Intermediate Jacobians of projective plane bundles over a smooth projective surface

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INTERMEDIATE JACOBIANS OF PROJECTIVE PLANE BUNDLES OVER A SMOOTH PROJECTIVE SURFACE

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ABSTRACT. Let $V \to X$ be a standard $\mathbb{P}^2$-bundle (Definition below) over a smooth projective surface X with the discriminant locus Δ and the associated cyclic cover $\phi : \tilde{\Delta} \to \Delta$ of degree three. The purpose of this paper is (i) to determine the etale l-adic cohomology groups of V (Theorem A), (ii) to give an isomorphism of the intermediate jacobian of V and the Prym variety associated to the triple cover ϕ as polarized abelian varieties (Theorem B), and (iii) to show the existence of a standard $\mathbb{P}^2$-bundle for a given cyclic cover of degree three over a normal crossing curve on X (Theorem D), under certain conditions of (X, Δ). An ideal basis of a standard $\mathbb{P}^2$-bundle over a regular local ring is determined (Theorem E).

1. Introduction

Let K be the function field of an algebraic variety defined over an algebraically closed field k of characteristic different from three and let V_K be a Severi-Brauer variety of dimesion two (a Severi-Brauer surface, for short) over K, i.e. $V_K \times_K \bar{K}$ is isomorphic to the projective plane $\mathbb{P}^2$ for an algebraic closure $\bar{K}$ of K.

Definition. *A proper flat morphism*

$$\tau : V \to X \tag{1.1}$$

is a standard $\mathbb{P}^2$-bundle associated to the Severi-Brauer surface V_K over a function field K if (i) V and X are smooth projective varieties with the generic fibre isomorphic to the given Severi-Brauer surface $V_K \to Spec(K)$, (ii) the locus Δ over which the fibres of τ are

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non-smooth is equal to the discriminant locus [A-M,p84] of the central simple algebra over K corresponding to the generic fibre V_K, and Δ is a normal crossing curve on X, (iii) the geometric fibre over a smooth point of Δ consists of three components Y_i ($i = 1,2,3$) with $Y_i \cong \mathbb{F}_1$ (one point blow-up of $\mathbb{P}^2$), $Y_i \cap Y_{i+1}$ (resp. $Y_i \cap Y_{i-1}$) is a fibre (resp. the (-1)-curve) of $Y_i \cong \mathbb{F}_1$ (where the suffix means mod 3) and $Y_1 \cap Y_2 \cap Y_3$ is one point, and the geometric fibre $\tau^{-1}(p)$ over a singular point p of Δ is non-reduced with the reduced part isomorphic to the cone over a rational twisted cubic in $\mathbb{P}^3$.

In [Ma1] is proved that there is a standard $\mathbb{P}^2$-bundle associated to any Severi-Brauer surface over a function field, which is a flat contraction morphism of an extremal ray, in particular the relative Picard number is equal to one. In Theorem B and C we assume the non-smooth locus Δ of a standard $\mathbb{P}^2$-bundle is *nonsingular*. Let W be the normalization of $\tau^{-1}(\Delta)$ in the function field of $\tau^{-1}(\Delta)$. The Stein factorization of the composite $W \to \tau^{-1}(\Delta) \to \Delta$ is given by

$$W \to \tilde{\Delta} \xrightarrow{\phi} \Delta \tag{1.2}$$

where $W \to \tilde{\Delta}$ is an $\mathbb{F}_1$-bundle and $\phi : \tilde{\Delta} \to \Delta$ is the cyclic cover of degree three associated to the tame symbol ${}_3\mathrm{Br}\ K \to \oplus\ \kappa_v^*/\kappa_v^{*3}$ [A-M,p84]. Here ${}_3\mathrm{Br}\ K$ is the 3-torsion part of the Brauer group $\mathrm{Br}\ K$ of K, and κ_v's are the residue fields of discrete valuations of K over k. Since Δ is equal to the discriminant locus of the central simple algebra associated to V_K and assumed nonsingular, ϕ is etale and nontrivial over each irreducible component of Δ. We call $\phi : \tilde{\Delta} \to \Delta$ the associated cyclic cover of the standard $\mathbb{P}^2$-bundle (1.1).

Theorem A. *Assume char k is equal to neither three nor a prime number l, and a standard $\mathbb{P}^2$-bundle (1.1) over a smooth irreducible projective surface X satisfies that (i) the etale l-adic cohomology groups $H^1(X,\mathbb{Z}_l) = H^3(X,\mathbb{Z}_l) = 0$ and $H^2(X,\mathbb{Z}_l)$ is torsion free, (ii) the discriminant locus Δ of τ consists of a disjoint union of n smooth curves on X. Let g be the arithmetic genus of Δ. Then the etale l-adic cohomology groups $H^q(V) = H^q(V,\mathbb{Z}_l)$ of V ($1 \le q \le 4$) are isomorphic to*

$$H^1(V) = 0, \qquad H^2(V) \cong H^2(X) \oplus \mathbb{Z}_l$$
$$H^3(V) \cong (\mathbb{Z}_l/3\mathbb{Z}_l)^{n-1} \oplus \mathbb{Z}_l^{4(g-n)}$$
$$H^4(V) \cong (\mathbb{Z}_l/3\mathbb{Z}_l)^{n-1} \oplus H^2(X) \oplus \mathbb{Z}_l^2.$$

Geometrically, $\mathbb{Z}_l^2$ in $H^4(V)$ corresponds to $\mathbb{F}_1$ in a fibre over a point of Δ and a subvariety $\tilde{X}$ of V such that $\tau : \tilde{X} \to X$ is generically finite of degree three. The torsion part of $H^3(V)$ (resp. $H^4(V)$) is generated by the differences of fibres of $\mathbb{F}_1$'s (resp. the differences of $\mathbb{F}_1$'s) which are mapped to points on different connected components of Δ. The free part $\mathbb{Z}_l^{4(g-n)}$ of $H^3(V)$ is isomorphic to $H^1(\tilde{\Delta})/\phi^* H^1(\Delta)$. In particular, if char k is differnt from three and if Δ is not connected (i.e. $n \geq 2$) then $H^3(V, \mathbb{Z}_3)$ has nontrivial torsion elements, so V is not a rational variety [A-M,p78,Prop.1]. Assume the base field is complex numbers $\mathbb{C}$. Since $\tau : V \to X$ is a contraction morphism of an extremal ray we see $H^3(V, \mathcal{O}_V) = H^3(X, \mathcal{O}_X)$. Hence $H^3(X, \mathbb{Z}) = 0$ guarantees $H^3(V, \mathbb{C}) = H^{12} \oplus H^{21}$, and H^{12} consists of primitive forms if $H^1(X, \mathbb{Z}) = 0$. Therefore an ample divisor of V defines a polarization Ξ_V on the second intermediate jacobian $J^2(V) = H^{12}/H_{\mathbb{Z}}$ [G,p8], where $H_{\mathbb{Z}}$ is the image in $H^{12}(V)$ under the natural homomorphism $H^3(V, \mathbb{Z}) \to H^3(V, \mathbb{C}) \to H^{12}(V)$. We take as an ample divisor $-K_V + \tau^* D$ for the anticanonical divisor $-K_V$ of V and a divisor D on X. As in the case of quadric bundles [B,p329,Th.2.1] we show

Theorem B. *Assume a standard $\mathbb{P}^2$-bundle (1.1) over a smooth projective surface X over $\mathbb{C}$ satisfies (i) $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is torsion free, (ii) the discriminant locus Δ is non-empty, nonsingular and irreducible. Then the second intermediate jacobian $(J^2(V), \Xi_V)$ defined above is isomorphic to the Prym variety (P, Ξ_P) of the associated cyclic cover (1.2) as polarized abelian varieties.*

Here the Prym variety P means the abelian subvariety of the jacobian $J(\tilde{\Delta})$ of $\tilde{\Delta}$ which is equal to the image of the endomorphism $1 - \iota$ of $J(\tilde{\Delta})$ for the covering automorphism ι of $\tilde{\Delta}$ over Δ and the polarization Ξ_P is the restriction to P of the theta divisor $\Xi_{\tilde{\Delta}}$ of $J(\tilde{\Delta})$ ([B,p316],[R,p60]). Contrary to the Prym variety associated to double covers, the polarization Ξ_P is not a multiple of a princiapl polarization. Indeed, if the genus of Δ is equal to g then the kernel of the polarization $\Xi_P : P \to \hat{P}$ is equal to $P \cap \phi_* J(\Delta) \cong (\mathbb{Z}/3\mathbb{Z})^{2g-2}$, hence the type is equal to $(1, ..., 1, 3, ..., 3)$ with 1 and 3 repeated $g-1$ times, respectively [R,p65]. Let $A^q(V)$ be the group of codimension q algebraic cycles of V algebraically equivalent to zero modulo rationally equivalent to zero. By arguments almost same as in the proof of Theorem B we see

Corollary C. *Assume a standard $\mathbb{P}^2$-bundle (1.1) over a smooth projective surface X over $\mathbb{C}$ satisfies (i) $A^q(X) = 0$ for any $q \in \mathbb{Z}$, (ii) Δ is non-empty, nonsingular and irreducible. Then there is a homomorphism θ from $A^1(\tilde{\Delta}) = Pic^0\ \tilde{\Delta}$ to $A^2(V)$, which induces the exact sequence*

$$0 \to \phi^* Pic^0\ \Delta \to Pic^0\ \tilde{\Delta} \xrightarrow{\theta} A^2(V) \to 0.$$

Combining Theorem B with Corollary C we see the Abel-Jacobi map from $A^2(V)$ to the intermediate jacobian $J^2(V)(\mathbb{C})$ is bijective. For a simply connected smooth projective surface X with the function field K there is an exact sequence [A-M,p84] :

$$\begin{aligned} 0 \to \mathrm{Br}\ X \to \mathrm{Br}\ K \to \oplus_{x \in X^{(1)}} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \\ \xrightarrow{\partial} \oplus_{x \in X^{(2)}} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0, \end{aligned} \qquad (1.3)$$

where $\mathrm{Br}\ X = H^2(X, \mathbb{G}_m)$ and $X^{(i)}$ is the set of codimension i subvarieties of X. Let $\phi : \tilde{\Delta} \to \Delta$ be a double cover over a normal crossing curve Δ on X. In [Sa,p388] is proved that if ϕ is mapped to zero under the homomorphism ∂ in (1.3) then there is a standard conic bundle over X whose associated double cover is equal to ϕ. By using a result in [A,p208] we show the same is true for a standard $\mathbb{P}^2$-bundle.

