Birational maps of standard projective plane bundles over algebraic surfaces

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BIRATIONAL MAPS OF STANDARD PROJECTIVE PLANE BUNDLES OVER ALGEBRAIC SURFACES

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in memory of Akira Takaku

ABSTRACT. Let X be a smooth algebraic surface with the function field K and let $\tau : V \to X$ be a standard $\mathbb{P}^2$-bundle over X, i.e. τ is a flat contraction morphism of an extremal ray of a smooth projective variety V with the generic fibre isomorphic to a K-form of $\mathbb{P}^2$, i.e. $V \times_X \bar{K} = \mathbb{P}^2$ for the algebraic closure $\bar{K}$ of K. In this paper, some birational maps from V to a standard $\mathbb{P}^2$-bundle W are represented by compositions of elementary birational morphisms, where W is a standrd $\mathbb{P}^2$-bundle over the blow-up of X at a point of the non-smooth locus Δ of τ. Let C be a smooth curve on X intersecting Δ transeversely at one point. A birational map from V to a standard $\mathbb{P}^2$-bundle over X which is isomorphic over $X - C$, is decomposed into elementary birational morphisms. These are generalizations of the results about standrd conic bundles by V. G. Sarkisov (Math. USSR. Izv. 20).

The purpose of this paper is to decompose three types of birational maps of standard $\mathbb{P}^2$-bunldes over smooth algebraic surfaces into elementary birational morphisms. Let K be a function field of an algebraic surface defined over an algebraically closed field k of characteristic not equal to 3 and let V_K be a K-form of $\mathbb{P}^2$, i.e. $V \times_K \bar{K} \cong \mathbb{P}^2$ for the algebraic closure $\bar{K}$ of K. Then it is constructed from V_K a *standrd $\mathbb{P}^2$-bundle*

$$\tau : V \to X, \tag{1}$$

(cf. [Ma]) i.e. V and X are smooth projective varieties and τ is a flat contraction morphism of an extremal ray with the generic fibre

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isomorphic to the given K-form $V_K \to \mathrm{Spec}(K)$. The non-smooth locus Δ of (1) is a simple normal crossing curve of X and the geometric fibre over a smooth point of Δ consists of three components H_i ($i = 1,2,3$) with $H_i \cong \mathbb{F}_1$ (one point blow up of $\mathbb{P}^2$), $H_i \cap H_{i+1}$(resp. $H_i \cap H_{i-1}$) is a fibre (resp. the (-1)-curve) on $H_i \cong \mathbb{F}_1$ (where the suffix means mod 3) and $H_1 \cap H_2 \cap H_3$ is a one point. The geometric fibre over a singular point of Δ is non-reduced with the reduced part isomorphic to the cone over a rational twisted cubic in $\mathbb{P}^3$.

Theorem. *(I) Let $Y \to X$ be the blow-up at a singular point of Δ. A birational map from the standard $\mathbb{P}^2$-bundle V of (1) to a standard $\mathbb{P}^2$-bundle W over Y is factored by elementary birational morphisms*

$$V \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \to\leftarrow V' \to\leftarrow V_4 \to V_5 \to W,$$

where $V_3 \to\leftarrow V'$ (resp. $V' \to\leftarrow V_4$) is a flop with the exceptional sets $\mathbb{F}_3 \to \mathbb{P}^1 \leftarrow \mathbb{F}_0$ (resp. $\mathbb{F}_2 \to \mathbb{P}^1 \leftarrow \mathbb{F}_5$). There are isomorphisms of the conormal bundles

$$\begin{aligned} \mathcal{C}_{\mathbb{F}_3/V_3} &\cong \mathcal{O}_{\mathbb{F}_3}(s+2f) \oplus \mathcal{O}_{\mathbb{F}_3}(s+2f), \\ \mathcal{C}_{\mathbb{F}_0/V'} &\cong \mathcal{O}_{\mathbb{F}_0}(s+2f) \oplus \mathcal{O}_{\mathbb{F}_0}(s-f), \\ \mathcal{C}_{\mathbb{F}_2/V'} &\cong \mathcal{O}_{\mathbb{F}_2}(s+3f) \oplus \mathcal{O}_{\mathbb{F}_2}(s-2f), \\ \mathcal{C}_{\mathbb{F}_5/V_4} &\cong \mathcal{O}_{\mathbb{F}_5}(s+3f) \oplus \mathcal{O}_{\mathbb{F}_5}(s+f), \end{aligned}$$

with the negative section s and a fibre f of the rational ruled surface $\mathbb{F}_n$ of degree n.

(II) Let $Y \to X$ be a blow-up at a smooth point of Δ. A birational map from the standard $\mathbb{P}^2$-bundle V of (1) to a standard $\mathbb{P}^2$-bundle W over Y is factored by

$$V \leftarrow V_1 \to\leftarrow V_3 \longleftrightarrow W,$$

where $V_1 \to\leftarrow V_3$ (resp. $V_3 \longleftrightarrow W$) is a flop (resp. a single blow up-and-down) with the exceptional sets $\mathbb{F}_1 \to \mathbb{P}^1 \leftarrow \mathbb{F}_1$, (resp. $\mathbb{P}^2 \leftarrow \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$).

(III) Let C be a smooth curve on X intersecting transversely at one smooth point of Δ. There is a birational map from the standard $\mathbb{P}^2$-bundle V of (1) to a standard $\mathbb{P}^2$-bundle W over X, which is factored by

$$V \leftarrow V_1 \longleftrightarrow V_3 \to\leftarrow V_5 \to W,$$

where $V_1 \longleftrightarrow V_3$ *(resp.* $V_3 \longrightarrow\!\longleftarrow V_5$*) is a single blow up-and-down (resp. a flop) with the exceptional sets* $\mathbb{P}^1 \leftarrow \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ *(resp.* $\mathbb{F}_0 \rightarrow \mathbb{P}^1 \leftarrow \mathbb{F}_1$*). If* C *is disjoint from* Δ *and* $V = \mathbb{P}[\mathcal{E}]$ *over a neibourhood of* C *with a rank three vector bundle* $\mathcal{E}$*, then the birational map (III) is nothing but an elementary transformation of the vector bundle* $\mathcal{E}$ *with center over* C*.*

The decompositions of birational maps of standard conic bundles corresponding to (I), (II), (III) above are appeared in [Sa,p368] (cf. (4.3)). The statements (I), (II), (III) in Theorem are proved in §1, §2, §3, respectively.

Throughout this paper, $\mathbb{F}_n$ is the rational ruled surface of degree n with a fibre f and the $(-n)$-curve s. The direct sum of line bundles is denoted by $\mathcal{O}_{\mathbb{F}_n}(s+f, s+2f) = \mathcal{O}_{\mathbb{F}_n}(s+f) \oplus \mathcal{O}_{\mathbb{F}_n}(s+2f)$, $\mathcal{O}(1,1) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ etc. The conormal bundle of a subvariety S in V is denoted by $\mathcal{C}_{S/V}$.

0. Preliminaries

(0.1) We recall the construction the standard $\mathbb{P}^2$-bundle over X from the K-form V_K (cf.[Ma]). Let A be the central simple algebra of rank 9 over K corresponding to V_K which is represented by an element of the one-dimensional Galois cohomology set $H^1(K, PGL_3)$. There are a smooth projective surface X with the function field isomorphic to K and a maximal $\mathcal{O}_X$-order Λ in A such that the discriminant curve $\Delta = \Delta(A, X)$ of A in X [A-M,p84] is a simple normal crossing curve on X, and $\Lambda \otimes R$ (where $R = \mathcal{O}_{X,p}$ is the local ring of X at a point p of X), is isomorphic to

(0.1.1)	$(\epsilon, \eta)_{3,R}$	if $p \in X - \Delta$,
(0.1.2)	$(\epsilon, g)_{3,R}$	if $p \in \Delta - \mathrm{Sing}(\Delta)$,
(0.1.3)	$(f, g)_{3,R}$	if $p \in \mathrm{Sing}(\Delta)$.

Here $\mathrm{Sing}(\Delta)$ is the singular locus of Δ, $\{\epsilon, \eta\}$ are units of R and $g = 0$ (resp. $fg = 0$) is a defining equation of Δ at p in (0.1.2) (resp. (0.1.3)), $(\epsilon, \eta)_{3,R}$ is the R-algebra generated by two elements x, y with relations $x^3 = \epsilon$, $y^3 = \eta$, $yx = \omega xy$ (where ω is a cube of unity). The standard $\mathbb{P}^2$-bundle V over X associated to V_K is constructed by gluing standard $\mathbb{P}^2$-bundles V_R over the local rings

$R = \mathcal{O}_{X,p}$ at each point $p \in X$, which are the intersection of $\mathbb{P}[E_R^\vee]$ and the grassmannian $G_3[\Lambda^\vee \otimes R]$ of 3-quotients of $\Lambda^\vee \otimes R$. Here E_R is the $(\Lambda \otimes R)^*$-subspace of $\wedge^3 \Lambda \otimes R$ (where $(\Lambda \otimes R)^*$ is the unit group of $\Lambda \otimes R$) with $E_R \otimes \bar{K}$ isomorphic to the third symmetric tensor representation space $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ of $(\Lambda \otimes \bar{K})^* \cong GL_3(\bar{K})$ for an algebraic closure $\bar{K}$ of K, and $E_R^\vee$ and $\Lambda^\vee \otimes R$ are the R-duals of E_R and $\Lambda \otimes R$, respectively.

(0.2) The following Lemma is used to describe the flop appearing in the birational maps (I)-(III) in Theorem.

Lemma. *Let $S \cong \mathbb{F}_n$ be a subvariety of a smooth four-fold V with the conormal bundle $\mathcal{C}_{S/V}$ of S in V. Assume $\mathcal{C}_{S/V}|_f \cong \mathcal{O}(1,1)$ for any fibre f of $S \cong \mathbb{F}_n$, and $S \subset V$ is flopped to $S^+ \cong \mathbb{F}_m \subset V^+$. Then*

(i) For an integre $a \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{C}_{S/V} &\cong \mathcal{O}_{\mathbb{F}_n}(s + af, s + (a-m)f), \\ \mathcal{C}_{S^+/V^+} &\cong \mathcal{O}_{\mathbb{F}_m}(s + af, s + (a-n)f). \end{aligned}$$

(ii) Assume there is a smooth divisor D of V containing S with the birational transform D^+ in V^+. Let $C^+ = D^+ \cap S^+$. If $\mathcal{C}_{S/D} \cong \mathcal{O}_{\mathbb{F}_n}(s + (a+b)f)$, then $(C^{+2})_{S^+} = m + 2b$.

(iii) Assume there is a smooth divisor F of V such that $C = F \cap S$ is a section of the ruled surface $S = \mathbb{F}_n$. If $\mathcal{C}_{S^+/F^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + (a+c)f)$, then $(C^2)_S = n + 2c$.

Proof. (i) The flopped variety V^+ is obtained by the blow-up $\sigma : W \to V$ along S followed by the blow-down $\tau : W \to V^+$ of the exceptional divisor E of σ to the other direction, so that $E = \mathbb{P}[\mathcal{C}_{S/V}]$ is isomorphic to the fibre product $S \times_{\mathbb{P}^1} S^+ \cong \mathbb{P}[\pi^* \mathcal{O}(0,-m)]$, where $\pi : S = \mathbb{F}_n \to \mathbb{P}^1$ is the projection. Hence $\mathcal{C}_{S/V}$ is isomorphic to $\pi^*\mathcal{O}(0,-m) = \mathcal{O}_{\mathbb{F}_n}(0, -mf)$ modulo Pic(S). Since $\mathcal{C}_{S/V}|_f \cong \mathcal{O}(1,1)$, we set $\mathcal{C}_{S/V} \cong \mathcal{O}_{\mathbb{F}_n}(s + af, s + (a-m)f)$ for an integer $a \in \mathbb{Z}$. Similarly, we put $\mathcal{C}_{S^+/V^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + bf, s + (b-n)f)$. Now we shall show $a = b$. Let $\tilde{s} \subset E$ be a curve which is mapped isomorphically onto the negative sections $s \subset S \cong \mathbb{F}_n$ and $s^+ \subset S^+ \cong \mathbb{F}_m$, i.e. $\tilde{s} = s \times_{\mathbb{P}^1} s^+$.

Then

$$(0.2.1)\qquad \begin{aligned}(K_V.s) &= c_1(\mathcal{C}_{S/V}|_s) + (K_S.s)\\ &= (-n+a) + (-n+a-m) + (n-2)\\ &= 2a-n-m-2,\end{aligned}$$

$$(0.2.2)\qquad \begin{aligned}(K_{V^+}.s^+) &= c_1(\mathcal{C}_{S^+/V^+}|_{s^+}) + (K_{S^+}.s^+)\\ &= (-m+b) + (-m+b-n) + (m-2)\\ &= 2b-n-m-2.\end{aligned}$$

On the other hand, $(K_V.s) = (K_{V^+}.s^+)$ because the canonical divisors are equal to $K_W = \sigma^*K_V + E = \tau^*K_{V^+} + E$ and $(E.\sigma^*s) = (E.\tau^*s^+) = 0$. Hence (0.2.1) = (0.2.2) implies $a = b$.

(ii) The intersection in W of E and the proper transform of D, is equal to $\mathbb{P}[\mathcal{C}_{S/D}]$ in $E = \mathbb{P}[\mathcal{C}_{S/V}]$. Then $C^+ \subset S^+$ is isomorphic to $\mathbb{P}[\mathcal{C}_{S/D}|_s] \subset \mathbb{P}[\mathcal{C}_{S/V}|_s]$. Since $\mathcal{C}_{S/V}|_s \cong \mathcal{O}(-n+a, -n+a-m) \to \mathcal{C}_{S/D}|_s \cong \mathcal{O}(-n+a+b)$, i.e. $\mathcal{O}(0,-m) \to \mathcal{O}(b)$, we see $(C^{+2})_{S^+} = m+2b$.

