 \\ d_2 & b_4 & f \end{pmatrix} \leq 1.$$

Therefore U is irreducible. By setting $f = g = 0$ we see $V_1^o \cap U$ in $\{a_4b_3c_1d_2 \text{ is non-zero}\} \cong \mathbb{A}_k^{12}$, is defined by the determinants

$$(c_1, a_3)(b_4, d_2), \quad a_2c_4, \quad b_3d_1$$

$$\begin{vmatrix} b_3 & a_3 \\ c_4 & c_1 \end{vmatrix}, \quad \begin{vmatrix} a_2 & a_3 \\ d_1 & c_1 \end{vmatrix}, \quad \begin{vmatrix} d_1 & c_4 \\ d_2 & b_4 \end{vmatrix}, \quad \begin{vmatrix} a_2 & b_3 \\ d_2 & b_4 \end{vmatrix}.$$

These quadrics define in $\{a_4b_1c_2d_3 \text{ is non-zero}\} \cong \mathbb{A}_k^9$ all the eight components corresponding to Z_i $(1 \le i \le 8)$. Therefore V_1 is irreducible. We see by a direct caclculation that V_1^o is nonsingular except at the point (2) (The proof is omitted).

(Proof of Theorem B(ii)) The Severi-Brauer scheme V_2 of Λ_2 is the closed subscheme of $(\mathbb{P}_R^3)^4$ defined by the minors of deg 2 of four matrices

$$B_1' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ fb_1 & b_2 & fb_3 & b_4 \\ gc_1 & gc_2 & c_3 & c_4 \\ fgd_1 & gd_2 & fd_3 & d_4 \end{pmatrix}, \quad B_2' = \begin{pmatrix} a_1 & fa_2 & a_3 & fa_4 \\ b_1 & b_2 & b_3 & b_4 \\ gc_1 & fgc_2 & c_3 & fc_4 \\ gd_1 & gd_2 & d_3 & d_4 \end{pmatrix}$$

$$B_3' = \begin{pmatrix} a_1 & a_2 & ga_3 & ga_4 \\ fb_1 & b_2 & fgb_3 & gb_4 \\ c_1 & c_2 & c_3 & c_4 \\ fd_1 & d_2 & fd_3 & d_4 \end{pmatrix}, \quad B_4' = \begin{pmatrix} a_1 & fa_2 & ga_3 & fga_4 \\ b_1 & b_2 & gb_3 & gb_4 \\ c_1 & fc_2 & c_3 & fc_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

The group $G = \langle \sigma, \tau \rangle \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ acts on V_2 by

$$\sigma : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} b_2 \\ b_1 \\ b_4 \\ b_3 \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} d_2 \\ d_1 \\ d_4 \\ d_3 \end{pmatrix}$$

$$\tau : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} c_3 \\ c_4 \\ c_1 \\ c_2 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} d_3 \\ d_4 \\ d_1 \\ d_2 \end{pmatrix}.$$

If $f=g=0$ then the minors of $\deg 2$ in $B'_1, \cdots, B'_4$ reduce to

$$(a_1,a_3)(b_2,b_4),\ (a_1,a_2)(c_3,c_4),\ (b_1,b_2)(d_3,d_4),\ (c_1,c_3)(d_2,d_4) \tag{3}$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}, \quad \begin{vmatrix} b_1 & b_2 \\ d_1 & d_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} \tag{4}$$

$$\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}, \quad \begin{vmatrix} a_3 & a_4 \\ c_3 & c_4 \end{vmatrix}, \quad \begin{vmatrix} b_3 & b_4 \\ d_3 & d_4 \end{vmatrix}, \quad \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix}. \tag{5}$$

Let F be one of the irreducible components of the closed fibre V_2^o with the defining ideal $\mathcal{P}$. We see from (3) that $\mathcal{P}$ contains one of the following set of three elements :

$$\{a_1,a_2,a_3\},\ \{b_1,b_2,b_4\},\ \{c_1,c_3,c_4\},\ \{d_2,d_3,d_4\}. \tag{6}$$

Since $G = \langle\sigma,\tau\rangle$ permutes (6) tarnsitively we assume $\mathcal{P}$ contains $\{a_1,a_2,a_3\}$. Then a_4 is not contained in $\mathcal{P}$ implies both b_2 and c_3 are contained in $\mathcal{P}$ by (3). Since $\mathcal{P}$ contains $(b_1,b_2)(d_3,d_4)$ and (c_1,c_3) (d_2,d_4) in (3) we see one of the following four cases occurs.

(i) If $b_1=b_2=c_1=c_3=0$ on F then $F=F_1$ is equal to

$$F_1 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & b_3 & b_4 \\ 0 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \mid \begin{vmatrix} b_3 & b_4 \\ d_3 & d_4 \end{vmatrix} = \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix} = 0\}.$$

(ii) If $b_1=b_2=d_2=d_4=0$ on F then $(c_1,b_4)d_3=0$ on F so $F=F'_2$ or F''_2 where

$$F'_2 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & 0 & 0 & 0 \end{pmatrix}\}, \qquad F''_2 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & b_3 & 0 \\ 0 & c_2 & 0 & c_4 \\ d_1 & 0 & d_3 & 0 \end{pmatrix}\}$$

which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively.

(iii) If $d_3=d_4=c_1=c_3=0$ on F then $(b_1,c_4)d_2=0$ on F so $F=F'_3$ or F''_3 where

$$F'_3 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ 0 & c_2 & 0 & c_4 \\ d_1 & 0 & 0 & 0 \end{pmatrix}\}, \qquad F''_3 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & b_3 & b_4 \\ 0 & c_2 & 0 & 0 \\ d_1 & d_2 & 0 & 0 \end{pmatrix}\}$$

which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively.