Theorem D. *Let X be a simply connected smooth projective surface over $\mathbb{C}$ with the function field K. Let Δ be a normal crossing curve on X and let $\phi : \tilde{\Delta} \to \Delta$ be a cyclic cover of degree three which is mapped to zero by the homomorphism ∂ in (1.3). Then there is a standard $\mathbb{P}^2$-bundle over X whose associated cyclic cover is equal to ϕ.*

The paper is organized as follows. Theorem A is proved in Section two using the trace map between H^1 of the associated cyclic cover (1.2). Section three is devoted to proving Theorem B and Corollary C by the idea in Chapter II and III of [B], respectively. Theorem D is proved in Section four after a K-theoretic simple proof of the exact sequence (1.3). In Section five an ideal basis of a standard $\mathbb{P}^2$-bundle over a regular local ring is determined explicitly (Theorem E), from which the results of section 2 in [Ma1] (V is a regular scheme etc.) easily follows.

2. The l-adic cohomology groups

In this section we prove Theorem A. We assume the discriminant locus Δ is a disjoint union of n smooth curves Δ_i with genus g_i ($1 \leq i \leq n$). We denote by $F = \mathbb{Z}/(l^m)$ for a prime number l different from char k and

$$H^*(S) = H^*_{et}(S, F_S),$$

the etale cohomology groups of a scheme S with coefficient the constant sheaf F_S on S. For a point $p \in \Delta$, the fibre $\tau^{-1}(p)$ consists of three components Y_i ($i = 1, 2, 3$) with $Y_i \cong \mathbb{F}_1$ and $Y_i \cap Y_{i+1}$ (resp. $Y_i \cap Y_{i-1}$) is a fibre (resp. the (-1)-curve) of $Y_i \cong \mathbb{F}_1$ (where the suffix means mod 3) and $Y_1 \cap Y_2 \cap Y_3$ consists of one point.

Lemma 1. *(i) For a point* $p \in X - \Delta$, $H^q(\tau^{-1}(p)) = F$ $(q = 0, 2, 4)$, 0 $(q \neq 0, 2, 4)$.

(ii) For a point $p \in \Delta$, $H^q(\tau^{-1}(p)) = F$ $(q = 0)$, F^3 $(q = 2, 4)$, 0 $(q \neq 0, 2, 4)$.

Proof. (i) follows from $\tau^{-1}(p) \cong \mathbb{P}^2$ for a point $p \in X - \Delta$. Since $Y_1 \cong \mathbb{F}_1$ we see $H^q(Y_1) = F$ $(q = 0, 4)$, and F^2 $(q = 2)$ and 0 $(q \neq 0, 2, 4)$. Since $Y_1 \cap Y_2 \cong \mathbb{P}^1$, considering the Mayer-Vietoris sequence for the pair (Y_1, Y_2) we see $H^q(Y_1 \cup Y_2) = F$ $(q = 0)$, F^3 $(q = 2)$, F^2 $(q = 4)$ and 0 $(q \neq 0, 2, 4)$. Similarly, $(Y_1 \cup Y_2) \cap Y_3 = (Y_1 \cap Y_3) \cup (Y_2 \cap Y_3)$ is a union of two $\mathbb{P}^1$'s intersecting at one point, so $H^q((Y_1 \cup Y_2) \cap Y_3) = F$ $(q = 0)$, F^2 $(q = 2)$, 0 $(q \neq 0, 2)$. The isomorphisms in (ii) follow from applying the Mayer-Vietoris sequence for the pair $(Y_1 \cup Y_2, Y_3)$. □

Since Δ is smooth, the associated cyclic cover $\phi : \tilde{\Delta} \to \Delta$ of (1.2) is etale, hence the arithmetic genus $\tilde{g}$ of $\tilde{\Delta}$ is equal to

$$\tilde{g} = \Sigma_{i=1}^n (3g_i - 2) = 3g - 2n \tag{2.1}$$

by Hurwitz's formula. Let A be the kernel of the trace homomorphism from $\phi_* F_{\tilde{\Delta}}$ to F_Δ. From the exact sequence

$$0 \to A \to \phi_* F_{\tilde{\Delta}} \overset{trace}{\to} F_\Delta \to 0 \tag{2.2}$$

we see $H^*(A) = H^*(\Delta, A)$ are isomorphic to

$$H^0(A) = (F/3F)^n, \qquad H^2(A) = (F/3F)^n \tag{2.3}$$
$$H^1(A) = (F/3F)^{2n} \oplus F^{2(\tilde{g}-g)} = (F/3F)^{2n} \oplus F^{4(g-n)}$$

and $H^q(A) = 0$ for any $q > 2$. By taking inverse limit we obtain the l-adic cohomology groups.

$$
\begin{aligned}
&\varprojlim H^0(A) = 0, \qquad \varprojlim H^2(A) = (\mathbb{Z}_l/3\mathbb{Z}_l)^n \\
&\varprojlim H^1(A) = (\mathbb{Z}_l/3\mathbb{Z}_l)^n \oplus \mathbb{Z}_l^{4(g-n)}.
\end{aligned} \tag{2.4}
$$

We use the following Lemma for the proof of Theorem A.

Lemma 2. *$R^2\tau_*F_V \cong R^4\tau_*F_V$ and there are commutative diagrams with exact rows and columns for $q = 2, 4$:*

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & i_*A & = & i_*A & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_!F_U & \longrightarrow & R^q\tau_*F_V & \longrightarrow & i_*\phi_*F_{\tilde{\Delta}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle trace} & & \\
0 & \longrightarrow & j_!F_U & \longrightarrow & F_X & \longrightarrow & i_*F_{\Delta} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.5}
$$

where $i : \Delta \subset X$ (resp. $j : U := X - \Delta \subset X$) is the closed (resp. open) immersion.

Assuming Lemma 2 we continue the proof of Theorem A. Let us consider the Leray spectral sequence for the morphism $\tau : V \to X$:

$$E_2^{pq} = H^p(R^q\tau_*F_V) \Rightarrow H^{p+q}(V).$$

We see $E_2^{p,2q+1} = 0$ for any q by Lemma 1, and $E_2^{2p+1,0} = H^{2p+1}(X) = 0$ for any p by the assumption. Thus there are exact sequences:

$$
\begin{aligned}
&0 \to E_2^{20} \to E^2 \to E_2^{02} \to 0, \\
&0 \to E^3 \to E_2^{12} \to E_2^{40} \xrightarrow{f} \mathcal{F}^2E^4 \to E_2^{22} \to 0 \\
&0 \to \mathcal{F}^2E^4 \to E^4 \to E_2^{04} \to E_2^{32}
\end{aligned} \tag{2.6}
$$

From $E_2^{10} = E_2^{01} = 0$ we see $E^1 = H^1(V) = 0$. Now we calculate $E_2^{p2} = H^p(R^2\tau_* F_V)$. From the middle column of (2.5) we see

$$0 \to H^0(A) \to E_2^{02} \to H^0(X) \to H^1(A) \to E_2^{12} \to H^1(X) \to$$
$$\to H^2(A) \to E_2^{22} \to H^2(X) \to H^3(A) \to E_2^{32} \to H^3(X),$$

where $H^0(X) = F$ and $H^1(X) = H^3(X) = 0$ by the assumption. Hence we see from (2.3)

$$E_2^{02} = E_2^{04} = (F/3F)^{n-1} \oplus F, \qquad E_2^{12} = (F/3F)^{2n-1} \oplus F^{4(g-n)},$$
$$E_2^{22} = (F/3F)^n \oplus H^2(X), \qquad E_2^{32} = 0.$$

Substituting these isomorphisms into (2.6) we obtain

$$E^2 = E_2^{20} \oplus E_2^{02} = H^2(X) \oplus F \oplus (F/3F)^{n-1} \tag{2.7}$$
$$0 \to E^3 \to (F/3F)^{2n-1} \oplus F^{4(g-n)} \to$$
$$\to H^4(X) \xrightarrow{f} \mathcal{F}^2 E^4 \to (F/3F)^n \oplus H^2(X) \to 0, \tag{2.8}$$
$$0 \to \mathcal{F}^2 E^4 \to E^4 \to F \oplus (F/3F)^{n-1} \to 0. \tag{2.9}$$

The image in $E^4 = H^4(V)$ of a generator of $H^4(X) \cong F$ under the homomorphism f in (2.8) is represented by the fibre $\tau^{-1}(p)$ of a closed point p of X. If p is contained in Δ then $\tau^{-1}(p)$ consists of three components, hence (2.8) implies

$$E^3 = (F/3F)^{2n-2} \oplus F^{4(g-n)} \tag{2.10}$$
$$\mathcal{F}^2 E^4 = (F/3F)^{n-1} \oplus H^2(X) \oplus F.$$

Hence we see from (2.11)

$$E^4 = (F/3F)^{2n-2} \oplus H^2(X) \oplus F^2. \tag{2.11}$$

In view of (2.4) we obtain the isomorphisms of Theorem A by taking inverse limit in the expressions $E^q = H^q(V)$ $(q = 2, 3, 4)$ in (2.7), (2.10) and (2.11).

(Proof of Lemma 2) By the proper base change theorem[Mi,p225] we have the exact sequence

$$0 \to j_!(R^q\tau_* F_{\tau^{-1}(U)}) \to R^q\tau_* F_V \to i_*(R^q\tau_* F_{\tau^{-1}(\Delta)}) \to 0. \tag{2.12}$$

Here the locally constant sheaf $R^q\tau_* F_{\tau^{-1}(U)}$ is isomorphic to the constant sheaf F_U because $\tau^{-1}(U) \to U$ is a $\mathbb{P}^2$-bundle in etale topology. We shall show isomorphisms $R^q\tau_* F_{\tau^{-1}(\Delta)} \cong \phi_* F_{\tilde{\Delta}}$ for $q = 2, 4$ and (2.12) fits in the middle row of (2.5).