(iii) The flopped surface S^+ is the exceptional divisor of the blow-up $F^+ \to F$ along C, so that $(S^+.f)_{F^+} = -1$ for a fibre f of $S^+ = \mathbb{F}_m$. Hence we put $\mathcal{C}_{S^+/F^+} \cong \mathcal{O}_{\mathbb{F}_m}(s + (a+c)f)$ for an integre $c \in \mathbb{Z}$. The intersection in W of E and the proper transform of F, is equal to $\mathbb{P}[\mathcal{C}_{S^+/F^+}]$ in $\mathbb{P}[\mathcal{C}_{S^+/V^+}]$. Then $C \subset S$ is isomorphic to $\mathbb{P}[\mathcal{C}_{S^+/F^+}|_s] \subset \mathbb{P}[\mathcal{C}_{S^+/V^+}|_s]$. Since $\mathcal{C}_{S^+/V^+}|_s \cong \mathcal{O}(-m+a, -m+a-n) \to \mathcal{C}_{S^+/F^+}|_s \cong \mathcal{O}(-m+a+c)$, i.e. $\mathcal{O}(0,-n) \to \mathcal{O}(c)$, we see $(C^2)_S = n+2c$. □

(0.3) Let $T \cong \mathbb{P}^2$ be a subvariety of a smooth 4-fold V with $\mathcal{C}_{T/V} \cong \mathcal{O}_{\mathbb{P}^2}(1,1)$. Then there are birational maps

$$V \overset{\sigma_1}{\leftarrow} V_1 \overset{\sigma_2}{\to} V_2,$$

where σ_1 is the blow-up along T with the exceptional divisor $E \cong \mathbb{P}^2 \times \mathbb{P}^1$, and σ_2 is the blow-down of E onto $\mathbb{P}^1$ (the projection to the second factor). Assume there is a smooth subvariety S of V of dimension 2 intersectiong T transeversely at one point p. Then

(i) the birational transform S_2 in V_2 of S is the blow up of S at the point $p = S \cap T$ with the exceptional line $e = \sigma_2(E)$,

(ii) $\mathcal{C}_{S_2/V_2} \cong \sigma_{2*}\sigma_1^*\mathcal{C}_{S/V} \otimes \mathcal{O}_{S_2}(-e)$.

Conversely, let S' be a smooth subvariety of a smooth four-fold W with $\mathcal{C}_{S'/W}|_e \cong \mathcal{O}(1,1)$ for a (-1)-curve e on S'. Then, from the exact sequence $0 \to \mathcal{C}_{S'/W}|_e \to \mathcal{C}_{e/W} \to \mathcal{C}_{e/S'} \to 0$, we see $\mathcal{C}_{e/W} \cong \mathcal{O}(1,1,1)$. Hence there are birational maps

$$W \overset{\sigma_1}{\leftarrow} W_1 \overset{\sigma_2}{\rightarrow} W_2,$$

where σ_1 is the blow-up along e with the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^2$, and σ_2 is the blow-down of E onto $\mathbb{P}^2$ (the projection to the second factor). The birational transform S_2' in W_2 of S' is the blow-down of S' along e with $\sigma_{1*}\sigma_2^*\mathcal{C}_{S_2'/W_2} \cong \mathcal{C}_{S'/W} \otimes \mathcal{O}_{S'}(e)$.

1. The birational map (I)

(1.1) In this section we consider the birational map (I) in Theorem. Let $\tau : V \to \mathrm{Spec}(R)$ be the standard $\mathbb{P}^2$-bundle over the local ring R of X at a singular point of Δ constructed from the R-order $(f,g)_{3,R}$ of (0.1.3) (cf.§4). For the blow-up $\sigma : Y = Y_0 \cup Y_1 \to \mathrm{Spec}(R)$ at the origin with $Y_0 = \mathrm{Spec}(R[g/f])$ and $Y_1 = \mathrm{Spec}(R[f/g])$, there is a standard $\mathbb{P}^2$-bundle $\tau_1 : W \to Y$ constructed from a maximal $\mathcal{O}_Y$-order $\tilde{\Lambda}$ with $\tilde{\Lambda}_{Y_0} = (f, g/f)_{3,Y_0}$ and $\tilde{\Lambda}_{Y_1} = (f/g, g)_{3,Y_1}$. In this section we decompose a birational map from V to W over $\mathrm{Spec}(R)$ into elementary birational morphisms.

(1.2) Let $\sigma_1 : V_1 \to V$ be the blow-up at the vertex of the central fibre $\tau^{-1}(o)$ (the cone over the rational twisted cubic in $\mathbb{P}^3$, i.e. the surface contracted along the (-3)-curve on $\mathbb{F}_3$). Let $\sigma_2 : V_2 \to V_1$ be the blow-up along the proper transform $Q_1 \cong \mathbb{F}_3$ of $\tau^{-1}(o)$. We will prove the following Lemma in (4.2).

Lemma. $\mathcal{C}_{\sigma_1(l)/V} \cong \mathcal{O}(2,0,-1)$ *for any fibre* l *of* $Q_1 \cong \mathbb{F}_3$.

Assume the above Lemma. Since σ_1 is the blow-up at a point on $\sigma_1(l)$, Lemma implies $\mathcal{C}_{l/V_1} \cong \mathcal{C}_{\sigma_1(l)/V} \otimes \mathcal{O}(1) \cong \mathcal{O}(3,1,0)$. From the exact sequence $0 \to \mathcal{C}_{Q_1/V_1}|_l \to \mathcal{C}_{l/V_1} \to \mathcal{C}_{l/Q_1} \cong \mathcal{O} \to 0$, we see

$$\mathcal{C}_{Q_1/V_1}|_l \cong \mathcal{O}(3,1). \tag{1.2.1}$$

(1.3) Let $H_2 \subset V_2$ be the exceptional divisor of σ_2. From (1.2.1), the restriction $H_l = \sigma_2^{-1}(l)$ of $\sigma_2 : H_2 \to Q_1$ to a fibre l of $Q_1 \cong \mathbb{F}_3$ is isomorphic to $\mathbb{F}_2$. Let b_l be the (-2)-curve on H_l and let $Q_2 \cong \mathbb{F}_3$ be the section of $\sigma_2 : H_2 \to Q_1$ defined by

(1.3.1) $\quad Q_2 =$ the union of b_l's for all fibres l of $Q_1 \cong \mathbb{F}_3$.

We define σ_3 as the blow up along $Q_2 \cong \mathbb{F}_3$.

Lemma. *The exact sequence*

(1.3.2) $$0 \to \mathcal{C}_{H_2/V_2}|_{Q_2} \to \mathcal{C}_{Q_2/V_2} \to \mathcal{C}_{Q_2/H_2} \to 0$$

splits with isomorphisms $\mathcal{C}_{H_2/V_2}|_{Q_2} \cong \mathcal{O}_{F_3}(s-f)$ *and* $\mathcal{C}_{Q_2/H_2} \cong \mathcal{O}_{\mathbb{F}_3}(2s+4f)$.

Proof. We will show the two isomorphisms in (1.3.2), i.e. $(H_2.s)_{V_2} = 4$, $(H_2.f)_{V_2} = -1$, $(Q_2.s)_{H_2} = 2$, $(Q_2.f)_{H_2} = -2$. Then Lemma follows from $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{F}_3}(2s+4f), \mathcal{O}_{\mathbb{F}_3}(s-f)) = H^1(\mathcal{O}_{\mathbb{F}_3}(-s-5f)) \cong H^1(\mathcal{O}_{\mathbb{F}_3}(-s)) = 0$ by Serre duality. We see $(H_2.f)_{V_2} = \mathcal{O}_{H_2}(-1)|_f = -1$ because the (-2)-curve $f = b_l$ on $H_l = \mathbb{P}[\mathcal{O}(3,1)]$ is defined by the surjection $\mathcal{O}(3,1) \to \mathcal{O}(1)$, and $(Q_2.f)_{H_2} = (Q_2|_{H_l}.b_l)_{H_l} = (b_l^2)_{\mathbb{F}_2} = -2$ because $Q_2 \cap H_l = b_l$. Let $E_1 = \mathbb{P}^3$ be the exceptional divisor of σ_1 and let E_2 be the proper transform of E_1 by σ_2. Then the restriction of σ_2 to E_2 is the blow-up of $E_1 = \mathbb{P}^3$ along the twisted cubic $C_1 = Q_1 \cap E_1$ with the exceptional divisor $S_2 = H_2 \cap E_2$ isomorphic to $\mathbb{F}_0$ because $\mathcal{C}_{C_1/E_1} \cong \mathcal{O}(-5,-5)$. We will show in Lemma(1.5)(ii),

(1.3.3) the (-3)-curve $C_2 = Q_2 \cap S_2$ on $Q_2 \cong \mathbb{F}_3$ is a $(+2)$-curve on $S_2 \cong \mathbb{F}_0$.

If we assume (1.3.3), then $(Q_2.s)_{H_2} = (Q_2|_{S_2}.s)_{S_2} = (C_2^2)_{S_2} = 2$ and $(H_2.s)_{V_2} = \mathcal{O}_{H_2}(-1)|_s = \mathcal{O}_{S_2}(-1)|_{C_2} = 4$ because the $(+2)$-curve C_2 on $S_2 \cong \mathbb{F}_0$ is defined by a surjection $\mathcal{O}(-5,-5) \to \mathcal{O}(-4)$. □

(1.4) For the twisted cubic C_1 in $E_1 = \mathbb{P}^3$, let $\phi : C_1 \to \mathbb{P}^1$ be the cyclic cover of degree three ramified at two points $\{p_0, p_\infty\} \subset \mathbb{P}^1$, and let $f_{p,0} \cup f_{p,1} \cup f_{p,2} \subset E_1$ be the three lines joining the three points $\phi^{-1}(p)$ for each point $p \in \mathbb{P}^1 - \{p_0, p_\infty\}$. Let

$$M_1 = \text{ the closure in } E_1 \text{ of } \cup_{p \in \mathbb{P}^1 - \{p_0, p_\infty\}} \cup_{i=0}^2 f_{p,i}.$$

We see the tangent lines at p_0 and p_∞ to C_1 are contained in M_1.

Lemma. *(i) $M_1 \subset E_1 \cong \mathbb{P}^3$ is a non-normal quartic surface with multiplicity two along the twisted cubic C_1.*

(ii) The proper transform M_2 of M_1 in V_2 is isomorphic to $\mathbb{F}_2$.

Proof. (i) Let $F(z_0, ..., z_3) = 0$ be the defining equation of M_1 in $\mathbb{P}^3$. For a point $p = (g : 1) \in \mathbb{P}^1$, we assume $\phi^{-1}(p) = \{p_0, p_1, p_2\}$ with $p_i = (g : \omega^{2i}\beta^2 : \omega^i\beta : 1)$, $(i = 0, 1, 2)$ for $\beta^3 = g$. Then the three lines $f_{p,i}$ $(i = 0, 1, 2)$ are contained in the plane $\{z_0 = gz_3\} \cong \mathbb{P}^2$, where the line $f_{p,i}$ is defined by the equation $l_i = z_1 + \omega^i\beta z_2 + \omega^{2i}\beta^2 z_3 = 0$. This means $F(gz_3, z_1, z_2, z_3)$ is divided by the product

$$\begin{aligned} l_0 l_1 l_2 &= z_1^3 + g z_2^3 + g^2 z_3^3 - 3g z_1 z_2 z_3 \\ &= z_1^3 + (z_0/z_3) z_2^3 + (z_0/z_3)^2 z_3^3 - 3 z_0 z_1 z_2. \end{aligned}$$

Hence $F(z_0, ..., z_3)$ is equal to

$$F(z_0, ..., z_3) = z_1^3 z_3 + z_0 z_2^3 + z_0^2 z_3^2 - 3 z_0 z_1 z_2 z_3. \tag{1.4.1}$$

We see easily the singular locus of $\{F = 0\}$ is equal to the twisted cubic $C_1 = \{z_0 z_2 - z_1^2 = z_0 z_3 - z_1 z_2 = z_1 z_3 - z_2^2 = 0\}$ with multiplicity two.

(ii) The quartic surface $M_1 = \{F = 0\}$ contains the line $s = \{z_0 = z_3 = 0\}$, so the conormal sheaf $\mathcal{C}_{s/M_1}$ is isomorphic to $\mathcal{O}(2)$ by the exact sequence $0 \to \mathcal{C}_{M_1/\mathbb{P}^3}|_s \to \mathcal{C}_{s/\mathbb{P}^3} \to \mathcal{C}_{s/M_1} \to 0$. Since s is disjoint from the singular locus C_1 of M_1 and since M_2 is nonsingular, we conclude M_2 is isomorphic to $\mathbb{F}_2$. □

(1.5) Next we investigate the 1-dimensional subscheme $M_2 \cap H_2$ in V_2.

Lemma. *(i) $M_2 \cap H_2$ consists of two sections C_2 and C_2' of $\sigma_2 : S_2 \cong \mathbb{F}_0 \to C_1$,*

(ii) C_2 is linearly equivalent to C_2' on both S_2 and on M_2, and $(C_2^2)_{S_2} = (C_2^2)_{M_2} = 2$.