(iv) If $d_2 = d_3 = d_4 = 0$ on F then $F = F_4$ is equal to

$$F_4 = \{\begin{pmatrix} 0 & 0 & 0 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & 0 & 0 & 0 \end{pmatrix}\} \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

Thus $F_2'', F_3'' \subset F_1$ and $F_2', F_3' \subset F_4$. Considering the action of $G = \langle \sigma, \tau \rangle$ we see the irreducible components of the closed fibre V_2^o consists of just six components, which are the G-orbits of the above F_1 and F_4.

If we set $a_4 = b_3 = c_2 = d_1 = 1$ in $B_1', \cdots, B_4'$ then the defining equations of V_2 reduce to

$$\begin{aligned} f &= a_3b_4 = c_1d_2, & a_1 &= a_3b_1 = a_2c_1, & c_3 &= a_3c_4 = c_1d_3 \\ g &= a_2c_4 = b_1d_3, & b_2 &= a_2b_4 = b_1d_2, & d_4 &= b_4d_3 = c_4d_2. \end{aligned}$$

Thus, $\{f, g, a_1, b_2, c_3, d_4\}$ are expressed by the eight elements $\{a_2, a_3, b_1, b_4, c_1, c_4, d_2, d_3\}$ with the relations

$$\operatorname{rank} \begin{pmatrix} a_2 & d_3 & d_2 & a_3 \\ b_1 & c_4 & b_4 & c_1 \end{pmatrix} \leq 1. \tag{7}$$

Therefore the openset U of V_2 where $a_4b_3c_2d_1$ is not zero is irreducible.

Setting $f = g = 0$ we see $U \cap V_2^o$ is the closed set in $\mathbb{A}_k^8$ with the affine coordinates $(a_2, a_3, b_1, b_4, c_1, c_4, d_2, d_3)$, which is defined by $a_3b_4 = c_1d_2 = a_2c_4 = b_1d_3 = 0$ together with (7). Hence we see $U \cap V_2^o$ decomposes into the six irreducible components corresponding to F_i $(1 \leq i \leq 6)$. Therefore V_2 is irreduicble. We see by a direct calculation that V_4^o is nonsingular except at the one point

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

where V_4^o is defined by (7).

3. Right ideals in $A = (0,0)_{4,k}$

Let $(R,(f),k)$ be a DVR and let $(f,g)_{4,R}$ be the cyclic R-algebra generated by x, y with the relations $x^4 = f, y^4 = g$ and $yx = \zeta xy$ ($\zeta = \sqrt{-1}$). If $g = 1$ then $(f,g)_{4,R}$ is isomorphic to the R-order Λ_1 in Section 2 setting $g = 1$ so the closed fibre of the Severi-Brauer scheme of $(f,1)_{4,R}$ consists of four components. For an element

$$\begin{aligned} z = y^3 + {} & (a_0 + a_1 x + a_2 x^2 + a_3 x^3) y^2 \\ & + (a_4 + a_5 x + a_6 x^2 + a_7 x^3) y + (a_8 + a_9 x + a_{10} x^2 + a_{11} x^3) \end{aligned} \tag{1}$$

of $(0,g)_{4,k}$ with $a_i \in k$ $(0 \le i \le 11)$ we consider the condition that the principal right ideal $z\Lambda_g$ of $\Lambda_g = (0,g)_{4,k}$ is rank 4. From $x^4 = 0$ we see ${}^t(z, zx, zx^2, zx^3)$ is equal to

$$A \cdot {}^t(1, x, x^2, x^3, y, xy, \cdots, y^2, \cdots, y^3, xy^3, x^2y^3, x^3y^3)$$

with the 4×16-matrix A equal to (where $\zeta = \sqrt{-1}$)

$$\begin{pmatrix} a_8 & a_9 & a_{10} & a_{11} & a_4 & a_5 & a_6 & a_7 & a_0 & a_1 & a_2 & a_3 & 1 & 0 & 0 & 0 \\ 0 & a_8 & a_9 & a_{10} & 0 & \zeta a_4 & \zeta a_5 & \zeta a_6 & 0 & \zeta^2 a_0 & \zeta^2 a_1 & \zeta^2 a_2 & 0 & \zeta^3 & 0 & 0 \\ 0 & 0 & a_8 & a_9 & 0 & 0 & \zeta^2 a_4 & \zeta^2 a_5 & 0 & 0 & a_0 & a_1 & 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & a_8 & 0 & 0 & 0 & \zeta^3 a_4 & 0 & 0 & 0 & \zeta^2 a_0 & 0 & 0 & 0 & \zeta \end{pmatrix}$$

and zy is equal to

$$\begin{aligned} g + (a_8 + a_9 x + a_{10} x^2 + a_{11} x^3) y + (a_4 + a_5 x + a_6 + a_7 x^3) y^2 \\ + (a_0 + a_1 x + a_2 x^2 + a_3 x^3) y^3. \end{aligned}$$

Since $\{z, zx, zx^2, zx^3\}$ are linearly independent over k we see $z\Lambda_g = z \cdot (0,g)_{4,k}$ is rank 4 if and only if

$$zy = a_0 z + \zeta a_1 zx + \zeta^2 a_2 zx^2 + \zeta^3 a_3 zx^3. \tag{3}$$

Therefore $a_0^4 = g$ and a_i $(4 \le i \le 11)$ are expressed by a_i $(0 \le i \le 3)$:

$$a_4 = a_0^2 \qquad a_8 = a_0^3$$
$$a_5 = (1+\zeta^3)a_0a_1 \qquad a_9 = \zeta^3 a_0^2 a_1$$
$$a_6 = \zeta^3 a_1^2 \qquad a_{10} = \zeta^2 a_0 a_1^2 + a_0^2 a_2$$
$$a_7 = (1+\zeta)(\zeta^2 a_1 a_2 + a_0 a_3) \qquad a_{11} = \zeta a_1^3 + 2\zeta^3 a_0 a_1 a_2 + \zeta a_0^2 a_3.$$