Let $S = \mathrm{Sing}(\tau^{-1}(\Delta))$ be the singular locus of $\tau^{-1}(\Delta)$. If we denote by $\tau^{-1}(p) = \cup_{i=1}^{3} Y_i$ with $Y_i \cong \mathbb{F}_1$ for a closed point $p \in \Delta$ then $\tau^{-1}(p) \cap S = \cup_{i \neq j}(Y_i \cap Y_j)$ consists of three $\mathbb{P}^1$'s intersecting at one point $\cap_{i=1}^{3} Y_i$. Let $\nu : W \to \tau^{-1}(\Delta)$ be the normalization in the function field of $\tau^{-1}(\Delta)$ and $\phi : \tilde{\Delta} \to \Delta$ be the associated cyclic cover (1.2) :

$$\begin{array}{ccc}
S & \longleftarrow & \nu^{-1}(S) \\
\cap & & \cap \\
V \supset \tau^{-1}(\Delta) & \overset{\nu}{\longleftarrow} & W \\
{\scriptstyle\tau}\downarrow \quad \downarrow & & \downarrow{\scriptstyle\pi} \\
X \quad \supset \Delta & \overset{\phi}{\longleftarrow} & \tilde{\Delta}
\end{array}$$

Here the pull back $\nu^{-1}(S)$ of S in W is reducible with two irreducible components $S_{1,i}$ and $S_{2,i}$ over each irreducible component $\tilde{\Delta}_i$ of $\tilde{\Delta}$. For a point $p \in \tilde{\Delta}_i$, $\pi^{-1}(p) \cap S_{1,i}$ (resp. $\pi^{-1}(p) \cap S_{2,i}$) is a fibre (resp. the (-1)-curve) of $\pi^{-1}(p) \cong \mathbb{F}_1$, and $S_{1,i} \cap S_{2,i}$ is a section over $\tilde{\Delta}_i$. Let $S_1 = \cup_i S_{1,i}$ and $S_2 = \cup_i S_{2,i}$, so that $\nu^{-1}(S) = S_1 \cup S_2$. Then $S_1 \cap S_2$ is isomorphic to $\tilde{\Delta}$ and $\nu : S_1 \cap S_2 \to \nu(S_1 \cap S_2) \subset S$ is isomorphic to the associated cyclic cover $\phi : \tilde{\Delta} \to \Delta$ of (1.2).

(Proof of $R^4\tau_* F_{\tau^{-1}(\Delta)} \cong \phi_* F_{\tilde{\Delta}}$) We define B by the exact sequence

$$0 \to F_{\tau^{-1}(\Delta)} \to \nu_* F_W \to B \to 0. \tag{2.13}$$

The support of B is equal to S which is of relative dimension one over Δ. Hence $R^3\tau_* B = R^4\tau_* B = 0$ and an isomorphism

$$R^4\tau_* F_{\tau^{-1}(\Delta)} \cong R^4\tau_*(\nu_* F_W). \tag{2.14}$$

Recall $\pi : W \to \tilde{\Delta}$ is an $\mathbb{F}_1$-bundle (in Zariski toplogy). Since $H^4(\mathbb{F}_1) = \mathbb{Z}$ and $H^2(\mathbb{F}_1)$ is generated by the class of a fibre and the (-1)-curve we see the locally constant sheaf $R^q\tau_* F_W$ is constant :

$$R^2\tau_* F_W = F^2_{\tilde{\Delta}}, \qquad R^4\tau_* F_W = F_{\tilde{\Delta}} \tag{2.15}$$

Now we have isomorphisms

$$
\begin{aligned}
\phi_* F_{\tilde{\Delta}} &\cong \phi_*(R^4\pi_* F_W) && \text{by (2.15)} \\
&\cong R^4(\phi\pi)_* F_W && \text{since } \phi \text{ is a finite morphism} \\
&\cong R^4(\tau\nu)_* F_W && \\
&\cong R^4\tau_*(\nu_* F_W) && \text{since } \nu \text{ is a finite morphism} \\
&\cong R^4\tau_* F_{\tau^{-1}(\Delta)} && \text{by (2.14).}
\end{aligned}
$$

(Proof of $R^2\tau_* F_{\tau^{-1}(\Delta)} \cong \phi_* F_{\tilde{\Delta}}$) For a closed point $p \in \Delta$ we see $H^2(\tau^{-1}(p), F_{\tau^{-1}(\Delta)}) = F^3 \to H^2(\tau^{-1}(p), \tau_* F_W) = F^6$ is injective, and $R^3\tau_* F_{\tau^{-1}\Delta} = 0$ by Lemma 1. Hence we see from the exact sequence (2.13)

$$0 \to R^2\tau_* F_{\tau^{-1}(\Delta)} \to R^2\tau_*(\nu_* F_W) \to R^2\tau_* B \to 0. \tag{2.16}$$

For the proof of an isomorphism $R^2\tau_* F_{\tau^{-1}(\Delta)} \cong \phi_* F_{\tilde{\Delta}}$ we will show $R^2\tau_*(\nu_* F_W) \cong \phi_* F^2_{\tilde{\Delta}}$ and $R^2\tau_* B \cong \phi_* F_{\tilde{\Delta}}$. Since $R^2\pi_* F_W \cong F^2_{\tilde{\Delta}}$ by (2.15) we see

$$
\begin{aligned}
\phi_* F^2_{\tilde{\Delta}} &\cong \phi_*(R^2\pi_* F_W) \cong R^2(\phi\pi)_* F_W \\
&\cong R^2(\tau\nu)_* F_W \cong R^2\tau_*(\nu_* F_W).
\end{aligned} \tag{2.17}
$$

Next we will show $R^2\tau_* B \cong \phi_* F_{\tilde{\Delta}}$. The restriction of ν to the open set $W - \nu^{-1}(S)$ of W is an isomorphism, so the exact sequence (2.13) on $\tau^{-1}(\Delta)$ induces the exact sequence

$$0 \to F_S \to \nu_* F_{\nu^{-1}(S)} \to B \to 0 \tag{2.18}$$

on S, where $F_S \to \nu_* F_{\nu^{-1}(S)}$ is injective. Let $\nu^{-1}(S) = S_1 \cup S_2$ and let $U_0 = S - \nu(S_1 \cap S_2)$. Then $\nu^{-1}(U_0)$ is a disjoint union of two components U_1 and U_2, which are isomorphic to U_0 by ν. Restricting $\nu_* F_{\nu^{-1}(S)}$ to U_0 we see $j^*\nu_* F_{\nu^{-1}(S)} \cong \nu_* F_{\nu^{-1}(U_0)} \cong F^2_{U_0}$ (where $j : U_0 \subset S$ is the open immersion), hence $j^* B \cong F_{U_0}$ by (2.18). In the exact sequence

$$0 \to j_! j^* B \cong j_! F_{U_1} \to B \to i_* i^* B \to 0$$

on S (where $i : \nu(S_1 \cap S_2) \subset S$ is the closed immersion), $R^q\tau_*(i_*i^*B) = 0$ for $q = 1, 2$ because the support of i_*i^*B is equal to $\nu(S_1 \cap S_2)$, which is of relative dimension one over Δ (i.e. a section of $\tau : S \to \Delta$). Hence $R^2\tau_*B \cong R^2\tau_*(j_!F_{U_0})$. For the proof of the isomorphism $R^2\tau_*B \cong \phi_*F_{\tilde{\Delta}}$ we will show $R^2\tau_*(j_!F_{U_0}) \cong \phi_*F_{\tilde{\Delta}}$. Since $\nu : U_1 \to U_0$ is an isomorphism we may replace $U_0 \subset S$ by $U_1 \subset S_1$ [Milne,p227,Prop.(3.1)]. Therefore the restriction of τ to U_1 is factored by

$$\tau : U_1 \subset S_1 \xrightarrow{\pi} \tilde{\Delta} \xrightarrow{\phi} \Delta$$

with a $\mathbb{P}^1$ (resp. $\mathbb{A}^1$)-bundle $\pi : S_1 \to \tilde{\Delta}$ (resp. $U_1 \subset S_1 \to \tilde{\Delta}$). Let $i : Z = S_1 - U_1 \cong \tilde{\Delta} \subset S_1$ be the closed immersion. In the exact sequence

$$0 \to j_!F_{U_1} \to F_{S_1} \to i_*F_Z \to 0,$$

we see $R^q\tau_*(i_*F_Z) = 0$ for $q = 1, 2$, hence

$$R^2\tau_*(j_!F_{U_1}) \cong R^2\tau_*F_{S_1} \cong R^2(\phi\pi)_*F_{S_1}$$
$$\cong \phi_*(R^2\pi_*F_{S_1}) \cong \phi_*F_{\tilde{\Delta}}.$$

Thus $R^2\tau_*B \cong R^2\tau_*(j_!F_{U_0}) \cong \phi_*F_{\tilde{\Delta}}$. Combining this with the isomorphism $R^2\tau_*(\nu_*F_W) \cong \phi_*F^2_{\tilde{\Delta}}$ obtained in (2.17) we get the isomorphism $R^2\tau_*F_{\tau^{-1}(\Delta)} \cong \phi_*F_{\tilde{\Delta}}$ by the exact sequence (2.16).

(End of the proof of Lemma 2) The exact sequence (2.12) is equal to $0 \to j_!F_U \to R^q\tau_*F_V \to i_*\phi_*F_{\tilde{\Delta}} \to 0$. On the other hand, we have the canonical homomorphism $f : R^q\tau_*F_V \to j_*j^*R^q\tau_*F_V \cong j_*R^q\tau_*F_{\tau^{-1}(U)} \cong j_*F_U \cong F_X$. Therefore we have the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!F_U & \longrightarrow & R^q\tau_*F_V & \longrightarrow & i_*\phi_*F_{\tilde{\Delta}} & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow \mathit{trace} & & \\ 0 & \longrightarrow & j_!F_U & \longrightarrow & F_X & \longrightarrow & i_*F_\Delta & \longrightarrow & 0. \end{array}$$

Here the right-hand square is commutative, from which we obtain the diagram (2.5).