Proof. $\sigma_2 : E_2 \to E_1$ is the blow-up of $E_1 \cong \mathbb{P}^3$ alon the twisted cubic $C = \{z_0 z_2 - z_1^2 = z_0 z_3 - z_1 z_2 = z_1 z_3 - z_2^2 = 0\}$, so E_2 is defined by the equations

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.5.1}$$

in $E_1 \times \mathbb{P}^2$, where (z_i) (resp. (y_i)) is the homogeneous coordinates of $E_1 = \mathbb{P}^3$ (resp. $\mathbb{P}^2$). The projection of $E_1 \times \mathbb{P}^2$ to $\mathbb{P}^2$ defines the $\mathbb{P}^1$-bundle structure $\pi : E_2 \to \mathbb{P}^2$ and $E_2 \cong \mathbb{P}[\mathcal{E}]$ with the rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ defined by

$$(1.5.2) \qquad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}^4 \to \mathcal{E} \to 0,$$

where

$$A = \begin{pmatrix} y_0 & y_1 & y_2 & 0 \\ 0 & y_0 & y_1 & y_2 \end{pmatrix},$$

We see $\mathcal{E}(-1)$ is a rank two stable vector bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}(-1)) = 0$ and $c_2(\mathcal{E}(-1)) = 2$. Since the equation (1.4.1) is equal to

$$F(z_0, \ldots, z_3) = -(z_0z_2 - z_1^2)(z_1z_3 - z_2^2) + (z_0z_3 - z_1z_2)^2,$$

the proper transform $M_2 \subset E_2$ of M_1 is equal to the $\mathbb{P}^1$-bundle $\pi^{-1}(q)$ over the conic $q = \{y_0y_2 = y_1^2\}$ in $\mathbb{P}^2$, and $\mathcal{E}|_q \cong \mathcal{O}_{\mathbb{P}^1}(3,1)$ because $M_2 = \mathbb{P}[\mathcal{E}|_q] \cong \mathbb{F}_2$ and $c_1(\mathcal{E}) = 2$ by (1.5.2). The intersection of M_2 and the exceptional divisor $S_2 = H_2 \cap E_2$ of $\sigma_2 : E_2 \to E_1$ is defined in $\mathbb{P}^3_z \times \mathbb{P}^2_y$ by (1.5.1) together with

$$rank \begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} = 1, \qquad rank \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix} = 1,$$

hence we see $M_2 \cap S_2 = M_2 \cap H_2$ consists of two sections C_2 and C_2' over the twisted cubic $C_1 \subset E_1$, where

$$(1.5.3) \quad C_2 = \{(\lambda^3 : \lambda^2\mu : \lambda\mu^2 : \mu^3) \times (\omega^2\lambda^2 : \omega\lambda\mu : \mu^2) | (\lambda : \mu) \in \mathbb{P}^1\}$$

and C_2' is equal to C_2 replacing ω by ω^2. Hence C_2 and C_2' are linearly equivalent on both $S_2 \cong \mathbb{F}_0$ and $M_2 \cong \mathbb{F}_2$, and intersects at two points $(\lambda : \mu) = (1 : 0)$ and $(0 : 1)$. Therefore $(C_2^2)_{S_2} = (C_2^2)_{M_2} = 2$ and C_2 is equal to $\mathbb{P}[\mathcal{O}(3)]$ in $M_2 = \mathbb{P}[\mathcal{E}|_q]$ by a surjection $\mathcal{E}|_q \cong \mathcal{O}(3,1) \to \mathcal{O}(3)$. □

The intersection of E_2 and Q_2 (see (1.3.1)) is equal to one of C_2 and C_2', say C_2. Let us consider the elementary transformation of $\mathcal{E}$ along $C_2 = E_2 \cap Q_2$:

$$(1.5.4) \qquad 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{O}_q(3) \to 0,$$

with $C_2 = \mathbb{P}[\mathcal{O}_q(3)]$ in $M_2 = \mathbb{P}[\mathcal{E}|_q] \cong \mathbb{P}[\mathcal{O}(3,1)]$ and $E' = \mathbb{P}[\mathcal{E}']$.

(1.6) In the exact sequence (1.5.4) we will show

Lemma. $\mathcal{E}'|_q \cong \mathcal{O}(1,-1)$.

Proof. Let $(p_1,p_2) = (y_1/y_0, y_2/y_0)$ (resp. $(q_0,q_1) = (y_0/y_2, y_1/y_2)$) be the affine coordinates of the openset $U_0 = \{y_0 = 0\}$ (resp. $U_2 = \{y_2 = 0\}$) in $\mathbb{P}^2_y$. From the exact sequence (1.5.2), we see

$$\begin{aligned} z_2 &= -(y_0/y_2)z_0 - (y_1/y_2)z_1, \\ z_3 &= -(y_0/y_2)z_1 - (y_1/y_2)z_2 \\ &= -(y_0/y_2)z_1 + (y_1/y_2)\{(y_0/y_2)z_0 + (y_1/y_2)z_1\}, \end{aligned}$$

on $E_2 = \mathbb{P}[\mathcal{E}]$, so $\mathcal{E}$ has a free basis $\{z_2, z_3\}$ (resp. $\{z_1, z_0\}$) over U_0 (resp. U_1) with

$$\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -q_1 & -q_0 \\ q_1^2 - q_0 & q_0 q_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}.$$

From (1.5.2), the kernel $\mathcal{E}'$ in (1.5.4) is given by

$$\begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 & -\omega p_1 \\ 0 & p_2 - p_1^2 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \qquad \text{on } U_0,$$

$$\begin{pmatrix} w_1 \\ w_0 \end{pmatrix} = \begin{pmatrix} 1 & -\omega^2 q_1 \\ 0 & q_0 - q_1^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \qquad \text{on } U_2.$$

Therefore ${}^t(w_2, w_3) = A \cdot {}^t(w_1, w_0)$ with A equal to

$$\begin{aligned} &\begin{pmatrix} 1 & -\omega p_1 \\ 0 & p_2 - p_1^2 \end{pmatrix} \begin{pmatrix} -q_1 & -q_0 \\ q_1^2 - q_0 & q_0 q_1 \end{pmatrix} \begin{pmatrix} 1 & -\omega^2 q_1 \\ 0 & q_0 - q_1^2 \end{pmatrix}^{-1} \\ =&\begin{pmatrix} 1 & -\omega(q_1/q_0) \\ 0 & (1/q_0) - (q_1/q_0)^2 \end{pmatrix} \begin{pmatrix} -q_1 & -q_0 \\ q_1^2 - q_0 & q_0 q_1 \end{pmatrix} \begin{pmatrix} 1 & -\omega^2 q_1 \\ 0 & q_0 - q_1^2 \end{pmatrix}^{-1} \\ =&\begin{pmatrix} -q_1 + \omega(q_1/q_0)(q_0 - q_1^2) & a_{12} \\ -(q_0 - q_1^2)^2/q_0^2 & -\omega^2 q_1(q_0 - q_1^2)/q_0^2 + (q_1/q_0) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} (q_0 - q_1^2)a_{12} &= \{-q_1 + \omega(q_1/q_0)(q_0 - q_1^2)\}\omega^2 q_1 + (-q_0 - \omega q_1^2) \\ &= -(q_0 - q_1^2) - (q_1^2/q_0)(q_1^2 - q_0), \end{aligned}$$

hence $a_{12} = -1 + (q_1^2/q_0)$. Since $q_0 = q_1^2$ on the conic q, we see

$$A|_q = \begin{pmatrix} -q_1 & 0 \\ 0 & q_1^{-1} \end{pmatrix}.$$

This means $\mathcal{E}'|_q \cong \mathcal{O}(1,-1)$. □

(1.7) Let $F_3 \subset V_3$ be the exceptional divisor of the blow-up σ_3 of V_2 along Q_2. From Lemma(1.3), the restriction $F_l = \sigma_3^{-1}(b_l)$ of F_3 to a fibre b_l of $Q_2 \cong \mathbb{F}_3$ is isomorphic to $\mathbb{F}_1$. Let n_l be the (-1)-curve on F_l and let Q_3 be the section of $\sigma_3 : F_3 \to Q_2$ defined by

$$(1.7.1) \qquad Q_3 = \text{ the union of } n_l\text{'s for all fibres } l \text{ of } Q_1 \cong \mathbb{F}_3.$$

Since $\sigma_3 : E_3 \to E_2$ is the blow-up along $C_2 := Q_2 \cap E_2$ with the exceptional divisor $T_3 := F_3 \cap E_3$, Lemma(1.6) means $T_3 \cong \mathbb{P}[\mathcal{E}'|_q] \cong \mathbb{F}_2$.

Lemma. *(i)* $\mathcal{C}_{Q_3/V_3}|_{n_l} \cong \mathcal{O}(1,1)$,

(ii) the (-3)-curve $C_3 = Q_3 \cap E_3$ on $Q_3 \cong \mathbb{F}_3$ is the (-2)-curve on $T_3 = F_3 \cap E_3 \cong \mathbb{F}_2$, i.e. $C_3 \subset T_3$ is defined from the surjection $\mathcal{E}'|_q \cong \mathcal{O}(1,-1) \to \mathcal{O}(-1)$.

(iii) Q_3 is disjoint from M_3 in V_3.

Proof. (i) In the exact sequence

$$(1.7.2) \qquad 0 \to \mathcal{C}_{F_3/V_3}|_{n_l} \to \mathcal{C}_{Q_3/V_3}|_{n_l} \to \mathcal{C}_{Q_3/F_3}|_{n_l} \to 0,$$

we see $(F_3.n_l)_{V_3} = -\mathcal{O}_{F_3}(1)|_{n_l} = -1$ because n_l is the (-1)-curve on $F_l \cong \mathbb{P}[\mathcal{O}(2,1)]$ (see Lemma(1.3)), and $(Q_3.n_l)_{F_3} = (Q_3|_{F_l}.n_l)_{F_l} = (n_l)^2_{\mathbb{F}_1} = -1$ because $Q_3 \cap F_l = n_l$.

(ii) From $(C_2^2)_{S_2} = 2$, we see $C_2 \subset S_2 \cong \mathbb{P}[\mathcal{C}_{C_1/E_1}]$ is defined by a surjection $\mathcal{C}_{C_1/E_1} \cong \mathcal{O}(-5,-5) \to \mathcal{O}(-4)$, so that $(S_2.C_2)_{E_2} = -\mathcal{O}_{S_2}(1)|_{C_2} = 4$. Therefore we obtain

$$(1.7.3) \quad 0 \to \mathcal{C}_{S_2/E_2}|_{C_2} \cong \mathcal{O}(-4) \to \mathcal{C}_{C_2/E_2} \to \mathcal{C}_{C_2/S_2} \cong \mathcal{O}(-2) \to 0,$$

from which we have equalities

(1.7.4)
$$(T_3^2)_{E_2} = -\sigma_3^* C_2 + c_1(\mathcal{N}_{C_2/E_2}) r_3 = -\sigma_3^* C_2 + 6r_3, \qquad (T_3^3)_{E_2} = -6$$

for a fibre r_3 of $T_3 = \mathbb{F}_2 \to C_2$. Let $S_3 = H_3 \cap E_3$ be the proper transform in E_3 of S_2 and let

$$m_3 = M_3 \cap E_3 = M_3 \cap T_3, \qquad t_3 = H_3 \cap T_3 = S_3 \cap T_3$$

be two sections of $\sigma_3 : T_3 \cong \mathbb{F}_2 \to C_2$. We see from (1.7.4)

$$\begin{aligned}(m_3.t_3)_{T_3} =&(M_3|_{T_3}.S_3|_{T_3})_{T_3} = (M_3.S_3.T_3)_{E_3}\\ =&(\sigma_3^* M_2 - T_3)(\sigma_3^* S_2 - T_3)T_3\\ =&\sigma_3^*(M_2.S_2)T_3 - \sigma_3^*(M_2+S_2)T_3^2 + T_3^3\\ =&((M_2+S_2)C_2)_{E_2} - 6.\end{aligned}$$

From (1.7.3) we see $(M_2.C_2)_{E_2} = (\sigma_2^* M_1 - 2S_2)C_2 = (M_1.C_1)_{E_1} - 2(S_2.C_2) = 4.3 - 2.4 = 4$. Hence $(m_3.t_3)_{T_3} = 4+4-6 = 2$. On the other hand, (1.3.2) induces the split exact sequence

$$0 \to \mathcal{C}_{H_2/V_2}|_f \cong \mathcal{O}(1) \to \mathcal{C}_{Q_2/V_2}|_f \to \mathcal{C}_{Q_2/H_2}|_f \cong \mathcal{O}(2) \to 0,$$

for fibres $f = b_l$ of $Q_2 \cong \mathbb{F}_2$. This implies $F_3 \cap H_3$ is covered by (+1)-curves $\mathbb{P}[\mathcal{C}_{Q_2/H_2}|_f]$ in $\mathbb{P}[\mathcal{C}_{Q_2/V_2}|_f] \cong \mathbb{F}_1$, so $F_3 \cap H_3$ is disjoint from Q_3 (see (1.7.1)), hence $(C_3.t_3)_{T_3} = 0$. Thus the three sections C_3, m_3, t_3 of $T_3 \to C_2$ satisfy $(m_3.t_3)_{T_3} = 2$ and $(C_3.t_3)_{T_3} = 0$. This means C_3 is (resp. m_3 and t_3 are) the (-2)-curve (resp. $(+2)$-curves) on $T_3 \cong \mathbb{P}[\mathcal{O}(1,-1)] \cong \mathbb{F}_2$. Since $Q_3 \cap M_3$ is contained in $T_3 = F_3 \cap E_3$, (iii) follows from the fact that C_3 is disjoint from m_3. □

(1.8) The following Lemma implies $\mathcal{C}_{M_3/V_3}|_f \cong \mathcal{O}(1,1)$ for fibres f of $M_3 \cong \mathbb{F}_2$, so there is a flop $V_3 \to\leftarrow V^+$ along M_3.

Lemma. *The two exact sequences*

(1.8.1) $\qquad 0 \to \mathcal{C}_{E_2/V_2}|_{M_2} \to \mathcal{C}_{M_2/V_2} \to \mathcal{C}_{M_2/E_2} \to 0,$

(1.8.2) $\qquad 0 \to \mathcal{C}_{E_3/V_3}|_{M_3} \to \mathcal{C}_{M_3/V_3} \to \mathcal{C}_{M_3/E_3} \to 0$

split with the isomorphisms

$$\begin{aligned}&\mathcal{C}_{E_2/V_2}|_{M_2} \cong \mathcal{O}_{\mathbb{F}_2}(s+3f), \qquad &\mathcal{C}_{M_2/E_2} \cong \mathcal{O}_{\mathbb{F}_2}(-4f),\\ &\mathcal{C}_{E_3/V_3}|_{M_3} \cong \mathcal{O}_{\mathbb{F}_2}(s+3f), \qquad &\mathcal{C}_{M_3/E_3} \cong \mathcal{O}_{\mathbb{F}_2}(2-2f).\end{aligned}$$

Proof. $\mathcal{C}_{E_2/V_2}|_{M_2} \cong \mathcal{C}_{E_3/V_3}|_{M_3} \cong \mathcal{O}_{\mathbb{F}_2}(s+3f)$ follow from $(E_3.s)_{V_3} = (E_2.s)_{V_2} = (E_1.s)_{V_1} = \mathcal{O}_{E_1}(-1)|_s = -1$ (since the image $\sigma_1(s)$ of s in V_1 is a line on $E_1 = \mathbb{P}^3$), and $(E_3.f)_{V_3} = (E_2.f)_{V_2} = (E_1.f)_{V_1} = \mathcal{O}_{E_1}(-1)|_f = -1$. We saw $\sigma_2(s)$ is disjoint from C_1 in the proof of

Lemma(1.4)(ii), so $(M_3.s)_{E_3} = (M_2.s)_{E_2} = (M_1.s)_{E_1} = deg(M_1) = 4$. Then the isomorphisms $\mathcal{C}_{M_2/E_2} \cong \mathcal{O}_{\mathbb{F}_2}(-4f)$ and $\mathcal{C}_{M_3/E_3} \cong \mathcal{O}_{\mathbb{F}_2}(s-2f)$ follow from $(M_2.f)_{E_2} = (\sigma_2^* M_1 - 2S_2)f = (M_1.f)_{E_1} - 2(S_2.f)_{E_2} = 4-4 = 0$ and $(M_3.f)_{E_3} = (\sigma_3^* M_2 - T_3)f = (M_2.f) - (T_3.f)_{E_3} = 0-1 = -1$. Both (1.8.1) and (1.8.2) split because $H^1(\mathcal{O}_{\mathbb{F}_2}(s+7f)) = H^1(\mathcal{O}_{\mathbb{F}_2}(5f)) = 0$. □

(1.9) From Lemma(1.7)(i), (iii) and Lemma(1.8), we define V_4 as the flopped variety of V_3 along the disjoint union M_3 and Q_3. Since $\mathcal{C}_{M_3/V_3} \cong \mathcal{O}_{\mathbb{F}_2}(s+3f, s-2f)$ by (1.8.2), the flopped surface M_4 in V_4 of M_3 is isomorphic to $\mathbb{P}[\mathcal{C}_{M_3/V_3}|_s] \cong \mathbb{F}_5$ with $\mathcal{C}_{M_4/V_4} \cong \mathcal{O}_{\mathbb{F}_5}(s+3f, s+f)$ by Lemma(0.2)(i). Applying Lemma(0.2)(iii) to $S = M_3$ and $F = F_3$ we see $\mathcal{C}_{M_4/F_4} \cong \mathcal{O}_{\mathbb{F}_5}(s+3f)$ because $m_3 = F_3 \cap M_3$ is a (+2)-curve on M_3 by Lemma(1.5)(ii). Hence there is a split exact sequence

(1.9.1)

$$0 \to \mathcal{C}_{F_4/V_4}|_{M_4} \cong \mathcal{O}_{\mathbb{F}_5}(s+f) \to \mathcal{C}_{M_4/V_4} \to \mathcal{C}_{M_4/F_4} \cong \mathcal{O}_{\mathbb{F}_5}(s+3f) \to 0.$$

(1.10) Let E_4, H_4, F_4 be the birational transforms of E_3, H_3, F_3, respectively. These are obtained as follows.

