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COMPLETE NONCOMMUTATIVE INTEGRATION OF HAMILTONIAN SYSTEMS

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Abstract

We indicate a method of a complete integration of Hamiltonian system *sgrad* F on a symplectic maifold M when dim $\boldsymbol{g}$ + rank $\boldsymbol{g}$ < dim M, where $\boldsymbol{g}$ is a noncommutative Lie algebra on M. Suppose that the reduced system $E(F)$ is completely integrable in the noncommutative sense. This enables us to enlarge the number of the integral functions of the initial system on M.

1. Introduction

Let M be a $2n$-dimensional symplectic manifold with the form ω, and v = *sgrd* F a Hamiltonian field with a Hamiltonian function F on M. Suppose $f_1, f_2, \ldots, f_k$ are smooth functions on M, where $1 \leqq k < 2(n-1)$.
In various concrete situations, Hamiltonian systems possess a set of integral functions $f_1, f_2, \ldots, f_k$ that do not form a commutative Lie algebra, i.e., they are not in involution. Consider the linear space $\boldsymbol{g}$ (over real space R) spanned with functions $\{f_i, 1 \leqq i \leqq k\}$; then dim $\boldsymbol{g} = k$. Suppose that the linear space $\boldsymbol{g}$ is closed with respect to the Poisson bracket , i.e., $\{f_i, f_j\} = \sum_{l=1}^{k} c_l^{ij} f_l$, where the coefficients c_l^{ij} are constant real numbers. Let G be the simply connected Lie group corresponding to $\boldsymbol{g}$. Then $G = \exp \boldsymbol{g}$. Let $f_1, f_2, \ldots, f_k$ be functionally independent (i.e. , gradient systems *grad* f_i ,where $1 \leqq i \leqq k$, are linearly independent).
By the mapping $f_i \mapsto$ *sgrad* f_i ,we identify $\boldsymbol{g}$ with the space spanned with

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$\{ sgrad\ f_i \}$. Hence, G is presented as the group of diffeomorphisms of the manifold that preserves ω. Let $\boldsymbol{g}^*$ be the dual space to $\boldsymbol{g}$, i.e., the space of linear forms on $\boldsymbol{g}$. For a covector $\xi \in \boldsymbol{g}^*$, let M_ξ be the common level surface of functions $\{ f_i \}$, i.e., $M_\xi = \{x \in M,\ f_i(x) = \xi_i,\ 1 \le i \le k\}$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$, $\xi_i \in R$. Let $Ad_g^* : \boldsymbol{g}^* \to \boldsymbol{g}^*$ be the transformation conjugate to $Ad_g : \boldsymbol{g} \to \boldsymbol{g}$, also $ad_f^* : \boldsymbol{g}^* \to \boldsymbol{g}^*$ be the transformation conjugate to $ad_f : \boldsymbol{g} \to \boldsymbol{g}$. Let $\xi \in \boldsymbol{g}^*$ be a covector. Consider the subspace $Ann\xi = \left\{ f \in \mathrm{g} \,;\, ad_f^* \xi = 0 \right\}$ in $\boldsymbol{g}$. The subspace $Ann\xi$ is called the annihilator of the covector ξ. The rank of $\boldsymbol{g}$ is the dimension of the annihilator of a *generic covector.* Let the rank $\boldsymbol{g}$ be r. The Lie group G acting on $\boldsymbol{g}^*$ fibrates $\boldsymbol{g}^*$ into orbits $O^*(\xi) = Adg^*\xi = \{ Ad_f^* \xi \,;\, f \in \boldsymbol{g} \}$ of coadjoint representation. The tangent space $T_\xi O^*(\xi)$ is $\{ad_f^* \xi \,;\, f \in \boldsymbol{g}\}$. Then, $\dim\ T_\xi O^*(\xi) = k - r$. By the property of the alternating matrix $\left[(ad_{f_i}^* \xi)(f_j) \right]$ $(1 \le i, j \le k)$, the number of $k - r$ is even. Therefore $k + r$ is also even. This is important for us. In the next section, we remember the construction that makes it possible to convert the initial Hamiltonian system into a Hamiltonian system on a symplectic manifold of smaller dimension. In the section 3, we describe Proposition 3.4 of our purpose.

2. The Reduction of Hamiltonian systems

Suppose that a Hamiltonian system $v = sgrad\ F$ has the Lie algebra of integrals $\boldsymbol{g}$ whose additive generators are k functionally independent smooth functions $f_1, f_2, \ldots, f_k$, i.e., $\{F, f_i\} = 0$, $i = 1, 2, \ldots, k$. Thus the Hamiltonian system $v = sgrad\, F$ is tangent to the common level surface M_ξ, i.e., F is invariant with respect to $\boldsymbol{g}$. In fact, $(sgrad\, F)(f) = \{f, F\}$ $= 0$, for $f \in \boldsymbol{g}$.