3. Intermediate jacobians

In this section we work over $\mathbb{C}$ and prove Theorem B. We use the following notations. For a point $p \in \Delta$, let $Y_{p,i}$ $(i = 1, 2, 3)$ be the three components of $\tau^{-1}(p)$ and set the closed subschemes

$$\Delta_0 = \cup_{p\in\Delta} \cap_{i=1}^{3} Y_{i,p}, \qquad M_0 = \cup_{p\in\Delta} \cup_{i\neq j} (Y_{i,p} \cap Y_{j,p})$$

of $\tau^{-1}(\Delta)$ with reduced induced structures. We see Δ_0 (resp. M_0) is the singular locus of $\tau^{-1}(\Delta)$ with multiplicity three (resp. two) at its generic point, and Δ_0 is a section of τ over Δ. Let $\sigma : V_1 \to V$ be the blow-up along Δ_0 with the exceptional divisor E_1 and the proper transform D_1 of $\tau^{-1}(\Delta)$. Let $\epsilon : V_2 \to V_1$ be the blow-up along the proper transform M_1 of M_0 with the exceptional divisor F and the proper transform D_2 (resp. E_2) of D_1 (resp. E_1).

$$\begin{array}{cccc}
V_2 & D_2 & F & E_2 \\
\epsilon\downarrow & \downarrow & \downarrow & \downarrow \\
V_1 & D_1 & M_1 & E_1 \\
\sigma\downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}$$
$$V \supset \tau^{-1}(\Delta) \supset M_0 \supset \Delta_0$$

Then $D_2 + F + E_2$ is a simple normal crossing divisor of V_2. The geometric fibres of $D_2 \xrightarrow{\epsilon} D_1 \xrightarrow{\sigma} \tau^{-1}(\Delta) \xrightarrow{\tau} \Delta$ and $M_1 \xrightarrow{\sigma} M_0 \xrightarrow{\tau} \Delta$ consist of three connected components and the Stein factorizations are given by

$$D_2 \xrightarrow{q} \tilde{\Delta} \xrightarrow{\phi} \Delta, \qquad M_1 \xrightarrow{r} \tilde{\Delta} \xrightarrow{\phi} \Delta, \tag{3.1}$$

with the associated cyclic cover $\phi : \tilde{\Delta} \to \Delta$ of (1.2) in Introduction. Here r is a $\mathbb{P}^1$-bundle and q is a fibre bundle in Zariski topology with fibre $Y'_{p,i}$ isomorphic to one point blow-up of $Y_{p,i} \cong \mathbb{F}_1$. The morphism $q : D_2 \to \tilde{\Delta}$ is factored by

$$D_2 \xrightarrow{q_1} D_3 \xrightarrow{q_2} D_4 \to \tilde{\Delta}. \tag{3.2}$$

Here q_1 is the blow-down of L, D_3 is isomorphic to the normalization of $\tau^{-1}(\Delta)$, q_2 is the blow-down of $q_1(M)$, and $D_4 \to \tilde{\Delta}$ is a $\mathbb{P}^2$-bundle. We set

$$D_2 \cap E_2 = L, \qquad D_2 \cap F = S \cup M \text{ (disjoint union)}\ . \tag{3.3}$$

The two-dimensional subvarieties L, S, M satisfy the following properties.

Lemma 3. *(i)* $L \cap Y'_{p,i}$ *is the exceptional line of the one point blow-up* $q_1 : Y'_{p,i} \to Y_{p,i}$.

(ii) The image of $S \cap Y'_{p,i}$ *(resp.* $M \cap Y'_{p,i}$*) by* q_1 *is the* (-1)*-curve (resp. a fibre) on* $Y_{p,i} \cong \mathbb{F}_1$.

We use the following notations of closed immersions.

$$\begin{array}{lll} i_1 : D_2 \subset V_2, & i_2 : F \subset V_2, & i_3 : E_2 \subset V_2 \\ l_1 : L \subset D_2, & s_1 : S \subset D_2, & m_1 : M \subset D_2, \\ l_3 : L \subset E_2, & s_2 : S \subset F, & m_2 : M \subset F. \end{array} \tag{3.4}$$

Then $\epsilon_F \circ s_2$ and $\epsilon_F \circ m_2$ are identities and if the diagram

$$\begin{array}{ccccc} S & \xrightarrow{s_2} & F & \xrightarrow{\epsilon_F} & M_1 \\ {\scriptstyle s_1}\downarrow & & & & \downarrow{\scriptstyle r} \\ D_2 & \xrightarrow{q} & \tilde{\Delta} & = & \tilde{\Delta}, \end{array} \tag{3.5}$$

is commutative, then we assume the diagram

$$\begin{array}{ccccc} M & \xrightarrow{m_2} & F & \xrightarrow{\epsilon_F} & M_1 \\ {\scriptstyle m_1}\downarrow & & & & \downarrow{\scriptstyle r} \\ D_2 & \xrightarrow{q} & \tilde{\Delta} & \xrightarrow{\iota} & \tilde{\Delta}, \end{array} \tag{3.6}$$

is commutative with a nontrivial automorphim $\iota : \tilde{\Delta} \to \tilde{\Delta}$ of degree three over Δ.

We denote by $H^*(V) = H^*(V, \mathbb{Z})$ the integral cohomology groups of V and define a homomorphism $\theta : H^1(\tilde{\Delta}) \to H^3(V)$ by the composite

$$H^1(\tilde{\Delta}) \overset{q^*}{\to} H^1(D_2) \overset{i_{1*}}{\to} H^3(V_2) \overset{\epsilon_*}{\to} H^3(V_1) \overset{\sigma_*}{\to} H^3(V). \tag{3.7}$$

Since $\tau \circ \sigma \circ \epsilon \circ i_1 = i_\Delta \circ \phi \circ q$ with the closed immersion $i_\Delta : \Delta \subset X$ we see for an element $\beta \in H^1(\Delta)$,

$$\theta(\phi^*\beta) = \sigma_*\epsilon_* i_{1*}\epsilon^*_{D_2}\sigma^*_{D_1}\tau^*_\Delta\beta = \tau^* i_{\Delta_*}\beta = 0 \tag{3.8}$$

by the assumption $H^3(X) = 0$. Here ϵ_{D_2} is the retriction to D_2 of ϵ etc. Thus, $\theta(\phi^* H^1(\Delta)) = 0$. We will show θ is surjective and the kernel is contained in $\phi^* H^1(\Delta)$. The following Lemma is shown along the same line as in the case of quadric bundles [B,p333,Lemma(2.5)].

Lemma 4. *(i)* $H^3(V - \tau^{-1}(\Delta)) = 0$, *(ii)* θ *is surjective.*

Proof. (i) Let $U = V - \tau^{-1}(\Delta)$ and consider the Leray spectral sequence for the $\mathbb{P}^2$-bundle $\tau : U \to X^* = X - \Delta$; $E_2^{pq} = H^p(X^*, R^q\tau_*\mathbb{Z}) \Rightarrow E^{p+q} = H^{p+q}(U)$. We see $R^q\tau_*\mathbb{Z}$ is equal to $\mathbb{Z}$ for $q = 0, 2$ and equal to zero for $q = 1, 3$. Hence $E_2^{pq} = 0$ for $q = 1, 3$ imply the exact sequence

$$0 \to E_\infty^{30} \to E^3 \to E_\infty^{12} \to 0. \tag{3.9}$$

In the Gysin sequence for the pair (X, Δ)

$$H^{i-2}(\Delta) \to H^i(X) \to H^i(X^*) \to H^{i-1}(\Delta) \tag{3.10}$$

we see $H^1(X) = H^3(X) = 0$ by the assumption and $H^i(\Delta) = \mathbb{Z} \to H^{i+2}(X)$ are injective for $i = 0, 2$ because Δ is connected. Hence we see from (3.10) that $H^1(X^*) = H^3(X^*) = 0$, i.e. $E_2^{12} = E_2^{30} = 0$. Therefore $E^3 = H^3(U) = 0$ by (3.9).

(ii) Recall $\tau^{-1}(\Delta)$ is a normal crossing divisor of V and the normalization $D_3 \to \tau^{-1}(\Delta)$ of its quotien field fits in the diagram

$$\begin{array}{ccc} D_3 & \longrightarrow & \tau^{-1}(\Delta) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau} \\ \tilde{\Delta} & \xrightarrow{\ \phi\ } & \Delta \end{array}$$

with an $\mathbb{F}_1$-bundle $f : D_3 \to \tilde{\Delta}$ (see(3.2)). Since $q^* : H^1(\tilde{\Delta}) \xrightarrow{f^*} H^1(D_3) \xrightarrow{q_2^*} H^1(D_2)$ is an isomorphism it is sufficient for the surjectivity of θ to show $\varphi_* : H^1(D_3) \to H^3(V)$ is surjective, where φ is the composite of the normalization $D_3 \to \tau^{-1}(\Delta)$ with the closed immersion $\tau^{-1}(\Delta) \subset V$. Let us consider the Leray spectral sequence for the open immersion $j : U = V - \tau^{-1}(\Delta) \subset V$; $E_2^{pq} = H^p(V, R^q j_*\mathbb{Z}) \Rightarrow H^{p+q}(U)$. We see

$$R^0 j_*\mathbb{Z} = \mathbb{Z}_V, \qquad R^1 j_*\mathbb{Z} = \mathbb{Z}_{D_3}, \qquad R^2 j_*\mathbb{Z} = \mathbb{Z}_S \oplus \mathbb{Z}_M.$$

Here $S \cup M$ is the normalization of the closure of the locus in $\tau^{-1}(\Delta)$ where the multiplicity is equal to two. Now $\phi_* : H^1(D_3) \to H^3(V)$ is given by the differential of E_2-terms

$$d_2 : E_2^{11} = H^1(R^1 j_*\mathbb{Z}) \to E_2^{03} = H^3(j_*\mathbb{Z}),$$

so we have to show $E_3^{30} = 0$. The differential $d_2 : E_2^{02} \to E_2^{21}$ is the Gysin homomorphism

$$H^0(R^2 j_* \mathbb{Z}) = H^0(S) \oplus H^0(M) \to H^2(R^1 j_* \mathbb{Z}) = H^2(D_3),$$

which is injective because the images of S and M are linearly independent in $H^2(D_3)$. Hence $E_3^{02} = 0$, so that $E_3^{30} = E_\infty^{30}$, which is equal to zero by (i). □

We need three Lemmas for the proof of the equality $\mathrm{Ker}(\theta) = \phi^* H^1(\Delta)$.