(a) E_4 is constructed by the elementary transformation (1.5.4) and a blow up ϵ_2:

(1.10.1) $$E_2 = \mathbb{P}[\mathcal{E}] \overset{\sigma_3}{\leftarrow} E_3 \overset{\epsilon_1}{\rightarrow} \mathbb{P}[\mathcal{E}'] \overset{\epsilon_2}{\leftarrow} E_4,$$

where

(i) σ_3 is the blow-up along the (+2)-curve $C_2 = \mathbb{P}[\mathcal{O}_q(1)]$ in $M_2 = \mathbb{P}[\mathcal{E}|_q] \cong \mathbb{P}[\mathcal{O}(3,1)]$ with the exceptional divisor $T_3 \cong \epsilon_1(T_3) = \mathbb{P}[\mathcal{E}'|_q] \cong \mathbb{P}[\mathcal{O}(1,-1)]$,

(ii) ϵ_1 is the blow-down of the proper transform M_3 of $M_2 = \mathbb{P}[\mathcal{E}|_q]$,

(iii) ϵ_2 is the blow-up along the (−2)-curve $\epsilon_1(C_3) = \mathbb{P}[\mathcal{O}_q(-1)]$ in $\mathbb{P}[\mathcal{E}'|_q] \cong \mathbb{P}[\mathcal{O}(1,-1)]$ with the birational transform T_4 in E_4 of T_2 isomorphic to $\epsilon_2(T_4) = \mathbb{P}[\mathcal{E}'|_q]$.

(b) H_4 is the blow-up of $H_3 \cong H_2$ along $C_3' \cong \sigma_3(C_3') = C_2'$ with the exceptional divisor equal to the flopped surfacae $M_4 = \mathbb{P}[\mathcal{C}_{M_3/V_3}|_{C_3'}] \cong \mathbb{F}_5$.

(c) F_4 is obtained from $F_3 \cong F_2$ by

(1.10.2) $$F_3 \overset{\epsilon_3}{\leftarrow} F' \overset{\epsilon_4}{\rightarrow} F_4,$$

where

(iv) ϵ_3 is the blow-up along $m_3 = M_3 \cap F_3$ with the exceptional divisor equal to the flopped surface $M_4 = \mathbb{P}[\mathcal{C}_{m_3/F_3}] \cong \mathbb{P}[\mathcal{C}_{M_3/V_3}|_{m_3}]$,

(v) ϵ_4 is the blow-down along $\epsilon_3^{-1}(Q_3) \cong Q_3 \cong \mathbb{F}_3$.

The $\mathbb{P}^1$-bundle structure $\sigma_3 : F_3 \to Q_2$ induces a $\mathbb{P}^1$-bundle structure $\pi : F_4 \to M_4 \cong \mathbb{F}_5$.

(1.11) From (1.7) we see

Proposition. *(i) The fibre f_4 of $\pi : F_4 \to M_4$ is an extremal rational curve on V_4 with $(-K_{V_4}.f_4) = 1$,*

(ii) E_4 is mapped to $\mathbb{P}^1 \times \mathbb{P}^2$ by the contraction morphism $\sigma_4 : V_4 \to V_5$ of f_4,

(iii) The flopped surface Q_4 on V_4 of Q_3 is isomrphically mapped to $\mathbb{P}^1 \times q \subset \sigma_4(E_4) \cong \mathbb{P}^1 \times \mathbb{P}^2$ with a conic q in $\mathbb{P}^2$.

Proof. In the exact sequence $0 \to \mathcal{C}_{F_4/V_4}|_{f_4} \to \mathcal{C}_{f_4/V_4} \to \mathcal{C}_{f_4/F_4} \cong \mathcal{O}(0,0) \to 0$, we will show $\mathcal{C}_{F_4/V_4}|_{f_4} \cong \mathcal{O}(1)$, i.e. $(F_4.f_4)_{V_4} = -1$. The fibre f_3 of $T_3 = F_3 \cap E_3$ is isomorphically transformed to the fibres of $T_4 = F_4 \cap E_4$, so we may assume f_4 is contained in E_4. Hence $(F_4.f_4)_{V_4} = (F_4|_{E_4}.f_4)_{E_4} = (T_4.f_4)_{E_4}$. We denote by $T' = \mathbb{P}[\mathcal{E}'|_q]$, by Q_4 the exceptional divisor of ϵ_2 of (1.9.1), and by f', q_4, m_3 the fibre of T', Q_4, M_3, respectively. Then we see

$$
\begin{aligned}
(T_3.f_3)_{E_3} &= (\epsilon_1^* T' - M_3)(\epsilon_1^* f' - m_4) = (T'.f')_{E'} - 1,\\
(T_4.f_4)_{E_4} &= (\epsilon_2^* T' - Q_4)(\epsilon_2^* f' - q_4) = (T'.f')_{E'} - 1.
\end{aligned}
$$

Hence $(F_4.f_4)_{V_4} = (T_4.f_4)_{E_4} = (T_3.f_3)_{E_3} = -1$ because T_3 is the exceptional divisor of $\sigma_3 : E_3 \to E_2$.

(ii) We see from (i) that the image $\sigma_4(E_4)$ is equal to the result of the elementary transformation of $E' = \mathbb{P}[\mathcal{E}']$ along $\epsilon_1(C_3) = \mathbb{P}[\mathcal{O}_q(-1)]$:

(1.11.1) $$0 \to \mathcal{E}_5 \to \mathcal{E}' \to \mathcal{O}_q(-1) \to 0,$$

where $\sigma_4(E_4) = \mathbb{P}[\mathcal{E}_5]$ and $\epsilon_1(C_3) = \mathbb{P}[\mathcal{O}_q(-1)] \subset T' = \mathbb{P}[\mathcal{E}'|_q] \cong \mathbb{P}[\mathcal{O}(1,-1)]$. Hence we will show $\mathcal{E}_5$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^2$. From (1.5.2) and (1.5.4), we see $c_1(\mathcal{E}') = 0$, $c_2(\mathcal{E}') = 2$ and $H^0(\mathcal{E}'(-4)) = 0$, so $h^0(\mathcal{E}'(1)) \geq \chi(\mathcal{E}'(1)) = 4$. Hence (1.11.1) implies $h^0(\mathcal{E}_5(1)) \geq 2$ and there is an inclusion $\iota : \mathcal{O}_{\mathbb{P}^2}(-1)^2 \to \mathcal{E}_5$. On the other hand, $c_1(\mathcal{E}_5(1)) = c_2(\mathcal{E}_5(1)) = 0$, so the inclusion ι is an isomorphism.

(iii) The flopped surface $Q_4 = \mathbb{P}[\mathcal{C}_{Q_3/V_3}|_{C_3}]$ on V_4 is equal to the exceptional divisor of the blow up of $\mathbb{P}[\mathcal{E}']$ along $\mathbb{P}[\mathcal{O}(-1)] \cong C_3$. Hence Q_4 is isomrphically mapped to $\mathbb{P}[\mathcal{E}_5|_q] \cong \mathbb{P}^1 \times q$ by the exact sequence (1). □

(1.12) Let r_5 =(point)×(line) in $E_5 = \sigma_4(E_4) \cong \mathbb{P}^1 \times \mathbb{P}^2$. To define $\sigma_5 : V_5 \to V_6 = W$ we show

Lemma. *(i) There is a split exact sequence*

$$0 \to \mathcal{C}_{E_5/V_5}|_{r_5} \cong \mathcal{O}(1) \to \mathcal{C}_{r_5/V_5} \to \mathcal{C}_{r_5/E_5} \cong \mathcal{O}(0,-1) \to 0,$$

(ii) r_5 is an extremal rational curve on V_5 and the associated morphism $\sigma_5 : V_5 \to V_6$ contracts $E_5 \cong \mathbb{P}^1 \times \mathbb{P}^2$ onto the first factor $\mathbb{P}^1$.

Proof. (i) We show $\mathcal{C}_{E_5/V_5}|_{r_5} \cong \mathcal{O}(1)$, i.e. $(E_5.r_5)_{V_5} = -1$. The surface $S_2 = H_2 \cap E_2 \cong \mathbb{F}_0$ in V_2 is transformed isomorphically onto $S_4 = H_4 \cap E_4$ in V_4 and the Stein fctorization of the composite $S_2 \cong S_4 \subset E_4 \xrightarrow{\sigma_4} E_5 \cong \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\pi_1} \mathbb{P}^1$ is given by

$$S_2 \xrightarrow{\sigma_1} C_1 \xrightarrow{\phi} \mathbb{P}^1,$$

where ϕ is the associated cyclic cover of degree three in (1.4). Let r_4 be the isomorphic image in S_4 of a fibre r_2 of $\sigma_1 : S_2 \cong \mathbb{F}_0 \to C_1$. Then $r_5 = \sigma_4(r_4)$ is equal to (point)×(line) in $E_5 = \mathbb{P}^1 \times \mathbb{P}^2$. Hence $(E_4.r_4)_{V_4} = (\sigma_4^* E_5.r_4)_{V_4} = (E_5.r_5)_{V_5}$. The left-hand side is equal to $(E_4.r_4)_{V_4} = (E_4|_{H_4}.r_4)_{H_4} = (S_4.r_4)_{H_4}$. We recall $M_2 \cap H_2 = M_2 \cap S_2 = C_2 \cup C_2'$ (Lemma(1.5)), and $H_4 \cdots \to H_2$ is the blow-up along C_2' with the exceptional divisor M_4 (see (1.10)(b)). Hence $(S_4.r_4)_{H_4} = (\sigma^* S_2 - M_4)r_4 = (S_2.r_2)_{H_2} - (M_4.r_4)_{H_4} = 0 - 1 = -1$ because $S_2 \cong \mathbb{F}_0$ is a $\mathbb{P}^1$-bundle over the twisted cubic C_1 with a fibre r_2. (ii) follows from (i). □

(1.13) Let h_6 be the image in $V_6 = W$ of a fibre h_2 of the $\mathbb{P}^1$-bundle $\sigma_1 : H_2 \to Q_1 \cong \mathbb{F}_3$. To see W is a standard $\mathbb{P}^2$-bundle over the blow-up Y at the origin of Spec(R), we show

Lemma. $(-K_W.h_6)_W = 1$.

Proof. We assume the birational transform h_4 on V_4 of h_2 is disjoint from $S_4 = H_4 \cap E_4$. Then $(H_4.h_4)_{V_4} = (H_3.h_3)_{V_3} = (\sigma_3^* H_2 - F_3)h_3 =$

$(H_2.h_2)_{V_2} - 1 = -2$ because H_2 is the exceptional divisor of σ_1 with the fibre h_2. Hence, from the exact sequence

$$0 \to \mathcal{C}_{H_4/V_4}|_{h_4} \cong \mathcal{O}(2) \to \mathcal{C}_{h_4/V_4} \to \mathcal{C}_{h_4/H_4} \cong \mathcal{O}(0,0) \to 0,$$

we see $\mathcal{C}_{h_4/V_4} \cong \mathcal{O}(2,0,0)$ and $(-K_{V_4}.h_4) = 0$. On the other hand, from $(F_4.h_4)_{V_4} = 0$ and $(E_5.h_5)_{V_5} = 0$, we see

$$\begin{aligned}(-K_{V_4}.h_4)_{V_4} =&(\sigma_4^*(-K_{V_5}) - F_4)h_4 = (-K_{V_5}.h_5)_{V_5} - 1\\ =&(\sigma_5^*(-K_{V_6}) - 2E_5)h_5 - 1 = (-K_{V_6}.h_6) - 1.\end{aligned}$$

Therefore $(-K_{V_6}.h_6) = 1$. □

(1.14) Next we determine the conormal bundles $\mathcal{C}_{Q_3/V_3}$ and $\mathcal{C}_{M_5/V_5}$.

Lemma. *(i) There is an exact sequence*

$$0 \to \mathcal{C}_{F_3/V_3}|_{Q_3} \cong \mathcal{O}_{\mathbb{F}_3}(s-f) \to \mathcal{C}_{Q_3/V_3} \to \mathcal{C}_{Q_3/F_3} \cong \mathcal{O}_{\mathbb{F}_3}(s+5f) \to 0,$$

(ii) $\mathcal{C}_{Q_3/V_3} \cong \mathcal{O}_{\mathbb{F}_3}(s+2f, s+2f)$, $\mathcal{C}_{M_5/V_5} \cong \mathcal{O}_{\mathbb{F}_5}(s+f, 2s+4f)$.