In order to define the Plücker coordinates of $z\Lambda_g$ we shall calculate $z\wedge zx\wedge zx^2\wedge zx^3$ i.e. the maximal minors of the above 4×16-matrix A. We number elements x^iy^j $(0 \le i,j \le 3)$ of a k-basis of $\Lambda_g = (0,g)_{4,k}$:

$$(0) = 1 \qquad (1) = x \qquad (2) = x^2 \qquad (3) = x^3$$
$$(4) = y \qquad (5) = xy \qquad (6) = x^2y \qquad (7) = x^3y$$
$$(8) = y^2 \qquad (9) = xy^2 \qquad (10) = x^2y^2 \qquad (11) = x^3y^2$$
$$(12) = y^3 \qquad (13) = xy^3 \qquad (14) = x^2y^3 \qquad (15) = x^3y^3$$

and denote by $(ijkl)$ $(0 \le i < j < k < l \le 15)$ the coefficient of $(i)\wedge(j)\wedge(k)\wedge(l)$ in $z\wedge zx\wedge zx^2\wedge zx^3$. We see from the 4×16-matrix A that the Λ_g-module generated by $\binom{16}{4} = 1820$ elements $\{(ijkl) \mid 0 \le i < j < k \le 15\}$ is of dimension 14 with the follwoing k-basis $\{x_i, y_j, z_k | 0 \le i \le 6, 0 \le j \le 3, 0 \le k \le 2\}$ (for simplicity we are calculating up to multiplication by non-zero constants) :

$$x_0 = (12,13,14,15) = 1$$
$$x_1 = (9,13,14,15) = a_1$$
$$x_2 = (6,13,14,15) = a_1^2$$
$$x_3 = (3,13,14,15) = a_1^3 + 2\zeta^2 a_0 a_1 a_2$$
$$x_4 = (3,10,14,15) = a_1^4 + (2\zeta^2+\zeta)a_0a_1^2a_2 + a_0^2(a_1a_3 + \zeta^3 a_2^2)$$
$$x_5 = (3,7,14,15) = a_1^5 + (\zeta + 3\zeta^2)a_0a_1^3a_2 + a_0^2\{(\zeta^3+2)a_1^2a_3 + (\zeta^3+1)a_2^2\} + (\zeta+\zeta^2)a_0^3a_2a_3$$
$$x_6 = (3,7,11,15) = a_1^6 + (2\zeta + 4\zeta^2)a_0a_1^4a_2 + a_0^2\{(4\zeta^3+2)a_1^2a_2^2 + (2\zeta^3+2)a_1^3a_3\} + 4\zeta a_0^3a_1a_2a_3,$$
$$y_0 = (10,13,14,15) = a_2$$
$$y_1 = (7,13,14,15) = a_1a_2 + \zeta^2 a_0 a_3$$
$$y_2 = (7,10,14,15) = a_1^2 a_2 + (\zeta+\zeta^2)a_0a_1a_3$$

$$
\begin{aligned}
y_3 &= (3,11,14,15) = a_1^3a_2 + a_0(2\zeta^2 a_1 a_2^2 + \zeta a_1^2 a_3) + (1+\zeta^3)a_0^2 a_2 a_3 \\
z_0 &= (11,13,14,15) = a_3 \\
z_1 &= (10,11,14,15) = a_1 a_3 + \zeta^2 a_2^2 \\
z_2 &= (7,11,14,15) = a_1^2 a_3 + (\zeta^2+\zeta^3)a_1 a_2^2 + (1+\zeta)a_0 a_2 a_3
\end{aligned}
$$

Thus the Severi-Brauer variety W of $\Lambda_g = (0,g)_{4,k}$ contains an openset U which is isomorphic to $\mathbb{A}_k^3$ with the affine coordinates (a_1, a_2, a_3), and which is embedded into $\mathbb{P}_k^{13}$ with the homogeneous coordinates (x_i, y_j, z_k) $(0 \le i \le 6, 0 \le j \le 3, 0 \le k \le 2)$. If g is non-zero then the closure of U in $\mathbb{P}_k^{13}$ is isomorphic to the closed fibre of the Severi-Brauer variety of Lemma 5(i) in Section 2.

Now let us consider the case $g = 0$ in the above calculations to describe the closure W_0 of $U \cong \mathbb{A}_k^3$ in $\mathbb{P}_k^{13}$. The above $\{x_i, y_j, z_k\}$ become

$$
\begin{aligned}
x_i &= a_1^i \quad (1 \le i \le 6), & z_0 &= a_3 \\
y_j &= a_2 a_1^j \quad (0 \le j \le 3), & z_1 &= a_1 a_3 + \zeta^2 a_2^2 \\
& & z_2 &= a_1^2 a_3 + (\zeta^2+\zeta^3)a_1 a_2^2.
\end{aligned} \tag{1}
$$

We see from this that the closure W_0 in $\mathbb{P}_k^{13}$ is defined by the following three kind of quadrics :

$$
\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & y_0 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y_1 & y_2 & y_3 \end{pmatrix} \le 1 \tag{2}
$$

$$
y_j z_2 - (1+\zeta)y_{j+1}z_1 + \zeta y_{j+2} z_0 = 0 \qquad (j = 0,1) \tag{3}
$$

$$
(z_0\ z_1\ 1)\cdot B_1 = (0,\cdots,0), \qquad (z_1\ z_2\ 1)\cdot B_2 = (0,\cdots,0) \tag{4}
$$

with the 3×6-matrices B_1 and B_2 equal to

$$
B_1 = \begin{pmatrix} -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_0^2 & y_0y_1 & y_1^2 & y_1y_2 & y_2^2 & y_2y_3 \end{pmatrix} \tag{5}
$$

$$
B_2 = \begin{pmatrix} -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ \zeta y_0y_1 & \zeta y_1^2 & \zeta y_1y_2 & \zeta y_2^2 & \zeta y_2y_3 & \zeta y_3^2 \end{pmatrix}. \tag{6}
$$

We see from (2)-(4) that $S=\{x_i=y_j=0\ (0\le i\le 6, 0\le j\le 3)\}\cong \mathbb{P}^2$ is singular on W_0. Moreover, we see from (4)

$$z_2x_i^2-(1+\zeta)z_1x_ix_{i+1}+\zeta z_0x_{i+1}^2$$
$$z_2y_j^2-(1+\zeta)z_1y_jy_{j+1}+\zeta z_0y_{j+1}^2$$

are equal to zero on W_0 for $0\le i\le 5$, $0\le j\le 2$ so that W_0 is non-normal along S. Let

$$\pi_0: W_0\subset \mathbb{P}^{13}\cdots\to\mathbb{P}^{10},\qquad (x:y:z)\to(x:y)$$

be the projection from $S=\mathbb{P}^2(\subset W_0)$. We see from (2) that the image $\pi_0(W_0)$ is isomorphic to $\mathbb{F}_3$. Let $\nu: X\to W_0$ be the blow-up along S.