Lemma 5. *$\epsilon^* \sigma^* \theta(\alpha)$ is expressed in $H^3(V_2)$ by*

$$\epsilon^* \sigma^* \theta(\alpha) = i_{1*} q^* \alpha + i_{2*} \epsilon_F^* r^* (1 + \iota^*) \alpha + i_{3*} \epsilon_{E_2}^* \sigma_{E_1}^* \phi_* \alpha. \qquad (3.11)$$

Proof. We show for $\alpha \in H^1(\tilde{\Delta})$

$$\sigma^* \theta(\alpha) = \epsilon_* i_{1*} q^* \alpha + j_{E_1 *} \sigma_{E_1}^* \phi_* \alpha \qquad \text{in } H^3(V_1) \qquad (3.12)$$

with the closed immersion $j_{E_1} : E_1 \subset V_1$. Let $x = \epsilon_* i_{1*} q^* \alpha$ be the first term of the right-hand side of (3.12). Since $\mathrm{codim}(\Delta_0, V) = 3$, by the formula [B,p312,(0.1.3)] we see

$$\begin{aligned} \sigma^* \theta(\alpha) &= \sigma^* \sigma_* x \\ &= x + j_{E_1 *} \{ \sigma_{E_1}^* \sigma_{E_1 *} (\gamma_1 \cdot j_{E_1}^* x) + h \cdot \sigma_{E_1}^* \sigma_{E_1 *} j_{E_1}^* x \}. \end{aligned}$$

Here h is the Chern class of $\mathcal{O}_{E_1}(1)$ in $H^2(E_1)$ and $\gamma_1 = h + \sigma_{E_1}^* c_1(\mathcal{N})$ with the normal bundle $\mathcal{N}$ of Δ_0 in V. The element $j_{E_1}^* x \in H^3(E_1)$ is mapped to zero by $\sigma_{E_1 *}$, hence

$$\begin{aligned} & j_{E_1 *} \sigma_{E_1}^* \sigma_{E_1 *} (\sigma_{E_1}^* c_1(\mathcal{N}) \cdot j_{E_1}^* x) = j_{E_1 *} \sigma_{E_1}^* \sigma_{E_1 *} (c_1(\mathcal{N}) \cdot \sigma_{E_1 *} j_{E_1}^* x) = 0, \\ & h \cdot \sigma_{E_1}^* \sigma_{E_1 *} j_{E_1}^* x = 0. \end{aligned}$$

On the other hand, we see $\sigma_{E_1 *}(h \cdot j_{E_1}^* x) = \phi_* \alpha \in H^1(\Delta)$ geometrically, so that we obtain the equality (3.12). Applying ϵ^* to (3.12) we see

$$\begin{aligned} \epsilon^* \sigma^* \theta(\alpha) &= \epsilon^* \epsilon_* (i_{1*} q^* \alpha) + \epsilon^* j_{E_1 *} \sigma_{E_1}^* \phi_* \alpha \qquad (3.13) \\ &= i_{1*} q^* \alpha + i_{2*} \epsilon_F^* \epsilon_{F*} i_2^* (i_{1*} q^* \alpha) + i_{3*} \epsilon_{E_1}^* \sigma_{E_1}^* \phi_* \alpha \end{aligned}$$

because $\operatorname{codim}(M_1, V_1) = 2$ and $\epsilon^* j_{E_1 *} = i_{3*}\epsilon^*_{E_2}$ on $H^1(E_1)$. By the commutativity of the diagrams (3.5) and (3.6), the second term of (3.13) is equal to

$$\begin{aligned} & i_{2*}\epsilon^*_F \epsilon_{F*}(s_{2*}s_1^* + m_{2*}m_1^*)q^*\alpha \\ = \; & i_{2*}\epsilon^*_F \epsilon_{F*}(s_{2*}s_2^*\epsilon^*_F r^* + m_{2*}m_2^*\epsilon^*_F r^* \iota^*)\alpha \\ = \; & i_{2*}\epsilon^*_F r^*(1+\iota^*)\alpha. \end{aligned} \tag{3.14}$$

The last equality follows because $\epsilon_F \circ s_2$ and $\epsilon_F \circ m_2$ are identities. Substituting (3.14) into (3.13) we obtain the equality of Lemma. □

Let $[L], [S], [M] \in H^2(D_2)$ be the Poincaré dual of the homology classes of the Cartier divisors L, S, M of D_2, respectively.

Lemma 6. *In $H^3(D_2)$, $i_1^*\epsilon^*\sigma^*\theta(\alpha)$ is equal to*

$$i_1^*\epsilon^*\sigma^*\theta(\alpha) = q^*(\phi^*\phi_* - 3)\alpha \cdot [L] + q^*(\iota^* - 1)\alpha \cdot [S] + q^*(\iota^{*2} - 1)\alpha \cdot [M].$$

Proof. Applying i_1^* to the equality (3.11) we see $i_1^*\epsilon^*\sigma^*\theta(\alpha)$ is equal to

$$i_1^* i_{1*}q^*\alpha + i_1^* i_{2*}\epsilon^*_F r^*(1+\iota^*)\alpha + i_1^* i_{3*}\epsilon^*_{E_2}\sigma^*_{E_1}\phi_*\alpha. \tag{3.15}$$

The first term of (3.15) is equal to $i_1^* i_{1*}q^*\alpha = q^*\alpha \cdot i_1^* D_2$ with

$$i_1^* D_2 = i_1^*(\sigma_2^* D_1 - 2F) = i_1^*\{\sigma_2^*(\sigma_1^*\tau^*\Delta - 3E_1) - 2F\}$$

because the multiplicity in $\tau^{-1}(\Delta)$ (resp. D_1) at the generic point of Δ_0 (resp. M_1) is equal to three (resp. two). Since $\tau \circ \sigma \circ \epsilon \circ i_1 = i_\Delta \circ \phi \circ q$ we see

$$q^*\alpha \cdot i_1^*\epsilon^*\sigma^*\tau^*\Delta = q^*\alpha \cdot q^*\phi^* i_\Delta^*\Delta = q^*(\alpha \cdot \phi^* i_\Delta^*\Delta) = 0$$

because $\alpha \cdot \phi^* i_\Delta^* \Delta = 0$ on $\tilde{\Delta}$. Hence we see from the definition (3.3)

$$\begin{aligned} i_1^* i_{1*}q^*\alpha &= -\,3q^*\alpha \cdot i_1^* E_1 - 2q^*\alpha \cdot i_1^* F \\ &= -\,3q^*\alpha \cdot [L] - 2q^*\alpha \cdot ([S] + [M]). \end{aligned} \tag{3.16}$$

By the commutativity of the diagrams (3.5) and (3.6), the second term of (3.15) is equal to

$$\begin{aligned} i_1^* i_{2*}\epsilon^*_F r^*(1+\iota^*)\alpha &= (s_{1*}s_2^* + m_{1*}m_2^*)\epsilon^*_F r^*(1+\iota^*)\alpha \\ &= s_{1*}s_2^*\epsilon^*_F r^*(1+\iota^*)\alpha + m_{1*}m_2^*\epsilon^*_F r^*(1+\iota^*)\alpha \\ &= s_{1*}s_1^* q^*(1+\iota^*)\alpha + m_{1*}m_1^* q^*(\iota^{-1})^*(1+\iota^*)\alpha \\ &= q^*(1+\iota^*)\alpha \cdot [S] + q^*((\iota^{-1})^* + 1)\alpha \cdot [M]. \end{aligned} \tag{3.17}$$

Since $\sigma_{E_1} \circ \epsilon_{E_2} \circ l_2 = \phi \circ q \circ l_1$, the third term of (3.15) is equal to

$$i_1^* i_{3*} \epsilon_{E_2}^* \sigma_{E_1}^* \phi_* \alpha = l_{1*} l_2^* \sigma_{E_2}^* \sigma_{E_1}^* \phi_* \alpha = l_{1*} l_1^* q^* \phi^* \phi_* \alpha = q^* \phi^* \phi_* \alpha \cdot [L]. \tag{3.18}$$

Substituting (3.16), (3.17) and (3.18) into (3.15) and using $(\iota^*)^{-1} = \iota^{*2}$ we obtain the expression of Lemma. □

Since the contraction morphism of an extremal ray $\tau : V \to X$ satisfies $R^i \tau_* \mathcal{O}_V = 0$ for any $i > 0$ we see $H^3(V, \mathcal{O}_V) = H^3(X, \mathcal{O}_X)$, which is equal to zero by the assumption $H^3(X, \mathbb{Z}) = 0$. Hence $H^3(V) = H^3(V, \mathbb{Z})$, which is torsion free since Δ is irreducible, determines a lattice $H_{\mathbb{Z}}$ in $H^{12}(V)$ by the natural homomorphism $H^3(V) \to H^3(V, \mathbb{C}) \to H^{12}(V)$ and we get the second intermediate jacobian $J^2(V) = H^{12}(V)/H_{\mathbb{Z}}$ [G,p8]. The Picard group of V is generated by that of X together with the anticanonical class of V; $\mathrm{Pic}(V) = \tau^* \mathrm{Pic}(X) \oplus \mathbb{Z}(-K_V)$. For, if $-K_V/3$ is contained in $\mathrm{Div}(V)$ then V is birationally equivalent to $X \times \mathbb{P}^2$ and the inverse images of the irreducible components of Δ are reducible. This contradicts the definition of standard $\mathbb{P}^2$-bundles in Introduction. Let

$$h = c_1(\mathcal{O}_V(-K_V + \tau^* D)) \in H^2(V)$$

be the first Chern class of an ample divisor $-K_V + \tau^* D$ on V. Let A_h be the integral skew symmetric form on $H^3(V)$ defined by the cup product with h.

$$A_h(a, b) = -(a \cdot b \cdot h)_V \qquad \text{for } \ a, b \in H^3(V). \tag{3.19}$$

Let φ and ϕ be the harmonic forms of type (1,2) which are the images of a and b under the homomorphism $H^3(V) \to H^{12}(V)$, respectively, and let

$$H(\varphi, \phi) = 2\sqrt{-1} \int_V \varphi \wedge \bar{\phi} \wedge \omega_h$$

with the Kähler form ω_h determined by h. Then H is the hermitian form on $H^{12}(V)$ satisfying

$$A_h(a, b) = -\mathrm{Im}\ H(\varphi, \phi).$$

Moreover $H^1(V) = 0$ guarantees $H^{12}(V)$ consists of primitive forms in the Lefshetz decomposition of $H^{12}(V)$. Therefore the hermitian form H on $H^{12}(V)$ is positive definte and (3.19) defines a polarization on the intermediate jacobian $J^2(V)$ [G,p7].

Lemma 7. $A_h(\theta(\alpha),\theta(\beta)) = ((1-\iota^*)\alpha,(1-\iota^*)\beta)_{\tilde{\Delta}}$ *for* $\alpha,\beta \in H^1(\tilde{\Delta})$.