Proof. (i) Let f (resp. s) be a fibre (resp. the (-3)-curve) on $Q_3 \cong \mathbb{F}_3$. For the isomorphism $\mathcal{C}_{F_3/V_3}|_{Q_3} \cong \mathcal{O}_{\mathbb{F}_2}(s-f)$, we show $(F_3.f)_{V_3} = -1$ and $(F_3.s)_{V_3} = 4$. By the definition (1.7.1), f is the (-1)-curve on $F_l = \mathbb{P}[\mathcal{C}_{Q_2/V_2}|_l]$ with $\mathcal{C}_{Q_2/V_2}|_l \cong \mathcal{O}(1,2)$ by (1.3.2). Hence $(F_3.f)_{V_3} = \mathcal{O}_{\mathbb{F}_3}(-1)|_f = -1$. We saw in Lemma(1.7)(ii) that $s = C_3$ is the (-1)-curve on $T_3 = \mathbb{P}[\mathcal{C}_{Q_2/V_2}|_{C_2}]$ with $\mathcal{C}_{Q_2/V_2}|_{C_2} \cong \mathcal{O}(-4,-2)$ by (1.3.2). Hence $(F_3.s)_{V_3} = \mathcal{O}_{\mathbb{F}_3}(-1)|_s = \mathcal{O}_{T_3}(-1)|_{C_3} = 4$. Similarly, $(Q_3.f)_{F_3} = (Q_3|_{F_l}.f)_{F_l} = (f^2)_{F_l} = -1$ and $(Q_3.s)_{F_3} = (Q_3|_{T_3}.C_3)_{T_3} = (C_3^2)_{T_3} = -2$, hence $\mathcal{C}_{Q_3/F_3} = \mathcal{O}_{\mathbb{F}_3}(s+5f)$.

(ii) Since $Q_4 \cong \mathbb{F}_0$ by Lemma(1.10)(iii), $\mathcal{C}_{Q_3/V_3} \cong \mathcal{O}_{\mathbb{F}_3}(s+af, s+af)$ for an integre $a \in \mathbb{Z}$. The exact sequence proved in (i) implies $a = 1$. The fibre f_4 of $M_4 \cong \mathbb{F}_5$ is the (-1)-curve on $\mathbb{P}[\mathcal{C}_{M_5/V_5}|_f] = \mathbb{F}_1$ with $f = \sigma_4(f_4)$, so that $f_4 \subset \mathbb{P}[\mathcal{C}_{M_5/V_5}|_f]$ is defined by the surjection $\mathcal{O}(a, a+1) \to \mathcal{O}(a)$ for an integer $a \in \mathbb{Z}$. Here $a = \mathcal{O}_{F_4}(1)|_{f_4} = \mathcal{C}_{F_4/V_4}|_{f_4} = 1$ by (1.9.1). Next we apply Lemma(0.2)(ii) to $S = M_3$, $D = E_3$ and $C^+ = s_4 := M_4 \cap E_4$. Then $\mathcal{C}_{M_3/E_3} = \mathcal{O}_{\mathbb{F}_2}(s-2f)$ in (1.8.2) implies $(s_4^2)_{M_4} = 5 - 2.5 = -5$. This (-5)-curve s_4 on $M_4 = \mathbb{F}_5$ is the $(+2)$-curve on $\mathbb{P}[\mathcal{C}_{M_5/V_5}|_s] = E_4 \cap F_4 \cong \mathbb{F}_2$ with $s = \sigma_4(s_4)$, so that $s_4 \subset \mathbb{P}[\mathcal{C}_{M_5/V_5}|_s]$ is defined by a surjection $\mathcal{O}(b-2, b) \to \mathcal{O}(b)$

for an integer $b \in \mathbb{Z}$. Here $b = \mathcal{O}_{F_4}(1)|_{s_4} = \mathcal{C}_{F_4/V_4}|_s = -4$ by (1.9.1). Thus we see

$$(1.14.1) \qquad \mathcal{C}_{M_5/V_5}|_f = \mathcal{O}(1,2), \qquad \mathcal{C}_{M_5/V_5}|_s = \mathcal{O}(-6,-4).$$

Now the canonical surjection $\sigma_4^*\mathcal{C}_{M_5/V_5} \to \mathcal{O}_{F_4}(1)$ induces the surjection $\phi : \sigma_4^*\mathcal{C}_{M_5/V_5}|_{M_4} = \mathcal{C}_{M_5/V_5} \to \mathcal{O}_{F_4}(1)|_{M_4} \cong \mathcal{C}_{F_4/V_4}|_{M_4} = \mathcal{O}_{\mathbb{F}_5}(s+f)$ by (1.9.1). Then (1.14.1) implies $Ker(\phi) = \mathcal{O}_{\mathbb{F}_5}(2s+4f)$ and there is an exact sequence

$$(1.14.2)\ 0 \to \mathcal{O}_{\mathbb{F}_5}(2s+4f) \to \mathcal{C}_{M_5/V_5} \xrightarrow{\phi} \mathcal{C}_{F_4/V_4}|_{M_4} = \mathcal{O}_{\mathbb{F}_5}(s+f) \to 0.$$

Here $H^1(\mathcal{O}_{\mathbb{F}_5}(s+3f)) = H^1(\mathcal{O}(3,-2)) \cong k$, but $\mathcal{C}_{M_5/V_5}|_s = \mathcal{O}(-6,-4)$ means that the restriction of (1.14.2) to s splits. Hence (1.14.2) itself splits and $\mathcal{C}_{M_5/V_5} \cong \mathcal{O}_{\mathbb{F}_5}(s+f, 2s+4f)$. □

(1.15) Let $C_4 = Q_4 \cap F_4$, $C_5 = \sigma_4(C_4)$ and $C_6 = \sigma_5(C_5) = \sigma_5(E_5)$. We show

Lemma. $\mathcal{C}_{C_6/V_6} \cong \mathcal{O}(1,1,1)$.

Proof. We apply Lemma(0.2)(iii) to $S = Q_3$ and $F = E_3$. Since $C_3 = Q_3 \cap E_3$ is the (-3)-curve on $Q_3 = \mathbb{F}_3$, we see $\mathcal{C}_{Q_4/E_4} \cong \mathcal{O}_{\mathbb{F}_0}(s-f)$. On the other hand, $\mathcal{C}_{Q_3/V_3}\mathcal{O}_{\mathbb{F}_3}(s+2f, s+2f)$ in Lemma(1.14)(ii) means $\mathcal{C}_{Q_4/V_4} \cong \mathcal{O}_{\mathbb{F}_0}(s+2f, s-f)$ by Lemma(0.2)(i). Hence, from the exact sequence

$$(1.15.1) \qquad 0 \to \mathcal{C}_{E_4/V_4}|_{Q_4} \to \mathcal{C}_{Q_4/V_4} \to \mathcal{C}_{Q_4/E_4} \to 0,$$

with the isomorphisms $\mathcal{C}_{Q_4/V_4} \cong \mathcal{O}_{\mathbb{F}_0}(s+2f, s-f)$ and $\mathcal{C}_{Q_4/E_4} \cong \mathcal{O}_{\mathbb{F}_0}(s-f)$, we obtain $\mathcal{C}_{E_4/V_4}|_{Q_4} \cong \mathcal{O}_{\mathbb{F}_0}(s+2f)$. Therefore $\mathcal{C}_{E_5/V_5}|_{C_5} \cong \mathcal{C}_{E_4/V_4}|_{C_4} \cong \mathcal{O}_{\mathbb{F}_0}(s+2f)|_{s+3f} = \mathcal{O}(5)$. We saw in (1.4.1) that $\mathcal{C}_{C_5/E_5} \cong \mathcal{C}_{M_5/V_5}|_{C_5} = \mathcal{O}(-6,-4)$. Hence the exact sequence $0 \to \mathcal{C}_{E_5/V_5}|_{C_5} = \mathcal{O}(5) \to \mathcal{C}_{C_5/V_5} \to \mathcal{C}_{C_5/E_5} = \mathcal{O}(-6,-4) \to 0$ implies $(K_{V_5}.C_5) = 5-10-2 = -7$. Since $\sigma_5 : C_5 \to C_6$ is the cyclic cover of degree three (cf.(1.4)), $(K_{V_5}.C_5) = (\sigma_5^*K_{V_6} + 2E_5)C_5 = 3(K_{V_6}.C_6) + 2(E_5.C_5)$, hence $(K_{V_6}.C_6) = +1$. We saw $E_5 = \mathbb{P}[\mathcal{C}_{C_6/V_6}]$ is ismorphic to $\mathbb{F}_0$ in Lemma(1.11)(i), so $\mathcal{C}_{C_6/V_6} = \mathcal{O}(a,a,a)$ for an integre $a \in \mathbb{Z}$. then $(K_{V_6}.C_6) = 1$ means $a = 1$. □

(1.16) We write down the conormal bundles of the ruled surfaces Q, M, T, S.

$$
\begin{aligned}
\mathcal{C}_{Q_2/V_2} &= \mathcal{O}_{\mathbb{F}_3}(s-f, 2s+4f), & \mathcal{C}_{Q_3/V_3} &= \mathcal{O}_{\mathbb{F}_3}(2s+4f, 2s+4f),\\
\mathcal{C}_{Q_4/V_4} &= \mathcal{O}_{\mathbb{F}_0}(s+2f, s-f), & \mathcal{C}_{Q_5/V_5} &= \mathcal{O}_{\mathbb{F}_0}(s+2f, -2f),\\
\mathcal{C}_{M_2/V_2} &= \mathcal{O}_{\mathbb{F}_2}(s+3f, -4f), & \mathcal{C}_{M_3/V_3} &= \mathcal{O}_{\mathbb{F}_2}(s+3f, s-2f),\\
\mathcal{C}_{M_4/V_4} &= \mathcal{O}_{\mathbb{F}_5}(s+f, s+3f), & \mathcal{C}_{M_5/V_5} &= \mathcal{O}_{\mathbb{F}_5}(2s+4f, s+f),\\
\mathcal{C}_{T_3/V_3} &= \mathcal{O}_{\mathbb{F}_2}(s-2f, 3f), & \mathcal{C}_{T_4/V_4} &= \mathcal{O}_{\mathbb{F}_2}(s-4f, 5f),\\
\mathcal{C}_{S_2/V_2} &= \mathcal{O}_{\mathbb{F}_0}(3f, s-5f), & \mathcal{C}_{S_3/V_3} &= \mathcal{O}_{\mathbb{F}_0}(3f, 2s-5f),\\
\mathcal{C}_{S_4/V_4} &= \mathcal{O}_{\mathbb{F}_0}(s+3f, s-5f).
\end{aligned}
$$

(1.17) The effective cones of V_i over $X = \mathrm{Spec}(R)$ $(1 \le i \le 6)$ and the intersection numbers with generators of the Picard group, are give as follows. Here q_i, m_i, s_i, t_i are the fibres of the ruled surfaces Q_i, M_i, S_i, T_i, respectively.

(1) $\mathrm{NE}(V_1/X) = \mathbb{R}[q_1] \oplus \mathbb{R}[m_1]$, $\mathrm{Pic}(V_1/X) = \mathbb{Z}(-K_{V_1}) \oplus \mathbb{Z}E_1$.

$$
\begin{aligned}
(-K_{V_1}.q_1) &= -2, & (-K_{V_1}.m_1) &= +3,\\
(E_1.q_1) &= +1, & (E_1.m_1) &= -1.
\end{aligned}
$$

(2) $\mathrm{NE}(V_2/X) = \mathbb{R}[q_2] \oplus \mathbb{R}[m_2] \oplus \mathbb{R}[s_2]$,
$\mathrm{Pic}(V_2/X) = \mathbb{Z}(-K_{V_2}) \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}H_2$.

	q_2	m_2	s_2
$-K_{V_2}$	-1	1	1
E_2	1	2	0
H_2	-1	2	-1

(3) $\mathrm{NE}(V_3/X) = \mathbb{R}[q_3] \oplus \mathbb{R}[m_3] \oplus \mathbb{R}[s_3] \oplus \mathbb{R}[t_3]$,
$\mathrm{Pic}(V_3/X) = \mathbb{Z}(-K_{V_3}) \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}H_3 \oplus \mathbb{Z}F_3$.

	q_3	m_3	s_3	t_3
$-K_{V_3}$	0	0	0	1
E_3	1	2	0	0
H_3	0	1	-1	1
F_3	-1	1	1	-1

(4) $\mathrm{NE}(V_4/X) = \mathbb{R}[q_4] \oplus \mathbb{R}[m_4] \oplus \mathbb{R}[s_4] \oplus \mathbb{R}[t_4]$,
$\mathrm{Pic}(V_4/X) = \mathbb{Z}(-K_{V_4}) \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}H_4 \oplus \mathbb{Z}F_4$.

	q_4	m_4	s_4	t_4
$-K_{V_4}$	0	0	0	1
E_4	-1	1	-1	0
H_4	0	5	2	1
F_4	1	-1	2	-1

(5) $\mathrm{NE}(V_5/X) = \mathbb{R}[q_5] \oplus \mathbb{R}[m_5] \oplus \mathbb{R}[s_5]$,
$\mathrm{Pic}(V_5/X) = \mathbb{Z}(-K_{V_5}) \oplus \mathbb{Z}E_5 \oplus \mathbb{Z}H_5$.

	q_5	m_5	s_5
$-K_{V_5}$	1	-1	2
E_5	-1	1	-1
H_5	1	3	2

(6) $\mathrm{NE}(V_6/X) = \mathbb{R}[q_6] \oplus \mathbb{R}[m_6]$, $\mathrm{Pic}(V_6/X) = \mathbb{Z}(-K_{V_6}) \oplus \mathbb{Z}H_6$.

	q_6	m_6
$-K_{V_6}$	-1	1
H_6	-2	0

Here $q_6 = \sigma_5(q_5)$ is equal to C_6 in (1.15), and $m_6 = \sigma_5(m_5)$.