Lemma 6. *π_0 induces a $\mathbb{P}^1$-bundle stucture $\pi: X\to\mathbb{F}_3\subset\mathbb{P}^{10}$.*

Proof. Let $(\xi_0:\cdots:\xi_6:\eta_0:\cdots:\eta_3)$ be the homogeneous coordinates of $\mathbb{P}^{10}$. We see from (2)-(4) that blow-up X of W_0 is the closed subscheme of $\mathbb{P}^{13}\times\mathbb{F}_3$ defined by

$$\operatorname{rank}\begin{pmatrix} x_0 & \cdots & x_6 & y_0 & \cdots & y_3\\ \xi_0 & \cdots & \xi_6 & \eta_0 & \cdots & \eta_3\end{pmatrix}=1 \tag{7}$$

together with

$$\eta_jz_2-(1+\zeta)\eta_{j+1}z_1+\zeta\eta_{j+2}z_0=0\quad (j=0,1) \tag{8}$$
$$(z_0\ z_1\ y_0\ \cdots\ y_3)\cdot C_1=(0,\cdots,0) \tag{9}$$
$$(z_1\ z_2\ y_0\ \cdots\ y_3)\cdot C_2=(0,\cdots,0) \tag{10}$$

where C_1 and C_2 are 4×6-matrices equal to

$$C_1=\begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6\\ \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5\\ \eta_0 & \eta_1 & 0 & 0 & 0 & 0\\ 0 & 0 & \eta_1 & \eta_2 & 0 & 0\\ 0 & 0 & 0 & 0 & \eta_2 & \eta_3\end{pmatrix}$$
$$C_2=\begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6\\ \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5\\ \zeta\eta_0 & \zeta\eta_1 & \zeta\eta_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \zeta\eta_2 & \zeta\eta_3 & 0\\ 0 & 0 & 0 & 0 & 0 & \zeta\eta_3\end{pmatrix}.$$

We see $\pi^{-1}(p) \cong \mathbb{P}^1$ for any point $p \in \mathbb{F}_3$. $\square$

In the equations (7)-(10) of X in $\mathbb{P}^{10} \times \mathbb{P}^{13}$, $\xi_i = 0$ $(0 \leq i \leq 6)$ imply $x_i = y_j = 0$ for all $0 \leq i \leq 6$ and $0 \leq j \leq 3$. Hence, for the (-3)-curve $s = \{\xi_i = 0 \ (0 \leq i \leq 6)\}$ on $\mathbb{F}_3$ we see $\pi^{-1}(s)$ is isomorphic to the closed subscheme of $\mathbb{P}^3_\eta \times \mathbb{P}^2_z \subset \mathbb{P}^{10} \times \mathbb{P}^{13}$ defined by

$$\begin{aligned} &\operatorname{rank} \begin{pmatrix} \eta_0 & \eta_1 & \eta_2 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} = 1 \\ &z_2\eta_j^2 - (1+\zeta)z_1\eta_j\eta_{j+1} + \zeta z_0\eta_{j+1}^2 = 0 \quad (j = 0, 1, 2). \end{aligned}$$

Therefore $\nu : \pi^{-1}(s) \to \mathbb{P}^2 \subset \mathbb{P}^{13}$ is a double cover with the branch locus equal to the conic $q = \{z_1^2 = 2z_0z_2\}$. From this we see $\pi^{-1}(s)$ is isomorphic to $\mathbb{F}_0$. On the other hand, for any point $p = (1 : \lambda : \lambda^2)$ on another conic $C = \{z_1^2 = z_0z_2\}$, $\nu^{-1}(p)$ contains a line $\{p \times (1 : \lambda : \lambda^2 : \cdots : \lambda^6 : \mu : \mu\lambda : \mu\lambda^2 : \mu\lambda^3) | \ \mu \in k\}$. This implies $\nu^{-1}(C) \to C$ is a $\mathbb{P}^1$-bundle and $\nu^{-1}(C)$ is a section of the $\mathbb{P}^1$-bundle $\pi : X \to \mathbb{F}_3$.

Next we construct a rank two bundle $\mathcal{E}$ on $\mathbb{F}_3$ such that $X = \mathbb{P}[\mathcal{E}]$. Let $\{U_{ij}; i, j = 0, 1\}$ be an open cover of $\mathbb{F}_3$ with $U_{ij} \cong \mathbb{A}^2$ and the affine coordinates given by

$$\begin{array}{lll} u_0 = \xi_1/\xi_0, & v_0 = \eta_0/\xi_0 & \text{on } U_{00} \\ u_0 = \eta_1/\eta_0, & v_1 = \xi_0/\eta_0 & \text{on } U_{01} \\ u_1 = \xi_5/\xi_6, & w_0 = \eta_3/\xi_6 & \text{on } U_{10} \\ u_1 = \eta_2/\eta_3, & w_1 = \xi_6/\eta_3 & \text{on } U_{11}. \end{array}$$

The transition functions are obtained from the relations

$$\begin{aligned} u_0u_1 &= v_0v_1 = w_0w_1 = 1 \\ v_0 &= \eta_0/\xi_0 = \eta_3/\xi_3 = (\eta_3/\xi_6)/(\xi_3/\xi_6) \\ &= (\eta_3/\xi_6)/(\xi_1/\xi_6)^3 = w_0/u_0^3. \end{aligned} \tag{11}$$