Proof. Since $(\iota^*\alpha,\iota^*\beta)_{\tilde{\Delta}} = (\alpha,\beta)_{\tilde{\Delta}}$ for $\alpha,\beta \in H^1(\tilde{\Delta})$ the right-hand side of Lemma is equal to

$$(\alpha,\beta)_{\tilde{\Delta}} - (\alpha,\iota^*\beta)_{\tilde{\Delta}} - (\iota^*\alpha,\beta)_{\tilde{\Delta}} + (\iota^*\alpha,\iota^*\beta)_{\tilde{\Delta}} = ((2-\iota^*-\iota^{*2})\alpha,\beta)_{\tilde{\Delta}}. \tag{3.20}$$

By the projection formula we see

$$\begin{aligned} -A_h(\theta(\alpha),\theta(\beta)) &= \theta(\alpha)\theta(\beta)h = \theta(\alpha)\cdot\sigma_*\epsilon_*i_{1*}q^*\beta\cdot h \\ &= i_1^*\epsilon^*\sigma^*\theta(\alpha)\cdot q^*\beta\cdot i_1^*\epsilon^*\sigma^*h. \end{aligned}$$

From Lemma 6 this is equal to

$$\begin{aligned} &\{q^*(\phi^*\phi_*-3)\alpha\cdot[L] + q^*(\iota^*-1)\alpha\cdot[S] + q^*((\iota^{-1})^*-1)\alpha\cdot[M]\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times q^*\beta\cdot i_1^*\epsilon^*\sigma^*h \\ &= ((\phi^*\phi_*-3)\alpha\cdot\beta)_{\tilde{\Delta}}\cdot(\xi\cdot[L]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2} \\ &\quad + ((\iota^*-1)\alpha\cdot\beta)_{\tilde{\Delta}}\cdot(\xi\cdot[S]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2} \\ &\quad + ((\iota^{*2})-1)\alpha\cdot\beta)_{\tilde{\Delta}}\cdot(\xi\cdot[M]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2}, \end{aligned}$$

where ξ is the class in $H^2(D_2)$ of the fibre $q^{-1}(p)$ of $q : D_2 \to \tilde{\Delta}$ of a closed point p of $\tilde{\Delta}$. Compairing this with (3.20) it is sufficient for the proof of Lemma to show

$$(\xi\cdot[L]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2} = 0, \tag{3.21}$$

$$(\xi\cdot[S]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2} = (\xi\cdot[M]\cdot i_1^*\epsilon^*\sigma^*h)_{D_2} = 1 \tag{3.22}$$

Let $Y_i \cong \mathbb{F}_1$ $(i=1,2,3)$ be the three components of the fibre $\tau^{-1}(\phi(p))$ over the point $\phi(p) \in \Delta$. Then we will show

$$-K_V|_{Y_1} = s_0 + 2m_0 \tag{3.23}$$

in $\mathrm{Pic}(Y_1)$ for the (-1)-curve s_0 and a fibre m_0 on $Y_1 \cong \mathbb{F}_1$. For, let C be a smooth curve on X intersecting transversely with Δ at $\phi(p)$. Then there are exact sequences

$$0 \to \tau^*\mathcal{C}_{C/X}|_{Y_1} \to \Omega_V|_{Y_1} \to \Omega_{\tau^{-1}(C)}|_{Y_1} \to 0, \tag{3.24}$$

$$0 \to \mathcal{C}_{Y_1/\tau^{-1}(C)} \to \Omega_{\tau^{-1}(C)}|_{Y_1} \to \Omega_{Y_1} \to 0, \tag{3.25}$$

where $\mathcal{C}_{C/X}$ is the conormal sheaf of C in X. Since $\tau^*\mathcal{C}_{C/X}|_{Y_1} \cong \mathcal{O}_{Y_1}$ and $\mathcal{C}_{Y_1/\tau^{-1}(C)} \cong \mathcal{O}_{\tau^{-1}(C)}(-Y_1)|_{Y_1} \cong \mathcal{O}_{\tau^{-1}(C)}(Y_2+Y_3)|_{Y_1} \cong \mathcal{O}_{Y_1}(s_0 + m_0)$ we see from (3.24) and (3.25)

$$\begin{aligned} -K_V|_{Y_1} =& - K_{\tau^{-1}(C)/Y_1} = -K_{Y_1} - \mathcal{C}_{Y_1/\tau^{-1}(C)} \\ =&(2s_0 + 3m_0) - (s_0 + m_0) = s_0 + 2m_0. \end{aligned}$$

Thus (3.23) is proved. Let

$$[l] = [L]|_{q^{-1}(p)}, \qquad [s] = [S]|_{q^{-1}(p)}, \qquad [m] = [M]|_{q^{-1}(p)},$$

be the elements of $H^2(q^{-1}(p))$ restricted to $q^{-1}(p)$ of $[L], [S], [M] \in H^2(D_2)$, respectively. Since $q_1 q^{-1}(p)$ is equal to one of Y_i (see(3.2)) we see from Lemma 3

$$q_1^* s_0 = [s] + [l], \qquad q_1^* m_0 = [m] + [l]. \tag{3.26}$$

Therefore it follows from (3.23) and (3.26)

$$\begin{aligned} (\xi \cdot [L] \cdot i_1^* \epsilon^* \sigma^* h)_{D_2} &= ([L]|_{q^{-1}(p)} \cdot \epsilon^* \sigma^* (-K_V)|_{q^{-1}(p)})_{q^{-1}(p)} \\ &= ([l] \cdot q_1^*(-K_V|_{Y_1}))_{q^{-1}(p)} \\ &= ([l] \cdot q_1^*(s_0 + 2m_0))_{q^{-1}(p)} = 0, \\ (\xi \cdot [S] \cdot i_1^* \epsilon^* \sigma^* h)_{D_2} &= ([S]|_{q^{-1}(p)} \cdot \epsilon^* \sigma^* (-K_V)|_{q^{-1}(p)})_{q^{-1}(p)} \\ &= ([s] \cdot q_1^*(s_0 + 2m_0))_{q^{-1}(p)} = 1, \\ (\xi \cdot [M] \cdot i_1^* \epsilon^* \sigma^* h)_{D_2} &= ([M]|_{q^{-1}(p)} \cdot \epsilon^* \sigma^* (-K_V)|_{q^{-1}(p)})_{q^{-1}(p)} \\ &= ([m] \cdot q_1^*(s_0 + 2m_0))_{q^{-1}(p)} = 1. \end{aligned}$$

Thus (3.21) and (3.22), hence Lemma 7, is proved. □

From Lemma 4 and Lemma 7 we see

Lemma 8. *The kernel of the homomorphism $\theta : H^1(\tilde{\Delta}) \to H^3(V)$ defined by (3.7) is equal to Ker$(1 - \iota)$.*

Proof. Since the bilinear form (3.19) is nondegenerate on $H^3(V)$ we see $\theta(\alpha) = 0$ iff $A_h(\theta(\alpha), b) = 0$ for all $b \in H^3(V)$. Hence Lemma 4 and Lemma 7 imply $\theta(\alpha) = 0$ iff $((2 - \iota^* - \iota^{*2})\alpha \cdot \beta)_{\tilde{\Delta}} = 0$ for all $\beta \in H^1(\tilde{\Delta})$, i.e. $(2 - \iota^* - \iota^{*2})\alpha = 0$ in $H^1(\tilde{\Delta})$. Now $2 - \iota^* - \iota^{*2} = (1 - \iota^*)(2 + \iota^*)$

and $(2+\iota^*)(4-2\iota^*+\iota^{*2}) = 8+\iota^{*3} = 9$, so that $\mathrm{Ker}(2-\iota^*-\iota^{*2}) = \mathrm{Ker}(1-\iota^*)$. □

Let $(J(\tilde{\Delta}), \Xi_{\tilde{\Delta}})$ be the jacobian of $\tilde{\Delta}$ with the theta divisor $\Xi_{\tilde{\Delta}}$ and $\lambda_{\tilde{\Delta}} : J(\tilde{\Delta}) \to \hat{J}(\tilde{\Delta})$ be the principal polarization defined by $\Xi_{\tilde{\Delta}}$. The Prym variety P associated to the cyclic cover $\phi : \tilde{\Delta} \to \Delta$ is defined by

$$P = \lambda_{\tilde{\Delta}}^{-1}(\mathrm{Ker}(\hat{\imath})),$$

where $\hat{\imath}$ is the dual of the inclusion $i : \phi^* J(\Delta) \subset J(\tilde{\Delta})$ and the polarization Ξ_P is the restriction to P of the theta divisor of $J(\tilde{\Delta})$ [B,p316]. We see P is equal to the image of the homomorphism $1-\iota : J(\tilde{\Delta}) \to J(\tilde{\Delta})$, so that the skew symmetric form A_P associated to Ξ_P is given by

$$A_P((1-\iota)\alpha, (1-\iota)\beta) = ((1-\iota)\alpha, (1-\iota)\beta)_{\tilde{\Delta}} \qquad \text{for } \alpha, \beta \in H^1(\tilde{\Delta}) \tag{3.27}$$

with the intersection form $(\ ,\ )_{\tilde{\Delta}}$ on $H^1(\tilde{\Delta})$. Now we prove Theorem B. By Lemma 8 there is an exact sequence

$$0 \to \mathrm{Ker}(1-\iota) \to H^1(\tilde{\Delta}) \xrightarrow{\theta} H^3(V) \to 0, \tag{3.28}$$

so $1-\iota$ and θ induce an isomorphism ν from P to $J^2(V)$:

$$\begin{array}{ccccccc} J(\Delta) & \xrightarrow{\phi^*} & J(\tilde{\Delta}) & \xrightarrow{1-\iota} & P & \longrightarrow & 0 \\ \| & & \| & & \downarrow{\scriptstyle\nu} & & \\ J(\Delta) & \xrightarrow{\phi^*} & J(\tilde{\Delta}) & \xrightarrow{\theta} & J^2(V) & \longrightarrow & 0 \end{array} \tag{3.29}$$

Moreover, Lemma 7 and the definition(3.27) imply

$$A_h(\theta(\alpha), \theta(\beta)) = A_P((1-\iota)\alpha, (1-\iota)\beta).$$

Thus $(J_2(V), \Xi_V)$ is isomorphic to (P, Ξ_P) as polarized abelian varieties.

(Proof of Corollary C) As in (3.7) we define a homomorphism $\theta : A^1(\tilde{\Delta}) = \mathrm{Pic}^0\tilde{\Delta} \to A^2(V)$ by the composite

$$\theta : \tilde{\Delta} = A^1(\tilde{\Delta}) \xrightarrow{q^*} A^1(D_2) \xrightarrow{i_{1*}} A^2(V_2) \xrightarrow{\epsilon_*} A^2(V_1) \xrightarrow{\sigma_*} A^2(V). \tag{3.30}$$

The proof of Corollary C is reduced to showing that (i) θ is surjective and (ii) $\mathrm{Ker}(\theta) = \phi^* A^1(\Delta)$.

(i) θ is surjective: We see q^* is an isomorphism because $q : D_2 \to \tilde{\Delta}$ is a fibre bundle with a fibre isomorphic to one point blow-up of $\mathbb{F}_1$ (cf. [B,p337,Lemma (3.1.1)]). By [B,p338] the surjectivity of θ is reduced to showing $A^2(V - \tau^{-1}(\Delta)) = 0$. This follows because $\tau : V - \tau^{-1}(\Delta) \to X - \Delta$ is a $\mathbb{P}^2$-bundle with $A^q(X - \Delta) = 0$ for any $q \in \mathbb{Z}$ [B,p341,3.1.7].