2. The birational map (II)

(2.1) In this section we consider the birational map (II). Let $\tau : V \to \mathrm{Spec}(R)$ be a standard $\mathbb{P}^2$-bundle over the local ring R of X at a smooth point of the non-smooth locus constructed from the R-order (0.1.2). Let $\tau^{-1}(o) = R \cup S \cup T$ be the central fibre such that R, S, T isomorphic to $\mathbb{F}_1$, and

(2.1.1) $$r = R \cap T, \qquad s = S \cap R, \qquad t = T \cap S$$

are the (-1)-curves on R, S, T, respectively. For the the non-smooth locus $\Delta \subset \mathrm{Spec}(R)$ we assume $\tau^{-1}(\Delta)$ decomposes into three divisoris D, H, L of V such that

$$R = \tau^{-1}(o) \cap D, \qquad S = \tau^{-1}(o) \cap H, \qquad T = \tau^{-1}(o) \cap L.$$

Then V is obtained by twice blow-ups of $\mathbb{P}^2_R$ along $\mathbb{P}^1_\Delta \supset \mathbb{P}^0_\Delta \cong \Delta$:

$$V \xrightarrow{\sigma} V_0 \xrightarrow{\sigma_0} \mathbb{P}^2_R,$$

where σ_0 is the blow-up along $\mathbb{P}^1_\Delta$ with the exceptional divisor H_0, and σ is the blow-up along $L_0 = \sigma_0^{-1}(\mathbb{P}^0_\Delta)$. Then D (resp. H) is the proper transform of $\mathbb{P}^2_\Delta$ (resp. H_0) and L is the exceptional divisor of σ.

$$\begin{array}{ccccccc}
V & & D & & H & & L \\
{\scriptstyle\sigma}\downarrow & & \downarrow & & \| & & \downarrow \\
V_0 & & D_0 & & H_0 & \supset & L_0 \\
{\scriptstyle\sigma_0}\downarrow & & \| & & \downarrow & & \downarrow \\
\mathbb{P}^2_R & \supset & \mathbb{P}^2_\Delta & \supset & \mathbb{P}^1_\Delta & \supset & \mathbb{P}^0_\Delta \cong \Delta
\end{array}$$

(2.2) The center $L_0 = \sigma_0^{-1}(\mathbb{P}^0_\Delta)$ of the blow-up $\sigma : V \to V_0$ is isomorphic to $\mathbb{P}[\mathcal{C}_{\mathbb{P}^1_\Delta/\mathbb{P}^2_R}|_{\mathbb{P}^0_\Delta}] \cong \mathbb{P}^1_\Delta$.

Lemma. *(i)* $\mathcal{C}_{L_0/V_0} \cong \mathcal{O}_{\mathbb{P}^1_\Delta}(1,0)$, *(ii) There is a split exact sequence*

$$0 \to \mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s+f) \to \mathcal{C}_{T/V} \to \mathcal{C}_{T/L} \cong \mathcal{O}_{\mathbb{F}_1} \to 0.$$

Proof. (i) We see $\mathcal{C}_{H_0/V_0}|_{L_0} \cong \mathcal{O}_{H_0}(1)|_{L_0} \cong \mathcal{O}_{L_0}(1) \cong \mathcal{O}_{\mathbb{P}^1_\Delta}(1)$ and $\mathcal{C}_{L_0/H_0} \cong \sigma_0^*\mathcal{C}_{\mathbb{P}^0_\Delta/\mathbb{P}^1_\Delta} \cong \mathcal{O}_{L_0}$. Hence (i) follows from the exact sequence $0 \to \mathcal{C}_{H_0/V_0}|_{L_0} \to \mathcal{C}_{L_0/V_0} \to \mathcal{C}_{L_0/H_0} \to 0$.

(ii) Let $p = \mathbb{P}^0_\Delta \cap \tau^{-1}(o)$ and let $f = \sigma_0^{-1}(p) \cong \mathbb{P}^1_k$ be the fibre of the $\mathbb{P}^1$-bundle $\sigma_0 : L_0 \to \mathbb{P}^0_\Delta \cong \Delta$. Then $\mathcal{C}_{f/L_0} \cong \mathcal{O}$ and $\mathcal{C}_{L_0/V_0}|_f \cong \mathcal{O}(1,0)$ by (i). Hence the exact sequence $0 \to \mathcal{C}_{L_0/V_0}|_f \to \mathcal{C}_{f/V_0} \to \mathcal{C}_{f/L_0} \to 0$ splits. The surjection $\phi : \sigma^*(\mathcal{C}_{L_0/V_0}|_f) \cong (\sigma^*\mathcal{C}_{L_0/V_0})|_T \to \mathcal{C}_{L/V}|_T$ induces a commutative diagram with exact rows :

$$\begin{array}{ccccc}
0 \to \sigma^*(\mathcal{C}_{L_0/V_0}|_f) & \longrightarrow & \sigma^*\mathcal{C}_{f/V_0} & \longrightarrow & \sigma^*\mathcal{C}_{f/L_0} \to 0 \\
{\scriptstyle\phi}\downarrow & & \downarrow & & \| \\
0 \to \mathcal{C}_{L/V}|_T & \longrightarrow & \mathcal{C}_{T/V} & \longrightarrow & \mathcal{C}_{T/L} \to 0.
\end{array}$$

Since the first row splits, the second row also splits. Hence, for the proof of (ii), we will show $\mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$. The exceptinal divisor L of σ is equal to $\mathbb{P}[\mathcal{C}_{L_0/V_0}]$ with $\mathcal{C}_{L_0/V_0} \cong \mathcal{O}_{\mathbb{P}^1_\Delta}(1,0)$ by (i), so that $T = \mathbb{P}[\mathcal{C}_{L_0/V_0}|_f] \cong \mathbb{P}[\mathcal{O}(1,0)]$ and $\mathcal{C}_{L/V}|_T \cong \mathcal{O}_L(1)|_T \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$. □

By symmetry of R, S, T, (ii) implies $\mathcal{C}_{R/V} \cong \mathcal{C}_{S/V} \cong \mathcal{C}_{T/V} \cong \mathcal{O}_{\mathbb{F}_1}(s+f, 0)$.

(2.3) Let $\sigma_1 : V_1 \to V$ be the blow-up along R with the exceptional divisor E_1 and the proper transforms D_1, H_1, L_1, S_1, T_1 of D, H, L, S, T, respectively. Let M_1 (resp. N_1) be the exceptional divisor of the restriction $\sigma_1 : H_1 \to H$ (resp. $\sigma_1 : L_1 \to L$) of σ_1 and let

$$s_1 = M_1 \cap S_1 = \mathbb{P}[\mathcal{C}_{s/H}], \qquad r_1 = N_1 \cap T_1 = \mathbb{P}[\mathcal{C}_{r/L}].$$

Lemma. *(i)* $M_1 \cong \mathbb{F}_1$, $(s_1^2)_{M_1} = 1$, *(ii)* $N_1 \cong \mathbb{F}_0$, $(r_1^2)_{N_1} = 0$.

Proof. We see $(S.s)_H = (T.r)_L = 0$, $(s^2)_S = -1$, $(r^2)_{N_1} = 0$, hence we have exact sequences

$$0 \to \mathcal{C}_{S/H}|_s \cong \mathcal{O} \to \mathcal{C}_{s/H} \to \mathcal{C}_{s/S} \cong \mathcal{O}(1) \to 0,$$
$$0 \to \mathcal{C}_{T/L}|_r \cong \mathcal{O} \to \mathcal{C}_{r/L} \to \mathcal{C}_{r/T} \cong \mathcal{O} \to 0.$$

Lemma follows since $s_1 \subset M_1$ (resp. $r_1 \subset N_1$) is defined by the surjection $\mathcal{C}_{s/H} \to \mathcal{C}_{s/S}$ (resp. $\mathcal{C}_{r/L} \to \mathcal{C}_{r/T}$). □

(2.4) For the proper transforms S_1 (resp. T_1) of S (resp. T), we have

Lemma. *The exact sequences*

$$(2.4.1) \qquad 0 \to \mathcal{C}_{H_1/V_1}|_{S_1} \to \mathcal{C}_{S_1/V_1} \to \mathcal{C}_{S_1/H_1} \to 0,$$
$$(2.4.2) \qquad 0 \to \mathcal{C}_{L_1/V_1}|_{T_1} \to \mathcal{C}_{T_1/V_1} \to \mathcal{C}_{T_1/L_1} \to 0.$$

splits with isomorphisms $\mathcal{C}_{H_1/V_1}|_{S_1} \cong \mathcal{C}_{L_1/V_1}|_{T_1} \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$, $\mathcal{C}_{S_1/H_1} \cong \mathcal{O}_{\mathbb{F}_1}(s)$ *and* $\mathcal{C}_{T_1/L_1} \cong \mathcal{O}_{\mathbb{F}_1}(f)$.

Proof. We saw $\mathcal{C}_{H_1/V_1}|_{S_1} \cong \mathcal{C}_{H/V}|_S \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$ and $\mathcal{C}_{L_1/V_1}|_{T_1} \cong \mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$ in Lemma(2.2)(ii). Hence we show $\mathcal{C}_{S_1/V_1} \cong \mathcal{O}_{\mathbb{F}_1}(s)$ and $\mathcal{C}_{T_1/L_1} \cong \mathcal{O}_{\mathbb{F}_1}(f)$, i.e. $(S_1.f)_{H_1} = -1$, $(T_1.f)_{L_1} = 0$, $(S_1.s_1)_{H_1} = 1$ and $(T_1.t_1)_{L_1} = -1$. For, $(S_1.f)_{H_1} = (\sigma_1^* S - M_1)f = (S.f)_H - (M_1.f)_{H_1} = 0 - 1 = -1$. By Lemma(2.3)(i), $s_1 \subset M_1$ is defined by a surjection $\mathcal{C}_{R/V}|_f \cong \mathcal{O}(1,0) \to \mathcal{O}(1)$, so $(M_1.s_1)_{H_1} = \mathcal{O}_{M_1}(-1)|_{s_1} = -1$, hence $(S_1.s_1)_{H_1} = (\sigma_1^* S - M_1)s_1 = (S.s)_H - (M_1.s_1)_{H_1} = 0 - (-1) = 1$. Similarly, $(T_1.f)_{L_1} = (\sigma_1^* T - N_1)f = (T.f)_L - (N_1.f)_{L_1} = 0 - 0 = 0$ and $(T_1.t_1)_{L_1} = (\sigma_1^* T - N_1)t_1 = (T.t_1)_L - (N_1.t_1)_{L_1} = 0 - 1 = -1$. The splitting of (2.4.1) and (2.4.2) follows from $\mathrm{Ext}(\mathcal{O}_{\mathbb{F}_1}(s), \mathcal{O}_{\mathbb{F}_1}(s+f)) = H^1(\mathcal{O}_{\mathbb{F}_1}(f)) = H^1(\mathcal{O}_{\mathbb{P}^1}(1)) = 0$ and $H^1(\mathcal{O}_{\mathbb{F}_1}(s)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(0,-1)) = 0$. □

(2.5) From (2.4.1) we see $\mathcal{C}_{S_1/V_1}|_f \cong \mathcal{O}(1,1)$, hence $S_1 \subset V_1$ is flopped, i.e. there are birational maps

$$V_1 \overset{\sigma_2}{\leftarrow} V_2 \overset{\sigma_3}{\rightarrow} V_3,$$

where σ_2 is the blow-up along S_1 with the exceptional divisor F_2, and σ_3 is the blow-down of F_2 to the other direction. Let $S_3 = \sigma_3(F_2)$. Let $M_2, E_2, T_2, \ldots$ (resp. $M_3, E_3, T_3, \ldots$) be the birational transforms on V_2 (resp. V_3) of $M_1, E_1, T_1, \ldots$, respectively.

Lemma. *$M_3 \cap S_3$ is the (-1)-curve on $S_3 = \mathbb{F}_1$.*

Proof. The (-1)-curve $s_1 = S_1 \cap M_1$ on $S_1 = \mathbb{F}_1$ is the $(+1)$-curve on $M_1 = \mathbb{F}_1$ by Lemma(2.3)(i); $\mathcal{C}_{s_1/M_1} \cong \mathcal{O}(-1)$. Hence, from (2.4.1), we have a surjection $\mathcal{C}_{s_1/E_1} \cong \mathcal{C}_{S_1/V_1}|_{s_1} \cong \mathcal{O}(0,-1) \to \mathcal{C}_{s_1/M_1} \cong \mathcal{O}(-1)$. This defines the closed immersion $M_2 \cap F_2 = \mathbb{P}[\mathcal{C}_{s_1/M_1}] \subset E_2 \cap F_2 = \mathbb{P}[\mathcal{C}_{s_1/E_1}]$, which is ismorpohic to $M_3 \cap S_3 \subset S_3$ by σ_3. □

(2.6) We apply Lemma(0.3)(iii) to $S = S_1 \cong \mathbb{F}_1$ and $F = E_1$. Since E_1 intersects S_1 with the (-1)-curve on $S_1 \cong \mathbb{F}_1$, $\mathcal{C}_{S_1/V_1} \cong \mathcal{O}_{\mathbb{F}_1}(s+f, s)$ impilies $\mathcal{C}_{S_3/E_3} \cong \mathcal{O}_{\mathbb{F}_1}(s)$ and there is a split exact sequence
(2.6.1)

$$0 \to \mathcal{C}_{E_3/V_3}|_{S_3} \cong \mathcal{O}_{\mathbb{F}_1}(s+f) \to \mathcal{C}_{S_3/V_3} \to \mathcal{C}_{S_3/E_3} \cong \mathcal{O}_{\mathbb{F}_1}(s) \to 0.$$

Lemma. *The exact sequences*

$$0 \to \mathcal{C}_{L_2/V_2}|_{T_2} \to \mathcal{C}_{T_2/V_2} \to \mathcal{C}_{T_2/L_2} \to 0, \tag{2.6.2}$$

$$0 \to \mathcal{C}_{L_3/V_3}|_{T_3} \to \mathcal{C}_{T_3/V_3} \to \mathcal{C}_{T_3/L_3} \to 0. \tag{2.6.3}$$

splits with isomorphisms $\mathcal{C}_{L_2/V_2}|_{T_2} \cong \mathcal{C}_{T_2/L_2} \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$ and $\mathcal{C}_{L_3/V_3}|_{T_3} \cong \mathcal{C}_{T_3/L_3} \cong \mathcal{O}_{\mathbb{P}^2}(1)$.