Now we see from (7)-(10)

$$(z_0 \ z_1 \ z_2 \ y_0) \begin{pmatrix} -\xi_1/\xi_0 & 0 \\ 1 & -\xi_1/\xi_0 \\ 0 & 1 \\ \eta_0/\xi_0 & \zeta\eta_1/\xi_0 \end{pmatrix} = (0, 0) \qquad \text{on } U_{00}$$

$$(z_0\ z_1\ z_2\ y_0)\begin{pmatrix} -\xi_1/\xi_0 & 0 \\ \xi_0/\eta_0 & -(1+\zeta)\eta_1/\eta_0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (0,0) \quad \text{on } U_{01}$$

$$(z_0\ z_1\ z_2\ y_3)\begin{pmatrix} -1 & 0 \\ \xi_5/\xi_6 & -1 \\ 0 & \xi_5/\xi_6 \\ \eta_2/\xi_6 & \zeta\eta_3/\xi_6 \end{pmatrix} = (0,0) \quad \text{on } U_{10}$$

$$(z_0\ z_1\ z_2\ y_3)\begin{pmatrix} 0 & \zeta \\ -\xi_6/\eta_3 & -(1+\zeta)\eta_2/\eta_3 \\ \xi_5/\eta_3 & \eta_1/\eta_3 \\ \zeta & 0 \end{pmatrix} = (0,0) \quad \text{on } U_{11}$$

Therefore $X = \mathbb{P}[\mathcal{E}]$ for a rank two bundle $\mathcal{E}$ on $\mathbb{F}_3$ such that

$$\mathcal{E}|_{U_{00}} = \mathcal{O}_{U_{00}}z_0 + \mathcal{O}_{U_{00}}y_0, \qquad \mathcal{E}|_{U_{10}} = \mathcal{O}_{U_{10}}z_2 + \mathcal{O}_{U_{10}}y_3$$
$$\mathcal{E}|_{U_{01}} = \mathcal{O}_{U_{01}}z_0 + \mathcal{O}_{U_{01}}z_1, \qquad \mathcal{E}|_{U_{11}} = \mathcal{O}_{U_{11}}z_1 + \mathcal{O}_{U_{11}}z_2.$$

The section $\nu^{-1}(C)$ of $\pi : X \to \mathbb{F}_3$ considered above, is defined by

$$y_0 = 0 \quad \text{over } U_{00}, \qquad y_3 = 0 \quad \text{over } U_{10}$$
$$u_0z_0 = z_1 \quad \text{over } U_{01}, \qquad z_1 = u_1z_2 \text{ over } U_{11}.$$

Therefore $\nu^{-1}(C) = \mathbb{P}[\mathcal{L}]$ for a line bundle $\mathcal{L}$ on X defined by the exact sequence $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \to 0$ with

$$0 \to \mathcal{O}_{U_{00}} \xrightarrow{y_0} \mathcal{E}|_{U_{00}} \to \mathcal{O}_{U_{00}}z_0 \to 0$$
$$0 \to \mathcal{O}_{U_{01}} \xrightarrow{u_0z_0-z_1} \mathcal{E}|_{U_{01}} \to \mathcal{O}_{U_{01}}z_0 \to 0$$
$$0 \to \mathcal{O}_{U_{10}} \xrightarrow{y_3} \mathcal{E}|_{U_{10}} \to \mathcal{O}_{U_{10}}z_2 \to 0$$
$$0 \to \mathcal{O}_{U_{11}} \xrightarrow{z_1-u_1z_2} \mathcal{E}|_{U_{11}} \to \mathcal{O}_{U_{11}}z_2 \to 0.$$

The relations (11) imply

$$\begin{aligned} z_1 &= z_0(\xi_1/\xi_0) - y_0(\eta_0/\xi_0) = z_0u_0 - y_0v_0 \\ z_2 &= z_1(\xi_1/\xi_0) - y_0(\zeta\eta_1/\xi_0) = (z_0u_0 - y_0v_0)u_0 - \zeta y_0u_0v_0 \\ &= z_0u_0^2 - (1+\zeta)y_0u_0v_0 \end{aligned}$$

so there are identities

$$\begin{aligned} u_0 z_0 \ - z_1 &= \ v_0 y_0 = (1/v_1) y_0 && \text{on } U_{00} \cap U_{01} \\ y_3 &= (\eta_1/\eta_0)^3 y_0 = u_0^3 y_0 = (1/u_1^3) y_0 && \text{on } U_{00} \cap U_{10} \\ z_2 &\equiv z_0 u_0^2 = (1/u_1^2) z_0 \quad \mod y_0. \end{aligned} \tag{12}$$

Now the (-3)-curve s on $\mathbb{F}_3$ is defined by $v_1 = 0$ on U_{01}, and $u_1 = 0$ defines a fibre on U_{10}, so that (12) implies $\mathcal{M} \cong \mathcal{O}_{\mathbb{F}_3}(s + 3f)$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{F}_3}(2f)$, i.e. $\mathcal{E}$ fits into an exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_3}(s + 3f) \to \mathcal{E} \to \mathcal{O}_{\mathbb{F}_3}(2f) \to 0 \tag{13}$$

and $\nu^{-1}(C) = \mathbb{P}[\mathcal{O}_{\mathbb{F}_3}(2f)]$. Restricting (13) to the (-3)-curve s we get

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}|_s \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

We have shown $\mathbb{P}[\mathcal{E}|_s] \cong \mathbb{F}_0$ so $\mathcal{E}|_s = \mathcal{O}(1,1)$ and (13) does not split.