(ii) $\mathrm{Ker}(\theta) = \phi^* A^1(\Delta)$: The calculations of integral cohomology groups hold almost all by replacing $H^{2q-1}(*)$ with $A^q(*)$. For $\beta \in A^1(\Delta)$ the same equalities hold as in (3.8) because $A^2(X) = 0$, so $\theta(\phi^* A^1(\Delta)) = 0$. From Lemma 5 and Lemma 6 we have the same formula for $i_1^* \epsilon^* \sigma^* \theta(\alpha)$ in $A^2(D_2)$. The morphism $q : D_2 \to \tilde{\Delta}$ is factored by $D_2 \xrightarrow{q_1} D_3 \xrightarrow{q_2} D_4 \xrightarrow{q_3} \tilde{\Delta}$ with a $\mathbb{P}^2$-bundle $q_3 : D_4 \to \tilde{\Delta}$ (see (3.2)). From Lemma 3 and Lemma 6 we see in $A^2(D_4)$

$$q_{2*} q_{1*} i_1^* \epsilon^* \sigma^* \theta(\alpha) = q_3^* (\iota^{*2} - 1)\alpha \cdot [M_4] \tag{3.31}$$

with $[M_4] = q_{2*} q_{1*} [M]$ is the class of the tautological line bundle $\mathcal{O}_{D_4}(1) \in \mathrm{Pic}(D_4)$. Since $A^2(D_4) = q_3^* A^1(\tilde{\Delta}) \cdot [M_4] \cong A^1(\tilde{\Delta})$ we see $\theta(\alpha) = 0$ implies $(\iota^{*2} - 1)\alpha = 0$ by (3.31), i.e. α is contained in $\phi^* \mathrm{Pic}^0 \Delta$.

4. Existence of standard projective plane bundles

Let X be a simply connected smooth projective surface over $\mathbb{C}$ with the function field K. First we give a simple proof of the exact seqence (1.3) using the Bloch-Ogus spectral sequence[Sr,191]

$$E_2^{pq} = H^p(X, R^q f_* \mu_n) \Rightarrow H^{p+q}(X, \mu_n), \tag{4.1}$$

where $f : X_{et} \to X_{Zar}$ is the identity, and the flasque resolution of the Zariski sheaf $R^q f_* \mu_n$ [ibid,p189] :

$$0 \to R^q f_* \mu_n \to i_* H^q(k(X), \mu_n) \to \oplus_{X^{(1)}} i_* H^{q-1}(\kappa(x), \mu_n) \to \cdots . \tag{4.2}$$

Indeed, $E_2^{pq} = H^p(X, R^q f_* \mu_n)$ is the p-th cohomology group of the complex obtained from (4.2), so that $E_2^{pq} = 0$ for $p > q$. Hence $E^1 = E_2^{01}$ and there are two exact sequences

$$0 \to E_2^{11} \to E^2 \to E_2^{02} \to 0 \tag{4.3}$$

$$0 \to E_2^{12} \to E^3 \to E_2^{03} \to E_2^{22} \to E^4. \tag{4.4}$$

Since the smooth projective surface X is assumed to be simply connected, $E^3 = H^3(X, \mu_n) = 0$, hence

$$E_2^{12} = H^1(X, R^2 f_* \mu_n) = 0$$

by (4.3). Now $E_2^{p2} = H^p(X, R^2 f_* \mu_n)$ $(p = 0, 1, 2)$ are the cohomology groups of the complex

$$H^2(k(X), \mu_n) \to \oplus_{X^{(1)}} H^1(\kappa(x), \mu_n) \to \oplus_{X^{(2)}} H^0(\kappa(x), \mu_n),$$

which is extended to the exact sequence

$$0 \to H^0(X, R^2 f_* \mu_n) \to H^2(k(X), \mu_n) \to \oplus_{X^{(1)}} H^1(\kappa(x), \mu_n)$$
$$\to \oplus_{X^{(2)}} \mathbb{Z}/n \to H^2(X, R^2 f_* \mu_n) \to 0 \tag{4.5}$$

because $E_2^{12} = 0$. The exact sequence (4.3) is written by

$$0 \to \mathrm{Pic}\ X/n \to H^2(X, \mu_n) \to H^0(X, R^2 f_* \mu_n) \to 0$$

hence $H^0(X, R^2 f_* \mu_n) = {}_nH^2(X, \mathbb{G}_m) := {}_n\mathrm{Br}\ X$. Next we shall show $H^2(X, R^2 f_* \mu_n) = \mathbb{Z}/n$. Let $\mathcal{K}_2$ be the Zariski sheaf on X associated to the presheaf $U \to K_2(\Gamma(U, \mathcal{O}_X))$. The exact sequence

$$0 \to \mathcal{K}_{2,n} \to \mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2 \to \mathcal{K}_2/n \to 0$$

provides us with the exact sequence

$$H^2(X, \mathcal{K}_2) \to H^2(X, n\mathcal{K}_2) \to H^3(X, \mathcal{K}_{2,n})$$
$$\to H^3(X, \mathcal{K}_2) \to H^3(X, n\mathcal{K}_2) \to H^4(X, \mathcal{K}_{2,n}).$$

Here $H^p(X, \mathcal{K}_2) = 0$ for $p > 2$ and $H^p(X, \mathcal{K}_{2,n}) = 0$ for $p > 1$, so $H^2(X, \mathcal{K}_2) \to H^2(X, n\mathcal{K}_2)$ is surjective and $H^3(X, n\mathcal{K}_2) = 0$. Hence we get an exact sequence

$$H^2(X, \mathcal{K}_2) \xrightarrow{n} H^2(X, \mathcal{K}_2) \to H^2(X, \mathcal{K}_2/n) \to H^3(X, n\mathcal{K}_2) = 0.$$

Here $H^2(X, \mathcal{K}_2) = CH^2(X) = CH_0(X)$ is the Chow group of zero cycles on X and $\mathcal{K}_2/n = R^2 f_* \mu_n$ by a Theorem of Merkurjev-Suslin [Sr,p191]. Thus we have $H^2(X, R^2 f_* \mu_n) = CH_0(X)/n$. The degree homomorphism $CH_0(X) \to \mathbb{Z}$ induces the isomorphism $CH_0(X)/n \cong \mathbb{Z}/n$. Taking the direct limit we obtain the exact sequence (1.3). □

We quote a result about Azumaya algebras over C_2-field in [A,p208].

Fact. *Let D be a central division algebra over a C_2-field K with order equal to $2^a 3^b$ in $Br\ K$. Then the index of D is equal to $2^a 3^b$.* $\square$

Using this fact we prove Theorem D. By the exact sequence (1.3) there is an element ξ of Br K of order three which is mapped to ϕ. The above Fact implies ξ is represented by a division algebra D with index three. We see D is a cyclic algebra of rank nine by a Theorem of Wedderburn [P,p288]. The maim theorem in [Ma1] implies there are a birational morphism $Y \to X$ from a smooth projective surface Y and a standard $\mathbb{P}^2$-bundle W over Y associated to the division algebra D. We see W descends over X using the inverse of the elementary transformations of type I and type II described in [Ma2]. $\square$

5. An ideal basis of a standard $\mathbb{P}^2$-bundle

Let k be a field containig a primitive cube ω of unity and let R be the localization of the polynomial ring $k[f,g]$ over k with indeterminates f, g at the maximal ideal (f,g). We denote by $\Lambda = (f,g)_{3,R}$ the cyclic algebra of rank nine over R, i.e. the R-algebra generated by two elements $\{x,y\}$ subject to the relations

$$x^3 = f, \quad y^3 = g, \quad yx = \omega xy. \tag{5.1}$$

We have constructed in [Ma1,§2] an irreducible regular scheme V projective over R with a contraction morphism of an extremal ray

$$\tau : V \to \text{Spec } R$$

such that

(i) the generic fibre V_K of τ is isomorphic to the Severi-Brauer variety corresponding to the central simple algebra $\Lambda \otimes_R K$ over the quotient field K of R,

(ii) V is embedded into $\mathbb{P}^9_R$ by the anticanonical divisor $-K_V$ of V.

In this section we determine the basis of the defining ideal of V in $\mathbb{P}^9_R$ by realizing V as an irreducible component of the scheme of left ideals of rank three of Λ. The case of a $\mathbb{P}^3$-bundle is treated similarly, which will appear in a forthcoming paper. Let $(x_i, y_i, z_0, z_1\ ;\ 0 \le i \le 3)$ be the homogeneous coordinates of $\mathbb{P}^9_R$.

Theorem E. *The closed subscheme V of $\mathbb{P}^9_R$ with the properties (i)-(ii), is defined by the following linearly independent 27 quadrics F_i, G_i, H_j $(1 \le i \le 10, 1 \le j \le 7)$.*

$$\begin{aligned}
F_1 &= x_0y_0 - x_1y_1 + \omega^2 z_0z_1 & G_1 &= x_1x_2 + z_1x_0 - fz_0^2\\
F_2 &= x_2y_2 + \omega^2 x_3y_1 + \omega x_1y_3 & G_2 &= y_1y_2 + z_1y_0\omega^2 + gz_0^2,\\
F_3 &= x_0y_1 - x_1^2 + z_0x_2 & G_3 &= x_0x_3 - x_2^2 + fz_0x_1\\
F_4 &= x_1y_0 - y_1^2 + \omega z_0y_2 & G_4 &= y_0y_3 - y_2^2 - \omega g z_0y_1\\
F_5 &= x_2y_0 - x_1y_2 + z_0y_3 & G_5 &= x_2y_3 + z_1x_3 + \omega^2 fz_0y_2\\
F_6 &= x_2y_0 + z_1y_1 - \omega^2 z_0y_3 & G_6 &= x_3y_2 + z_1y_3 - \omega^2 gz_0x_2\\
F_7 &= x_2y_1 + z_1x_1 + z_0x_3 & G_7 &= x_1y_3 - z_1^2 - \omega fz_0y_0\\
F_8 &= x_0y_2 + z_1x_1 - \omega^2 z_0x_3 & G_8 &= x_3y_1 - z_1^2 + gz_0x_0\\
F_9 &= x_1x_3 + \omega^2 x_0y_3 - \omega z_1x_2 & G_9 &= x_1x_3 - x_0y_3 - \omega^2 fz_0y_1\\
F_{10} &= y_1y_3 + \omega x_3y_0 - \omega^2 z_1y_2 & G_{10} &= y_1y_3 - x_3y_0 + \omega^2 gz_0x_1\\
H_1 &= x_2x_3 + f(x_1y_1 + z_0z_1) - gx_0^2 & H_5 &= z_1x_3 - fy_1^2 + gx_0x_1\\
H_2 &= y_2y_3 + fy_0^2 - g(x_1y_1 + \omega z_0z_1) & H_6 &= z_1y_3 - fy_0y_1 + gx_1^2\\
H_3 &= x_3^2 - fz_1x_1 - gx_0x_2 & H_7 &= x_3y_3 + fy_1y_2 - gx_1x_2\\
H_4 &= y_3^2 + fy_0y_2 + gz_1x. \quad \square
\end{aligned}$$