Proof. We see $\mathcal{C}_{L_2/V_2}|_{T_2} \cong \mathcal{C}_{L/V}|_T \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$, hence we will show $\mathcal{C}_{T_2/V_2} \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$, i.e. $(T_2.f)_{L_2} = -1$ and $(T_2.t_2)_{L_2} = 0$. Let $Q_2 =$

$\mathbb{P}[\mathcal{C}_{S_1/V_1}|_{t_1}] \cong \mathbb{P}[\mathcal{C}_{t_1/L_1}]$ be the exceptional divisor of $\sigma_2 : L_2 \to L_1$. Then $(T_2.f)_{L_2} = (\sigma_2^* T_1 - Q_2)f = (T_1.f)_{L_1} - (Q_2.f)_{L_2} = 0 - 1 = -1$. Since the (-1)-curve t_1 on $T_1 \cong \mathbb{F}_1$ is a fibre on S_1, $\mathcal{C}_{S_1/V_1}|_{t_1} \cong \mathcal{O}_{\mathbb{F}_1}(s+f, s)|_f \cong \mathcal{O}(1,1)$ by (2.4.1). Hence $t_2 = T_2 \cap Q_2 \subset Q_2$ is defined by a surjection $\mathcal{C}_{t_1/L_1} \cong \mathcal{C}_{S_1/V_1}|_{t_1} \cong \mathcal{O}(1,1) \to \mathcal{C}_{t_1/T_1} \cong \mathcal{O}(1)$, so $(Q_2.t_2) = \mathcal{O}_{Q_2}(-1)|_{t_2} = -1$ and $(T_2.t_2)_{L_2} = (\sigma_2^* T_1 - Q_2)t_2 = (T_1.t_1)_{L_1} - (Q_2.t_2) = -1 - (-1) = 0$ since $\mathcal{C}_{T_1/L_1} \cong \mathcal{O}_{\mathbb{F}_1}(f)$ by (2.4.2). The exact sequence (2.6.3) follows from $(T_3.f)_{L_3} = (\sigma_3^* T_3.f)_{L_2} = (T_2.f)_{L_2} = -1$ and $(L_3.f)_{V_3} = (\sigma_3^* L_3.f)_{V_2} = (L_2.f)_{V_2} = -1$ by (2.6.2). □

(2.7) From (2.6.3) we see $\mathcal{C}_{T_3/V_3} \cong \mathcal{O}_{\mathbb{P}^2}(1,1)$, so there are birational maps

$$V_3 \overset{\sigma_4}{\leftarrow} V_4 \overset{\sigma_5}{\rightarrow} V_5,$$

where σ_4 is the blow-up along $T_3 \cong \mathbb{P}^2$ with the exceptional divisor $G_4 = \mathbb{P}[\mathcal{C}_{T_3/V_3}] \cong \mathbb{P}^2 \times \mathbb{P}^1$, and σ_5 is the blow-down of G_4 onto the second factor $\mathbb{P}^1$. The birational transform S_5 on V_5 of $S_3 \cong \mathbb{F}_1$ is the blow-up of S_3 at the point $p = S_3 \cap T_3$. we recall the relative Picard number of V_5 over Spec(R) is equal to two. Let f_3 be the fibre of $S_3 = \mathbb{F}_1$ intersectiong at the point $p = S_3 \cap T_3$ and let f_4 (resp. f_5) be the birational transform of f_3 on S_4 (resp. S_5). The two extremal rays of V_5 over X are generated by f_5 and the image $e_5 = \sigma_5(G_4)$. We see

$$\begin{aligned}(-K_{V_4}.f_4)_{V_4} &= (\sigma_4^*(-K_{V_3}) - G_4)f_4 = (-K_{V_3}.f_3)_{V_3} - (G_4.f_4)_{V_4}, \\ &= (\sigma_5^*(-K_{V_5}) - 2G_4)f_4 = (-K_{V_5}.f_5)_{V_5} - 2(G_4.f_4)_{V_4}\end{aligned}$$

with $(-K_{V_3}.f_3) = 0$ and $(G_4.f_4) = 1$, hence $(-K_{V_5}.f_5)_{V_5} = 1$. Let

(2.7.1) $$\tau_1 : V_5 \to X_1$$

be the contraction morphism of f_5.

Lemma. $(E_5.f_5)_{V_5} = 0$.

Proof. From (0.3) we see the exact sequence (2.6.1) induces the split exact sequecne

(2.7.2) $$0 \to \mathcal{C}_{E_5/V_5}|_{S_5} \to \mathcal{C}_{S_5/V_5} \to \mathcal{C}_{S_5/E_5} \to 0.$$

with isomorphisms $\mathcal{C}_{E_5/V_5}|_{S_5} \cong \mathcal{O}_{S_5}(s+f-e)$ and $\mathcal{C}_{S_5/E_5} \cong \mathcal{O}_{S_5}(s-e)$. Since T_3 is disjoint from M_3, Lemma(2.5) means the point $p = S_3 \cap T_3$ is not on the (-1)-curve on $S_3 = \mathbb{F}_1$, so that $f_5 \equiv s+f-e$ on S_5. Hence we see from (2) that $(E_5.f_5) = -\mathcal{C}_{E_5/V_5}|_{f_5} = -(s+f-e)^2 = 0$. □

(2.8) The above Lemma(2.7) implies that any irreducible curve C on V_5 with $(E_5.C) = 0$ is contracted by the morphism (2.7.1). We show E_5 is covered by such curves. Let f be a fibre of $R \cong \mathbb{F}_1$ on V. From Lemma(2.2)(ii), we see $\mathcal{C}_{R/V}|_f \cong \mathcal{O}(1,0)$, so that $E_{1,f} := \sigma_1^{-1}(f) = \mathbb{P}[\mathcal{C}_{R/V}|_f]$ is isomorphic to $\mathbb{F}_1$. Let C be a section of $E_{1,f}$ containing the point $E_{1,f} \cap T_1$ (such C exists with 1-parameter family for each fibre f of R).

Lemma. $(E_5.C_5)_{V_5} = 0$ *for the birational transform* C_5 *of* C *on* V_5.

Proof. For simplicity we use the same letter C for the birational transforms of C on V_i $(1 \le i \le 5)$. Since the $(+1)$-curve $C \subset E_{1,f}$ is defined by a surjection $\mathcal{C}_{R/V}|_f \cong \mathcal{O}(1,0) \to \mathcal{O}(1)$, we see $(E_1.C) = \mathcal{O}_{E_1}(-1)|_C = -1$. Then $(E_3.C) = -1$ because $E_2 = \sigma_2^* E_1 - F_2 = \sigma_3^* E_3 - F_2$. From $E_4 = \sigma_4^* E_3 - G_4 = \sigma_5^* E_5 - 2G_4$ together with $(G_4.C) = 1$, we see $(E_5.C) = (E_3.C) + (G_4.C) = -1 + 1 = 0$. □

By the above Lemma, there are 1-parameter family of the curves C with $(E_5.C) = 0$ for each point on $e = \sigma_5(G_4)$, so the image of the morphism (2.7.1) is 2-dimensional. This means $\tau_1 : V_5 \to X_1$ is a standard $\mathbb{P}^2$-bundle and the structure morphism $X_1 \to \mathrm{Spec}(R)$ is the blow-up at the origin. The statement (II) of Theorem is proved.

(2.9) Let X be a smooth algebraic surface and let $Y \to X$ be the blow up at a point of X with the exceptional line e. Let $\tau_1 : W \to Y$ be a standard $\mathbb{P}^2$-bundle with the non-smooth locus intersecting e transeversely at one point $p_0 = \Delta \cap e$. If we find a smooth subvariety S_5 of $\tau_1(e)$ such that

(2.9.1) S_5 is a $\mathbb{P}^1$-bundle over e away from the point $p_0 = e \cap \Delta$,
(2.9.2) there is a section e_5 on S_5 over e with $(e_5^2)_{S_5} = -1$,
then we obtain a standard $\mathbb{P}^2$-bundle V over X by applying the inverse of the birational map (II) described above. We see the existence of such subvarieties $e_5 \subset S_5$ by the following argument. As in (2.1), $\tau_1^{-1}(e)$ is obtained from a $\mathbb{P}^2$-bundle $\tau_2 : P \to e$ by twice blow-ups

$$\tau_1^{-1}(e) \xrightarrow{\epsilon_2} P_1 \xrightarrow{\epsilon_1} P,$$

where ϵ_1 is the blow-up of P along a line l in $\tau_2^{-1}(p_0) \cong \mathbb{P}^2$, and ϵ_2 is the blow-up of P_1 along a fibre f of the exceptional divisor $\epsilon_1^{-1}(l)$ of ϵ_1. We write $P = \mathbb{P}[\mathcal{E}]$ with a rank three vector bundle $\mathcal{E} \cong \mathcal{O}(0, a, b)$ on $e \cong \mathbb{P}^1$. Let $\phi : \mathcal{E}\mathcal{O}(c, c+1)$ be a surjection for an integer $c \in \mathbb{Z}$, and $e_0 = \mathbb{P}[\mathcal{O}(c)] \subset S_0 = \mathbb{P}[\mathcal{O}(c, c+1)]$ be the corresponding subvarieties of $P = \mathbb{P}[\mathcal{E}]$. We choose $\phi : \mathcal{E} \to \mathcal{O}(c, c+1)$ such that

(2.9.3) the line $S_0 \cap \tau_2^{-1}(p_0)$ is not equal to the center l of ϵ_1,
(2.9.4) $e_0 \cap \tau_1^{-1}(p_0)$, $S_0 \cap l$, $\epsilon_1(f)$ are distinct three points on $\tau_2^{-1}(p_0)$.

Then we see the proper transforms in $\tau_1^{-1}(e)$ of S_0 and e_0 satify (2.9.1) and (2.9.2).

3. The birational map (III)

(3.1) In this section we consider the birational map (III). Let $\tau : V \to X$ be a standard $\mathbb{P}^2$-bundle over a smooth algebraic surface X and let $C \subset X$ be a curve intersecting the non-smooth locus Δ of τ transversely at one smooth point p_0 of Δ. Let $C_0 \subset V$ be a curve which is isomoorphic to C by τ. Let $\tau^{-1}(p_0) = R \cup S \cup T$ with R, S, T isomorphic to $\mathbb{F}_1$ and assume $s = S \cap R$ is the (-1)-curve on S.

Lemma. *There is a split exact sequence*

$$0 \to \mathcal{C}_{\tau^{-1}(C)/V}|_S \cong \mathcal{O}_{\mathbb{F}_1} \to \mathcal{C}_{S/V} \to \mathcal{C}_{S/\tau^{-1}(C)} \cong \mathcal{O}_{\mathbb{F}_1}(s+f) \to 0.$$

Proof. $\mathcal{C}_{\tau^{-1}(C)/V}|_S \cong \tau^*\mathcal{O}_X(-C)|_S \cong \mathcal{O}_{\mathbb{F}_1}$ is clear. Since $R+S+T \equiv 0$ on $\tau^{-1}(C)$, we see $(S.f)_{\tau^{-1}(C)} = -(R.f) - (T.f) = -1 - 0 = -1$, and $(S.s)_{\tau^{-1}(C)} = 0$ because s is a fibre of R. Hence $\mathcal{C}_{S/\tau^{-1}(C)} \cong \mathcal{O}_{\mathbb{F}_1}(s+f)$. □

(3.2) Let $\sigma_1 : V_1 \to V$ be the blow-up along C_0 with the exceptional divisor E_1 and let R_1, S_1, T_1, D_1 be the proper transforms of $R, S, T, \tau^{-1}(C)$, respectively. The restriction to S_1 of σ_1 is the blow-up of S at the point $p = \Delta \cap C_0$. Let

$$e_1 = S_1 \cap E_1$$

be the exceptional line of $\sigma_1 : S_1 \to S \cong \mathbb{F}_1$.

Lemma. *The exact sequence*

$$(3.2.1)\qquad 0 \to \mathcal{C}_{D_1/V_1}|_{S_1} \to \mathcal{C}_{S_1/V_1} \to \mathcal{C}_{S_1/D_1} \to 0$$

splits with isomorphisms $\mathcal{C}_{D_1/V_1}|_{S_1} \cong \mathcal{O}_{S_1}(e_1)$ *and* $\mathcal{C}_{S_1/D_1} \cong \mathcal{O}_{S_1}(s_1 + f_1)$.

Proof. $\mathcal{C}_{D_1/V_1}|_{S_1} \cong \mathcal{O}_{S_1}(e_1)$ follows from

$$\begin{aligned}(D_1.s_1)_{V_1} &= (\sigma_1^* D - E_1)s_1 = (D.s)_V - (E_1.s)_{V_1} = 0 - 0 = 0,\\ (D_1.f_1)_{V_1} &= (\sigma_1^* D - E_1)f_1 = (D.f)_V - (E_1.f)_{V_1} = 0 - 0 = 0,\\ (D_1.e_1)_{V_1} &= (\sigma_1^* D - E_1)e_1 = -(E_1.e_1)_{V_1} = 0.\end{aligned}$$

Similarly, $\mathcal{C}_{S_1/D_1} \cong \mathcal{O}_{S_1}(s_1 + f_1)$ follows from the equalities $(S_1.s_1)_{D_1} = (\sigma_1^* S.s_1) = (S.s)_{\tau^{-1}(C)} = 0$, $(S_1.f_1)_{V_1} = (\sigma_1^* S.f_1) = (S.f)_{\tau^{-1}(C)} = -1$ and $(S_1.e_1)_{D_1} = (\sigma_1^* S.e_1) = 0$. Since $H^1(\mathcal{O}_{S_1}(-s_1 - f_1 + e_1)) = H^1(\mathcal{O}_{S_1}(-s_1 - 2f_1))$ (Serre duality) $= H^1(\mathcal{O}_{\mathbb{F}_1}(-s - 2f)) = 0$, the exact sequence (3.2.1) splits. □

(3.3) There is a blow-down $S_1 \to \mathbb{F}_0$ with the exceptional line $e \equiv f_1 - e_1$. We take

$$(3.3.1)\qquad s = f_1, \qquad f = s_1 + f_1 - e_1$$

as the section and the fibre of S_1 induced from those of $\mathbb{F}_0$. Then $s - e = e_1$ and $s + f - e = s_1 + f_1$, so the exact sequence (3.2.1) is equal to

$$(3.3.2)\qquad 0 \to \mathcal{C}_{D_1/V_1}|_{S_1} \to \mathcal{C}_{S_1/V_1} \to \mathcal{C}_{S_1/D_1} \to 0.$$

with isomorphisms $\mathcal{C}_{D_1/V_1}|_{S_1} \cong \mathcal{O}_{S_1}(s-e)$ and $\mathcal{C}_{S_1/D_1} \cong \mathcal{O}_{S_1}(s+f-e)$. Hence $\mathcal{C}_{S_1/V_1}|_e \cong \mathcal{O}(1,1)$, so there are birational maps

$$V_1 \overset{\sigma_2}{\leftarrow} V_2 \overset{\sigma_3}{\to} V_3,$$

where σ_2 is the blow-up of e with the exceptional divisor $B_3 \cong \mathbb{P}^1 \times \mathbb{P}^2$, and σ_3 is the blow-down of B_3 to the other direction. Let $D_3, S_3, \ldots$ be the proper transforms on V_3 of $D_1, S_1, \ldots$, respectively. We see from (0.3) that the exact sequence (3.3.2) induces

$$(3.3.3)\qquad 0 \to \mathcal{C}_{D_3/V_3}|_{S_3} \to \mathcal{C}_{S_3/V_3} \to \mathcal{C}_{S_3/D_3} \to 0.$$

with isomorphisms $\mathcal{C}_{D_3/V_3}|_{S_3} \cong \mathcal{O}_{\mathbb{F}_0}(s)$ and $\mathcal{C}_{S_3/D_3} \cong \mathcal{O}_{\mathbb{F}_0}(s+f)$. Therefore $\mathcal{C}_{S_3/V_3}|_f \cong \mathcal{O}(1,1)$, so that $S_3 \subset V_3$ is flopped, i.e. there are birational maps

$$V_3 \overset{\sigma_4}{\leftarrow} V_4 \overset{\sigma_5}{\rightarrow} V_5,$$

where σ_4 is the blow-up of S_3 with the exceptional divisor F_4, and σ_5 is the blow-down of F_4 to the other direction. The flopped surface S_5 on V_5 is isomorphic to $\mathbb{P}[\mathcal{C}_{S_3/V_3}|_s] \cong \mathbb{P}[\mathcal{O}(-1,0)] = \mathbb{F}_1$ by (3.3.3), and satisfies $\mathcal{C}_{S_5/V_5} \cong \mathcal{O}_{\mathbb{F}_1}(s+f, s+f)$ by Lemma(0.3)(i).