(Proof of Theorem C(iv)) Let I be a rank 4 right ideal of $A = (0,0)_{4,k}$ contained in xyA. Suppose I contains an element z which has a nonzero coefficient in one of the three monomials xy, x^2y, xy^2. Then we see z generates a right ideal of rank greater than 4. Hence the rank 4 ideal I is contained in the rank 6 ideal generated by $x^i y^j$ with $i + j \geq 4$. Therefore I contains an element

$$z = b_1 xy^3 + b_2 x^2y^2 + b_3 x^3 y + b_4 x^2 y^3 + b_5 x^3 y^2 + b_6 x^3 y^3 \tag{14}$$

with b_1, b_2, b_3 not all equal to zero. The identities

$$\begin{pmatrix} zx\zeta^2 \\ zy \end{pmatrix} = \begin{pmatrix} \zeta^3 b_1 & b_2 & \zeta b_3 \\ b_2 & b_3 & b_5 \end{pmatrix} \cdot {}^t(x^2y^3 \ x^3y^2 \ x^3y^3)$$
$$(zx^2 \ zxy \ zy^2) = (\zeta^2 b_1 \ \zeta^2 b_2 \ b_3) \ x^3 y^3$$

imply that if $\zeta b_1 b_3 - b_2^2$ is not equal to zero then $\{z, zx, zy, x^3y^3\}$ is a k-basis of I and its Plücker coordinates are given by

$$\begin{aligned} b_1 &= xy^3 \wedge x^2y^3 \wedge x^3y^2 \wedge x^3y^3 = z_0 \\ b_2 &= x^2y^2 \wedge x^2y^3 \wedge x^3y^2 \wedge x^3y^3 = z_1 \\ b_3 &= x^3y \wedge x^2y^3 \wedge x^3y^2 \wedge x^3y^3 = z_2 \end{aligned}$$

while the other Plücker coordinates are all equal to zero. Thus I corresponds to a point of S. This also holds in the case $\zeta b_1 b_3 = b_2^2$ considering the closure of $\zeta b_1 b_3 - b_2^2$ not equal to zero. Therefore rank 4 right ideals in xyA corresponds bijectively to k-points of S. Moreover, I contains an element z in (14) with $\zeta b_1 b_3 = b_2^2$ if and only if $I = \langle z, x^2y^3, x^3y^2, x^3y^3 \rangle$, i.e. a non-princiapl ideal. The conic $\zeta b_1 b_3 = b_2^2$ is equal to C in (iii-b). □

4. Galois descent

Let A be a central simple algebra of degree n over a field F with the unit group A^*. Let K/F be a Galois splitting field of A with the Galois group $\mathcal{G}$ and let $\{\phi_\sigma\} \in Z^1(\mathcal{G}, PGL(V_K))$ be a 1-cocycle defining A ($\dim\ V_K = n$). From the cocycle condition $\phi_\sigma^\tau \cdot \phi_\tau = \phi_{\sigma\tau}$ if we define the K/F-semi-linear transformations $[\sigma]$ of $M_n(K)$ by $x^{[\sigma]} = \phi_\sigma^{-1} \cdot x^\sigma \cdot \phi_\sigma$ for $\sigma \in \mathcal{G}$ then we see $x^{[\sigma][\tau]} = x^{[\sigma\tau]}$ and A is realized as the F-subalgebra of End V_K fixed by this Galois group action on End V_K :

$$A = \{x \in \text{End } V_K \mid x^{[\sigma]} = x \quad \text{for all } \sigma \in \mathcal{G}\}.$$

We see from this

Lemma 7. *A K-homomorphism $\rho : GL(V_K) \to GL(W_K)$ descends to an F-homomorphism $\rho_0 : A^* \to GL(W_F)$ if and only if there is an element $\lambda \in GL(W_K)$ such that the diagram*

$$\begin{array}{ccc} GL(V_K) & \xrightarrow{\rho} & GL(W_K) \\ {\scriptstyle [\sigma]}\big\downarrow & & \big\downarrow{\scriptstyle [\sigma]'} \\ GL(V_K) & \xrightarrow{\rho} & GL(W_K) \end{array} \tag{1}$$

is commutative for all elements $\sigma \in \mathcal{G}$ where $[\sigma]'$ is defined by $y^{[\sigma]'} = \lambda^{-1} \cdot \lambda^\sigma \cdot y^\sigma \cdot (\lambda^\sigma)^{-1} \cdot \lambda$ for $y \in GL(W_K)$.

Proof. The K-homomorphism ρ descends over F if and only if there is a 1-cocycle $\{\theta_\sigma\} \in Z^1(\mathcal{G}, \text{Aut}_{K\text{-alg}}\, GL(W_K))$ such that the diagram (1) is commutative for $y^{[\sigma]'} = \theta_\sigma^{-1} \cdot y^\sigma \cdot \theta_\sigma$. Since θ defines the split group $GL(W)$ over F the 1-cocycle θ is equal to zero in $H^1(\mathcal{G}, PGL(W_K))$. This implies there is a 1-cocycle $\{\lambda_\sigma\} \in Z^1(\mathcal{G}, GL(W_K))$ such that $\theta_\sigma \equiv \lambda_\sigma$ in $PGL(W_K)$ or $\theta_\sigma \equiv (\lambda^\sigma)^{-1} \cdot \lambda$ for an element $\lambda \in GL(W_K)$ by Hilbert Theorem 90. □

Remark. If $\{w_1, \cdots, w_N\}$ is a K-basis of W_K (dim $W_K = N$) then the images $\{w_1^\lambda, \cdots, w_N^\lambda\}$ by $\lambda \in GL(V_K)$ in Lemma 7 is an F-basis of W. For, let ${}^t(w_1^\lambda, \cdots, w_N^\lambda) = M_\lambda \cdot {}^t (w_1, \cdots, w_N)$ for a matrix $M_\lambda \in GL_N(K)$. Then

$$\begin{aligned}{}^t(w_1^\lambda, \cdots, w_N^\lambda)^{\sigma\cdot(\lambda^\sigma)^{-1}\cdot\lambda} &= M_\lambda^\sigma \cdot {}^t (w_1, \cdots, w_N)^{(\lambda^\sigma)^{-1}\lambda} \\ &= {}^t(w_1, \cdots, w_N)^\lambda\end{aligned}$$

i.e. $\{w_i^\lambda\}$ is invariant under the K/F-semi-linear transformations $\sigma \cdot (\lambda^\sigma)^{-1}\lambda$ for all $\sigma \in \mathcal{G}$.