We first note that if $\iota : \mathbb{P}^2 \subset \mathbb{P}^9 = \mathbb{P}H^0(-K_{\mathbb{P}^2})$ is the Veronese embedding of degree three over a field then the vector space $H^0(-K_{\mathbb{P}^2}) = H^0(\mathcal{O}(3))$ is regarded as the representation space of the third symmetric tensor representation of $GL(3)$. The second symmetric tensor representation space $S^2H^0(\mathcal{O}(3))$ of $H^0(\mathcal{O}(3))$ is decomposed by

$$S^2H^0(\mathcal{O}(3)) = H^0(\mathcal{O}(6)) \oplus \{4, 2\}$$

where $\{4, 2\}$ is the irreducible representation of $GL(3)$ with signature $(4, 2)$, which is of dimension 27. The defining ideal of $\iota(\mathbb{P}^2)$ in $\mathbb{P}^9$ is generated by the quadrics consisting of all elements of the subspace $\{4, 2\}$. In view of this fact, if we start from the Severi-Brauer surface V_K embedded by the anticanonical divisor $-K_{V_K}$ of V_K then we see from descent theory that the defining ideal of V_K in $\mathbb{P}H^0(-K_{V_K}) = \mathbb{P}^9_K$, is generated by the quadrices in the subspace W of $S^2H^0(-K_{V_K})$

such that $W \otimes \bar{K} \cong \{4,2\}$ as $GL_3(\bar{K})$-module for an algebraic closure $\bar{K}$ of K. Let

$$\xi = (b_0 + b_1 x + b_2 x^2) + (b_3 + b_4 x + b_5 x^2) y + y^2$$

be an element of $\Lambda = (f, g)_{3,R}$. We see from (5.1)

$$\begin{pmatrix} \xi \\ x\xi \\ x^2\xi \end{pmatrix} = B \cdot {}^t(1\ x\ x^2\ y\ xy\ x^2y\ y^2\ xy^2\ x^2y^2)$$

with B equal to

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 1 & 0 & 0 \\ fb_2 & b_0 & b_1 & fb_5 & b_3 & b_4 & 0 & 1 & 0 \\ fb_1 & fb_2 & b_0 & fb_4 & fb_5 & b_3 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Since $\{\xi, x\xi, x^2\xi\}$ are linearly independent over R the element ξ generates a rank three left ideal of Λ iff $y\xi$ is a linear combination of $\xi, x\xi, x^2\xi$ over R iff

$$y\xi = b_3\xi + \omega b_4 x\xi + \omega^2 b_5 x^2 \xi. \tag{5.3}$$

We see from the relation (5.1)

$$y\xi = g + (b_0 + \omega b_1 x + \omega^2 b_2 x^2) y + (b_3 \omega b_4 x + \omega^2 b_5 x^2) y^2.$$

Hence (5.3) is equivalent to the following four equations.

$$\begin{aligned} b_0 &= b_3^2 - f b_4 b_5 \\ \omega b_1 &= \omega^2 (f b_5^2 - b_3 b_4) \\ \omega^2 b_2 &= \omega (b_4^2 - b_2 b_5) \\ g &= b_0 b_3 + \omega f b_2 b_4 + \omega^2 f b_1 b_5 \\ &= b_3(b_3^2 - f b_4 b_5) + f b_4 (b_4^2 - b_3 b_5) + f b_5^2 - b_3 b_4) \\ &= b_3^3 + f b_4^3 + f^2 b_5^3 - 3 f b_3 b_4 b_5. \end{aligned} \tag{5.4}$$

Under the condition (5.4) we have to calculate the Plücker coordinates of the rank three left ideal $\Lambda\xi$, i.e. the maximal minors of the 3×9-matrix (5.2). Among the $\binom{9}{3} = 84$ maximal minors there are only the following ten linearly independent ones over R (here (ijk) means the 3×3-minors formed by the i, j, k-th columns of (5.2) $(0 \le i < j < k \le 8)$.

$$
\begin{aligned}
x_0 &= (678) = 1 \\
x_1 &= (478) = b_4 \\
x_2 &= (378) = b_3 \\
x_3 &= (348) = b_3^2 - f b_4 b_5 \\
y_0 &= \omega(258) = \omega(b_2 b_4 - b_1 b_5) = b_4(b_4^2 - b_3 b_5) - \omega^2 b_5 (f b_5^2 - b_3 b_4) \\
&= b_4^3 + (\omega^2 - 1) b_3 b_4 b_5 - \omega^2 f b_5^3 \qquad (5.5) \\
y_1 &= (458) = b_4^2 - b_3 b_5 \\
y_2 &= -\omega^2(158) = -\omega^2(b_1 b_4 - b_0 b_5) \\
&= -b_4(f b_5^2 - b_3 b_4) + \omega^2 b_5 (b_3^2 - f b_4 b_5) \\
&= b_3 b_4^2 + \omega^2 b_3^2 b_5 + \omega f b_4 b_5^2 \\
y_3 &= \omega(256) = \omega(b_1 b_3 - b_0 b_4) = \omega^2 b_3 (f b_5^2 - b_3 b_4) - \omega b_4 (b_3^2 - f b_4 b_5) \\
&= b_3^2 b_4 + f(\omega^2 b_3 b_5^2 + \omega b_4^2 b_5) \\
z_0 &= (578) = b_5 \\
z_1 &= -(358) = f b_5^2 - b_3 b_4.
\end{aligned}
$$

We see from this that the affine 3-space $\mathbb{A}_R^3$ with the affine coordinates (b_3, b_4, b_5) over R, is isomorphically embedded into $\mathbb{P}_R^9$ with the homogeneous coordinates $(x_i, y_i, z_0, z_1\ ;\ 0 \le i \le 3)$.

$$\mathbb{A}_R^3 \cong U_{x_0} \subset \mathbb{P}_R^9.$$

Then the R-scheme V in Theorem E is obtained as the closure of U_{x_0} in $\mathbb{P}_R^9$. In order to simplify the calculations we shall consider another affine open subset of V by interchanging x with y. Let

$$\eta = (c_0 + c_3 y + c_5 y^2) + x(c_1 + c_4 y + c_6 y^2) + x^2$$

be an element of Λ. We see

$$\begin{pmatrix} \eta \\ y\eta \\ y^2\eta \end{pmatrix} = C \cdot {}^t(1\ x\ x^2\ y\ xy\ x^2y\ y^2\ xy^2\ x^2y^2)$$

where C is equal to

$$C = \begin{pmatrix} c_0 & c_1 & 1 & c_3 & c_4 & 0 & c_5 & c_6 & 0 \\ gc_5 & \omega gc_6 & 0 & c_0 & \omega c_1 & \omega^2 & c_3 & \omega c_4 & 0 \\ gc_3 & \omega^2 gc_4 & 0 & gc_5 & \omega^2 gc_6 & 0 & c_0 & \omega^2 c_1 & 0 \end{pmatrix} \tag{5.6}$$

Since $\{\eta, y\eta, y^2\eta\}$ are linearly independent over R the element η generates a rank three left ideal of Λ iff

$$x\eta = c_1\eta + \omega c_4 y\eta + \omega^2 c_6 y^2 \eta. \tag{5.7}$$

We see from (5.1)

$$x\eta = f + c_0 x + c_1 x^2 + (c_3 x + c_4 x^2)y + (c_5 x + c_6 x^2)y^2$$

so that (5.7) is equivalent to the four equations.

$$\begin{aligned} c_0 &= c_1^2 - gc_4c_6 \\ c_3 &= \omega(gc_6^2 - c_1c_4) \\ c_5 &= \omega^2(c_4^2 - c_1c_6) \\ f &= c_0c_1 + \omega gc_4c_5 + \omega^2 gc_3c_6 \\ &= c_1(c_1^2 - gc_4c_6) + gc_4(c_4^2 - c_1c_6) + gc_6(gc_6^2 - c_1c_4) \\ &= c_1^3 + gc_4^3 + g^2c_6^3 - 3c_1c_4c_6. \end{aligned} \tag{5.8}$$

As in (5.5) we obtain the ten maximnal minors of (5.6) which are linearly independent over R.

$$\begin{aligned} x_0 &= (678) = \omega^2 c_4c_5 - \omega c_3c_6 = \omega c_4(c_4^2 - c_1c_6) - \omega^2 c_6(gc_6^2 - c_1c_4) \\ &= \omega\{c_4^3 + (\omega - 1)c_1c_4c_6 - \omega gc_6^3\} \\ x_1 &= (478) = \omega^2(c_4^2 - c_1c_6) \\ x_2 &= (378) = \omega^2 c_3c_4 - \omega c_0c_6 = c_4(gc_6^2 - c_1c_4) - \omega c_6(c_1^2 - gc_4c_6) \\ &= -\{c_1c_4^2 + \omega c_1^2c_6 + \omega^2 gc_4c_6^2\} \\ x_3 &= (348) = \omega^2 c_1c_3 - \omega c_0c_4 = c_1(gc_6^2 - c_1c_4) - \omega c_4(c_1^2 - gc_4c_6) \\ &= \omega^2\{c_1^2c_4 + g(\omega c_1c_6^2 + \omega^2 c_4^2c_6)\} \end{aligned} \tag{5.9}$$

$$
\begin{aligned}
y_0 &= \omega(258) = \omega \\
y_1 &= (458) = c_4 \\
y_2 &= -\omega^2(158) = -\omega^2 c_1 \\
y_3 &= \omega(256) = c_0 = c_1^2 - g c_4 c_6 \\
z_0 &= (578) = -c_6 \\
z_1 &= -(358) = -c_3 = -\omega(g c_6^2 - c_1 c_4).
\end{aligned} \tag{5.9}
$$

Substituting (5.5) or (5.9) in the right-hand side of the quadrics in Theorem E we see they are all equal to zero. These 27 quadrics are linearly independent over R and define an irreducible regular scheme V in $\mathbb{P}^9_R$, in particular V is regular at the vertex of the closed fibre, i.e. at the closed point $(f = g = x_i = y_i = z_1 = 0\ ; 0 \leq i \leq 3)$.

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