(3.4) The extremal rays on V_5 over X. One is generated by the fibre f_5 of the flopped surface $S_5 \cong \mathbb{F}_1$, and the other is the birational transform l_5 of the line l in $\tau^{-1}(p) \cong \mathbb{P}^2$ with $l \cap C_0$ non-empty for a point $p \in C - (C \cap \Delta)$. We see $(-K_{V_5}.f_5) = 0$, $(-K_{V_5}.l_5) = 1$, $(E_5.f_5) = -1$ and $(E_5.l_5) = 1$. The birational transform D_5 on V_5 of $\tau^{-1}(C)$ has a $\mathbb{P}^1$-bundle structure over the surface $D_5 \cap E_5$ with fibre l_5, hence the contraction morphism of l_5

$$\sigma_6 : V_5 \to V_6,$$

is the blow-up of V_6 along the surface $\sigma_6(D_5)$. The structure morphism $V_6 \to X$ defines the standard $\mathbb{P}^2$-bundle over X.

4. Appendix

(4.1) The standard $\mathbb{P}^2$-bundle V over the local ring R of a singular point of Δ constructed from the R-algebra (0.1.3), is described as follows.

emma. *[M,(2.4)] (i) V is covered by three open sets U_3, U_{11}, U_{12}, which are isomorphic to the affine space $\mathbb{A}^4_k$ of dimension four with affine coordinates (f, x_1, x_2, x_3), (g, y_8, y_5, y_3), $(w_{12}, w_2, w_5, w_{11})$, respectively, such that the transition functions are given by*

(a) U_{11} to U_{12}:

$$\begin{aligned} &f = y_8^3 + g y_5^3 + g^2 y_3^3 - 3g y_8 y_5 y_3,\\ &x_1 = (y_3 y_8^2 - \omega y_5^2 y_8 - \omega^2 g y_3^2 y_5)/y_{12},\\ &x_2 = (y_5^2 + y_3 y_8)/y_{12}, \qquad x_3 = y_3/y_{12},\\ &\textit{with} \quad y_{12} = \omega^2 y_5^3 + \omega g y_3^3 + (\omega^2 - \omega) y_3 y_5 y_8, \end{aligned}$$

(b) U_{12} to U_{11}:

$$\begin{aligned} &g = x_1^3 + f x_2^3 + f^2 x_3^3 - 3f x_1 x_2 x_3,\\ &y_8 = (-x_1 x_2^2 - \omega x_1^2 x_3 - \omega^2 f x_2 x_3^2)/x_{11}\ , \end{aligned}$$

$$y_5 = \omega^2(x_2^2 - x_1x_3)/x_{11}, \qquad y_3 = x_3/x_{11},$$
$$\textit{with} \quad x_{11} = \omega x_2^3 - \omega^2 f x_3^3 + (\omega^2 - \omega)x_1x_2x_3,$$

(c) U_3 to U_{12}:
$$\begin{aligned} f &= w_2^3 - \omega w_{11}w_{12}^2 + (1-\omega)w_2w_5w_{12} \\ &= w_2(w_2^2 - \omega w_5 w_{11}) + w_{12}(w_2w_5 - \omega w_{11}w_{12}), \\ x_1 &= (w_2^2 - \omega w_5 w_{12})/w_{12}, \\ x_2 &= w_2/w_{12}, \qquad x_3 = 1/w_{12}, \end{aligned}$$

(d) U_{12} to U_3:
$$w_{12} = 1/x_3, \qquad w_2 = x_2/x_3,$$
$$w_5 = \omega^2(x_2^2 - x_1x_3)/x_3, \qquad w_{11} = x_{11}/x_3,$$

(e) U_3 to U_{11}:
$$\begin{aligned} -g &= w_5^3 - \omega^2 w_{11}^2 w_{12} + (\omega - 1)w_2w_5w_{11}, \\ &= w_5(w_5^2 - w_2w_{11}) + \omega w_{11}(w_2w_5 - \omega w_{11}w_{12}), \\ y_8 &= (w_2w_{11} - w_5^2)/w_{11}, \\ y_5 &= w_5/w_{11}, \qquad y_3 = 1/w_{11}, \end{aligned}$$

(f) U_{11} to U_3:
$$w_{12} = y_{12}/y_3, \qquad w_2 = (y_5^2 + y_3y_8)/y_3,$$
$$w_5 = y_5/y_3, \qquad w_{11} = 1/y_3,$$

(ii) The projection $\tau : V \to Spec(R)$ is given by

$$\begin{aligned} \tau(f, x_1, x_2, x_3) &= (f, x_1^3 + f x_2^3 + f^2 x_3^3 - 3f x_1x_2x_3) \qquad \textit{on } U_{12}, \\ \tau(g, y_8, y_5, y_3) &= (y_8^3 + g y_5^3 - g^2 y_3^3 + 3g y_3y_5y_8, g) \qquad \textit{on } U_{11}, \\ \tau(w_{12}, w_2, w_5, w_{11}) &= (w_2^3 - \omega w_{11}w_{12}^2 + (1-\omega)w_2w_5w_{12}, \\ &\quad - w_5^3 + \omega^2 w_{11}^2 w_{12} + (1-\omega)w_2w_5w_{11}) \ \textit{on } U_3, \end{aligned}$$

(iii) The central fibre $\tau^{-1}(p)$ with reduced structure is defined by the ideal

$$(f, x_1) \quad \textit{on } U_{12}, \qquad (g, y_8) \quad \textit{on } U_{11},$$
$$(\omega w_{12}w_5 - w_2^2,\ \omega w_{11}w_{12} - w_2w_5,\ w_2w_{11} - w_5^2) \qquad \textit{on } U_3,$$

and the vertex of $\tau^{-1}(p)_{red}$ is the origin of $U_3 \cong \mathbb{A}_k^4$.

(4.2) (Proof of Lemma(1.2)) By Lemma(4.1)(iii) we assume the fibre l is equal to $l = \{\lambda(1, a, \omega^2 a^2, \omega a^3) | \lambda \in k\}$ on $U_3 \cong \mathbb{A}^4$, $(w_{12}, w_2, w_5, w_{11})$, for a constant $a \in k$. Let

$$w_2' = w_2 - a w_{12}, \qquad w_5' = w_5 - \omega^2 a^2 w_{12}, \qquad w_{11}' = w_{11} - \omega a^3 w_{12},$$

Then we see (where $\equiv$ means modulo the ideal $(w_2', w_5', w_{11}')^2$)

$$\begin{aligned} w_2^2 - \omega w_5 w_{12} =& (w_2' + aw_{12})^2 - \omega(w_5' + \omega^2 a^2 w_{12})w_{12} \\ \equiv& (2aw_2' - \omega w_5')w_{12}, \\ w_2 w_5 - w_{12}w_{11} =& (w_2' + aw_{12})(w_5' + \omega^2 a^2 w_{12}) - \omega(w_{11}' + \omega a^3 w_{12})w_{12} \\ \equiv& \omega^2 a^2 w_2' + aw_5' - \omega w_{11}'. \end{aligned}$$

Hence, by Lemma(4.1)(i)(c),

$$\begin{aligned} f \equiv& (w_2' + aw_{12})(2aw_2' - \omega w_5')w_{12} + w_{12}^2(\omega^2 a^2 w_2' + aw_5' - \omega w_{11}') \\ \equiv& w_{12}^2\{(1-\omega)a^2 w_2' + a(1-\omega)w_5' - \omega w_{11}'\} \\ x_1 =& (w_2'^2 - \omega w_5' w_{11}')/w_{12} + (2aw_2' - \omega w_5'), \\ x_2 =& (w_2' + aw_{12})/w_{12}, \qquad x_3 = 1/w_{12}. \end{aligned}$$

Therefore, on the fibre $l = \{w_2' = w_5' = w_{11}' = 0\}$, ${}^t(df, dx_1, d(x_2 - a), dx_3)$ is equal to

$$\begin{pmatrix} -\omega w_{12}^2 & a(1-\omega)w_{12}^2 & (1-\omega)w_{12}^2 & 0 \\ 0 & -\omega & 2a & 0 \\ 0 & 0 & 1/w_{12} & 0 \\ 0 & 0 & 0 & -1/w_{12}^2 \end{pmatrix} \begin{pmatrix} dw_{11}' \\ dw_5' \\ dw_2' \\ dw_{12} \end{pmatrix}.$$

Then $C_{\sigma_1(l)/V} \cong \mathcal{O}(2, 0, -1)$ follows from the above transition matrix of (4.2.1).

(4.3) Lastly we explain briefly the three birational maps (I)-(III) of standard conic bundles (cf.[Sa]) corresponding to those of $\mathbb{P}^2$-bundles treating in this paper. Let $\tau : V \to X$ be a standard conic bundle over a smooth algebraic surface X.

(I) Let p be a singular point of the discriminant locus Δ of V and $\sigma : X_1 \to X$ be the blow up at p. The reduced fibre $l = \tau^{-1}(p)_{red} \subset V$ is isomorphic to $\mathbb{P}^1$ and the conormal bundle $C_{l/V} \cong \mathcal{O}_{\mathbb{P}^1}(2, -1)$. Let $\sigma_1 : V_1 \to V$ be the blow-up along l and let s be the (-3)-curve on the exceptional divisor $E = \mathbb{P}[C_{l/V}] \cong \mathbb{F}_3$. Then $C_{s/V_1} \cong \mathcal{O}(1, 1)$, so $s \subset V_1$ is flopped to $s^+ \subset W$. Now W has a conic bundle structure over X_1 with the non-smooth locus equal to the union of the exceptional divisor e and the proper transform Δ' of Δ. The birational map (I) is factored by

$$V \overset{\sigma_1}{\leftarrow} V_1 \to\leftarrow W. \tag{4.3.1}$$

The flopped curve $s^+ \subset W$ is the closure of the singulsr locus of $\tau^{-1}(e - \Delta')$.

(II) Let p be a smooth point of the discriminat locus Δ of V and $\sigma : X_1 \to X$ be the blow-up at p. The fibre $\tau^{-1}(p) = s \cup m$ is a union of two distinct lines s and m. Let $\sigma_1 : V_1 \to V$ be the blow-up along m and let $s_1 \subset V_1$ be the proper transform of s. Then $\mathcal{C}_{s_1/V_1} \cong \mathcal{O}(1,1)$, so $s_1 \subset V_1$ is flopped to $s^+ \subset W$. Now W has a conic bundle structure over X_1 with the non-smooth locus equal to the proper transform of Δ. The birational map (II) is factored as in (4.3.1). The flopped curve s^+ is a section over the exceptional curve e on X_1 with $\mathcal{C}_{s^+/\tau^{-1}(e)} \cong \mathcal{O}(1)$.

(III) Let $C \subset X$ be a smooth curve intersecting transversely at one smooth point p of Δ. Let $C_0 \subset V$ be a curve which is isomorphic to C by τ. The fibre $\tau^{-1}(p) = s \cup m$ is the union of two lines s and m, and we assume C_0 intersects s. Let $\sigma_1 : V_1 \to V$ be the blow-up along C_0 and let $s_1 \subset V_1$ be the proper transform of s. Then $\mathcal{C}_{s_1/V_1} \cong \mathcal{O}(1,1)$, so $s_1 \subset V_1$ is flopped to $s^+ \subset V_2$. The birational transform $F \subset V_2$ of $\tau^{-1}(C)$ is a $\mathbb{P}^1$-bundle over C and its fibre f is an extremal rational curve. Let $\sigma_2 : V_2 \to W$ be the contraction of f. Then W has a conic bundle structure over X with the same non-smooth locus Δ of V. The birational map (III) is factored by

$$V \overset{\sigma_1}{\leftarrow} V_1 \to\leftarrow V_2 \overset{\sigma_2}{\to} W.$$

If C is isomorphic to $\mathbb{P}^1$ with $\mathcal{C}_{C/X} \cong \mathcal{O}(a)$ and $\mathcal{C}_{C_0/\tau^{-1}(C)} \cong \mathcal{O}(b)$ for integers $a, b \in \mathbb{Z}$, then $\mathcal{C}_{\sigma_2(F)/\tau_1^{-1}(C)} \cong \mathcal{O}(a-b+1)$. In particular, if C is a (-1)-curve with $\mathcal{C}_{C_0/\tau^{-1}(C)} \cong \mathcal{O}(b)$, then $\mathcal{C}_{\sigma_2(F)/\tau_1^{-1}(C)} \cong \mathcal{O}(-b+2)$.

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