Let $A = (f, g)_{3,F}$ be a cyclic algebra of degree 3 over F. Then $K = F(\alpha)$ with $\alpha^3 = f$ is a Galois splitting field whose Galois group $\mathcal{G} = \mathbb{Z}/3$ is generated by $\sigma : \alpha \to \zeta\alpha$ with $\zeta = exp(2\pi\sqrt{-1}/3)$. Lemma 2 in Section 1 shows there is an F-homomorphism $\rho_0 : A^* \to GL(W)$ such that $W \otimes K$ is the third symmetric tensor representation $S_{(3)}V_K$ of $A^* \otimes K = GL(V_K)$. We will write down below an F-basis of W and an ideal basis of the Severi-Brauer variety in $\mathbb{P}[W] = \mathbb{P}_F^9$ of $A = (f, g)_{3,F}$. Let $\{v_1, v_2, v_3\}$ be a K-basis of V_K and let

$$(u) = {}^t(u_{111},\ u_{222},\ u_{333},\ u_{112},\ u_{223},\ u_{331},\ u_{122},\ u_{233},\ u_{331},\ u_{123})$$

be the column vector consisting of the K-basis of $S_{(3)}V_K$ where $u_{ijk} = v_i v_j v_k$. The element $\phi_\sigma \in PGL(V_K)$ with

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^{\phi_\sigma} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ g & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \phi_{\sigma^2} = \phi_\sigma^2, \quad \phi_1 = I_3.$$

is a 1-cocycle $\{\phi_\sigma\} \in Z^1(\mathcal{G}, PGL(V_K))$ defining $A = (f, g)_{3,F}$. We take a preimage $\theta_\sigma \in GL(S_{(3)}V_K)$ of $\rho(\phi_\sigma) \in PGL(S_{(3)}V_K)$ by $(u)^{\theta_\sigma} = B_\sigma \cdot (u)$. Here $B_\sigma \in GL_{10}(K)$ is equal to

$$\text{diagonal}(\begin{pmatrix} 0 & g^{-1} & 0 \\ 0 & 0 & g^{-1} \\ g^2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & g^{-1} & 0 \\ 0 & 0 & 1 \\ g & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & g^{-1} & 0 \\ 0 & 0 & g \\ 1 & 0 & 0 \end{pmatrix}, 1).$$

Next we shall find a $C \in GL_{10}(K)$ such that $B_\sigma = (C^\sigma)^{-1}C$. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix}, R = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}$$

$$S_{r_1,r_2,r_3} = P \cdot Q \cdot R = \begin{pmatrix} r_1 & r_2 & r_3 \\ \alpha^{-1}r_1 & \alpha^{-1}\zeta^2 r_2 & \alpha^{-1}\zeta r_3 \\ \alpha^{-2}r_1 & \alpha^{-2}\zeta r_2 & \alpha^{-2}\zeta^2 r_3 \end{pmatrix}$$

for $r_i \in F$. Then $(S^{\sigma}_{r_1,r_2,r_3})^{-1}\, S_{r_1,r_2,r_3}$ is equal to

$$R^{-1}Q^{-1}P^{-1}\begin{pmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{pmatrix}PQR = R^{-1}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}R$$
$$= \begin{pmatrix}0&r_1^{-1}r_2&0\\0&0&r_2^{-1}r_3\\r_3^{-1}r_1&0&0\end{pmatrix}.$$

Hence if we set C = diagonal$(S_{1,g^{-1},g^{-2}}, S_{1,g^{-1},g^{-1}}, S_{1,g^{-1},1}, 1)$ then $(C^{\sigma})^{-1}C = B_{\sigma}$. By the Remark after Lemma 7 an F-basis of W is given by $C\cdot(u_{ijk})$, i.e. the following ten elements; $w = u_{123}$ and

$$\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix} = PQ\begin{pmatrix}1&0&0\\0&g^{-1}&0\\0&0&g^{-2}\end{pmatrix}\begin{pmatrix}u_{111}\\u_{222}\\u_{333}\end{pmatrix}$$
$$\begin{pmatrix}y_0\\y_1\\y_2\end{pmatrix} = PQ\begin{pmatrix}1&0&0\\0&g^{-1}&0\\0&0&g^{-1}\end{pmatrix}\begin{pmatrix}u_{112}\\u_{223}\\u_{331}\end{pmatrix} \qquad (2)$$
$$\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix} = PQ\begin{pmatrix}1&0&0\\0&g^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}u_{122}\\u_{233}\\u_{311}\end{pmatrix}.$$

Proposition 8. *The following set of 27 quadric equations is an ideal basis of the Severi-Brauer variety $A = (f, g)_{3,F}$ in $\mathbb{P}^9_F$ with the homogeneous coordinates (x_i, y_i, z_i, w) $(i = 0, 1, 2)$ $(\zeta = exp(2\pi\sqrt{-1}/3))$.*

$$\begin{pmatrix}x_0&fx_2&fx_1\\x_1&x_0&fx_2\\x_2&x_1&x_0\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix} = \begin{pmatrix}y_0&fy_2&fy_1\\y_1&y_0&fy_2\\y_2&y_1&y_0\end{pmatrix}\begin{pmatrix}y_0\\y_1\\y_2\end{pmatrix}$$
$$g\begin{pmatrix}x_0&fx_2&fx_1\\x_1&x_0&fx_2\\x_2&x_1&x_0\end{pmatrix}\begin{pmatrix}y_0\\\zeta^2y_1\\\zeta y_2\end{pmatrix} = \begin{pmatrix}z_0&fz_2&fz_1\\\omega^2z_1&\zeta^2z_0&\zeta^2fz_2\\\omega z_2&\zeta z_1&\zeta z_0\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix}$$
$$\begin{pmatrix}y_0&fy_2&fy_1\\y_1&y_0&fy_2\\y_2&y_1&y_0\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix} = w\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}$$
$$g\begin{pmatrix}x_0&fx_2&fx_1\\x_1&x_0&fx_2\\x_2&x_1&x_0\end{pmatrix}\begin{pmatrix}x_0\\\zeta x_1\\\zeta^2x_2\end{pmatrix} = \begin{pmatrix}y_0&fy_2&fy_1\\y_1&y_0&fy_2\\y_2&y_1&y_0\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix}$$

$$g\begin{pmatrix} x_0 & fx_2 & fx_1 \\ x_1 & x_0 & fx_2 \\ x_2 & x_1 & x_0 \end{pmatrix}\begin{pmatrix} y_0 \\ \zeta y_1 \\ \zeta^2 y_2 \end{pmatrix} = \begin{pmatrix} z_0 & fz_2 & fz_1 \\ z_1 & z_0 & fz_2 \\ z_2 & z_1 & z_0 \end{pmatrix}\begin{pmatrix} z_0 \\ \zeta^2 z_1 \\ \zeta z_2 \end{pmatrix}$$

$$= w\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$$g\begin{pmatrix} x_0 & fx_2 & fx_1 \\ x_1 & x_0 & fx_2 \\ x_2 & x_1 & x_0 \end{pmatrix}\begin{pmatrix} z_0 \\ \zeta z_1 \\ \zeta^2 z_2 \end{pmatrix} = g\begin{pmatrix} y_0 & fy_2 & fy_1 \\ y_1 & y_0 & fy_2 \\ y_2 & y_1 & y_0 \end{pmatrix}\begin{pmatrix} y_0 \\ \zeta^2 y_1 \\ \zeta y_2 \end{pmatrix}$$

$$= w\begin{pmatrix} z_0 \\ \zeta^2 z_1 \\ \zeta z_2 \end{pmatrix}$$

$$g\begin{pmatrix} y_0 & fy_2 & fy_1 \\ y_1 & y_0 & fy_2 \\ y_2 & y_1 & y_0 \end{pmatrix}\begin{pmatrix} z_0 \\ \zeta z_1 \\ \zeta^2 z_2 \end{pmatrix} = \begin{pmatrix} w^2 \\ 0 \\ 0 \end{pmatrix}.$$

Proof. Note first that the Veronese embedding $\mathbb{P}[V] \to \mathbb{P}[S_{(3)}V]$ of degree three is defined by the quadrics in the subspace $S_{(4,2)}V$ in $S_{(2)}S_{(3)}V = S_{(6)}V \oplus S_{(4,2)}V$ so there are $\dim S_{(4,2)}V = 27$ linearly independent quadrics. We see from (2)

$$\begin{pmatrix} u_{111} \\ u_{222} \\ u_{333} \end{pmatrix} = \begin{pmatrix} X_0 \\ gX_1 \\ g^2X_2 \end{pmatrix}, \quad 3\begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}\begin{pmatrix} x_0 \\ \alpha x_1 \\ \alpha^2 x_2 \end{pmatrix}$$

$$\begin{pmatrix} u_{112} \\ u_{223} \\ u_{331} \end{pmatrix} = \begin{pmatrix} Y_0 \\ gY_1 \\ gY_2 \end{pmatrix}, \quad 3\begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}\begin{pmatrix} y_0 \\ \alpha y_1 \\ \alpha^2 y_2 \end{pmatrix}$$

$$\begin{pmatrix} u_{122} \\ u_{233} \\ u_{311} \end{pmatrix} = \begin{pmatrix} Z_0 \\ gZ_1 \\ Z_2 \end{pmatrix}, \quad 3\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}\begin{pmatrix} z_0 \\ \alpha z_1 \\ \alpha^2 z_2 \end{pmatrix}$$

The first three identities in Proposition 8 follow from $u_{111}u_{122} = u_{112}^2$:

$$9u_{111}u_{122} = 9X_0Z_0 = (x_0\ x_1\ x_2)\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}(1\ \alpha\ \alpha^2)\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

$$= (x_0\ x_1\ x_2) \begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & f \\ \alpha^2 & f & f\alpha \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

$$= (1\ \alpha\ \alpha^2) \begin{pmatrix} x_0 & fx_2 & fx_1 \\ x_1 & x_0 & fx_2 \\ x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}.$$

$$9u_{112}^2 = (3Y_0)^2 = (1\ \alpha\ \alpha^2) \begin{pmatrix} y_0 & fy_2 & fy_1 \\ y_1 & y_0 & fy_2 \\ y_2 & y_1 & y_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}.$$

The remaining eight sets of the three identities are obtained by the same way from the realtions $u_{111}u_{133} = u_{113}^2$ and

$$u_{111}u_{123} = u_{112}u_{113}, \qquad u_{111}u_{222} = u_{112}u_{122}$$
$$u_{111}u_{223} = u_{113}u_{122} = u_{112}u_{123}$$
$$u_{111}u_{233} = u_{112}u_{133} = u_{113}u_{123}$$
$$u_{112}u_{233} = u_{123}^2$$

respectively. □

The above calculations are generalized to those of a cyclic algebra of higher degree, e.g. it can be written by hand $465 = \dim S_{(2)}S_{(4)}V - \dim S_{(8)}V$ (where $\dim V = 4$) quadrics defining the Severi-Brauer variety in $\mathbb{P}^{34}$ associated with a cyclic algebra $(f, g)_{4,F}$ of degree four. However, if we set $f = g = 0$ in the 27 quadrics in Proposition 8 then the resulting algebraic set in $\mathbb{P}^9$ has dimension greater than two so these quadrics are not used for a flat fibre space with the generic fibre isomorphic to the Severi-Brauer variety of $(f, g)_{3,F}$.

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