Consider the action of the Lie group G on M. Let $G(x)$ be the orbit of the point $x \in M$, and $H_\xi(x)$ the orbit of x under the action of $H_\xi = \exp Ann\xi$. Then $T_x G(x) = \{sgrad\, f; f \in \mathrm{g}\}$, $T_x H_\xi(x) = \{sgrad\, h; h \in Ann\xi\}$. This implies that $G(x) \cap M_\xi = H_\xi(x)$. Suppose that in a small neighborhood of M_ξ this action has the same type of stationary subgroups, i.e., all orbits of G close to $G(x)$ are diffeomorphic. Consider the projection $p: M \to M/G$ of onto the space of orbits M/G. The space M/G is a smooth manifold of dimension $2n-k$ in a small neighborhood of the point $p(G(x))$. Note that the space M/G must not be symplectic since, for example, it can be odd-dimensional. The projection p, being restricted onto M_ξ, maps it onto $Q_\xi = M_\xi / H_\xi$. Therefore, the space M/G is fibrated onto surfaces Q_ξ. The dimension of the manifold Q_ξ is $2n-k-r$ (even), and Q_ξ has a symplectic structure ρ which is defined by $\omega\big|_{M_\xi} = p^*\rho$, where $p^*\rho$ is a pullback of ρ by p. And then, the following is known. Euler system $E(F)$ $= p_*(sgrad\, F) = sgrad_\rho \hat{F}$, where p_* is a differential map of p, $\hat{F}$ is a smooth function on Q_ξ, and $F\big|_{M_\xi} = p^*\hat{F}$. $(Q_\xi, E(F), \rho)$ is called the reduction of the initial Hamiltonian system $sgradF$. Moreover, the following statement holds ([1]).

Proposition 2.1

Let $\boldsymbol{g}$ be a finite dimensional Lie algebra of integrals of *sgrad F* on *M*. Let *E(F)* be the reduced system on each Q_ξ. Suppose that $\boldsymbol{g}'$ is the linear space of functions on *M/G* such that their restriction onto Q_ξ constitutes the finite dimensional Lie algebra of integrals of the vector field *E(F)*. Then, the space of functions $\boldsymbol{g} \oplus \boldsymbol{g}''$, where $\boldsymbol{g}'' = p^* \boldsymbol{g}'$, is the Lie algebra of integrals of the system *sgrad F*, and $\{\boldsymbol{g}, \boldsymbol{g}''\} = 0$.

This proposition describes that $\boldsymbol{g}''$ is also the Lie algebra. Let *F* and $\boldsymbol{g}''$ satisfy the above conditions. Then $\{F, \boldsymbol{g}''\} = 0$.
In fact, because the linear space $\boldsymbol{g}'$ consists of integrals of *E(F)*, $E(F)(f_i' \big| Q_\xi) = 0$, and let $F = p^* \hat{F}$, where $\hat{F}$ is a function on *M/G*, then

$$\{F, f''\} = \{p^* \hat{F}, p^* f'\} = p^* \{\hat{F}\big|Q_\xi, f'\big|Q_\xi\}_\rho = -p^*((sgrad_\rho \hat{F})(f'\big|Q_\xi))$$

$$= -p^* 0 = 0, \quad f'' \in \boldsymbol{g}''.$$

Furthermore, this proposition include various indication. For example, if $f \in \boldsymbol{g}$, $f'' \in \boldsymbol{g}''$, then f is constant on M_ξ, and f'' is constant on *G(x)*. We remember that $G(x) \cap M_\xi = H_\xi(x)$. Therefore, f, f'' are functionally independent.

We make a remark that the dimension of Q_ξ is *2n-k-r*. If *2n-k-r* = 0, then $H_\xi^r(x) = M_\xi^{2n-k}$. In this case, the Hamiltonian vector field *sgrad F* is completely integrable in the noncommutative sense (dim$\boldsymbol{g}$+rank$\boldsymbol{g}$=dim*M*).
In this paper, we consider the Lie algebra of integrals of *sgrad F* in the case of $2n > k+r$, and suppose that the reduced system is completely

integrable in the noncommutative sense on Q_ξ which dimension is less than $2n$. It may appear that we can treat simply integrals of Hamiltonian system on lower dimensional manifolds .

3. Complete integration

Let η' be a covector on a linear space $\boldsymbol{g}'$ defined on M/G. Uniquely we can define a covector η'' on a linear space $\boldsymbol{g}''$ defined on M such that $\eta''(f'') = (p^*\eta')(p^*f') = \eta'(f')$ by a projection p from M onto M/G, where $f'' = p^*f'$ and $\boldsymbol{g}'' = p^*\boldsymbol{g}'$.

Lemma 3.1

Let η' be a covector on $\boldsymbol{g}'$ and $Ann\,\eta'$ be its annihilator in the Lie algebra $\boldsymbol{g}'$. Similarly, let η'' be a covector on $\boldsymbol{g}''$ and $Ann\,\eta''$ be its annihilator in the Lie algebra $\boldsymbol{g}''$, where $\boldsymbol{g}'' = p^*\boldsymbol{g}'$ and $\eta'' = p^*\eta'$.
Then, $\dim Ann\,\eta'' = \dim Ann\,\eta'$.

Proof. If $f_i' \in Ann\,\eta'$, we indicate that $f_i'' \in Ann\,\eta''$.
Let $f_i' \in Ann\,\eta'$, then $ad_{f_i'}^*\eta' = 0$. For $f' \in \boldsymbol{g}'$,

$$0 = 0(f') = (ad_{f_i'}^*\eta')(f') = \eta'\left\{f_i', f'\right\}. \quad \text{For } f'' \in \boldsymbol{g}'',$$

$$\eta''\left\{f_i'', f''\right\} = (p^*\eta')\left\{p^*f_i', p^*f'\right\} = (p^*\eta')p^*\left\{f_i', f'\right\} = \eta'\left\{f_i', f'\right\} = 0$$

Therefore, $f_i'' \in Ann\,\eta''$. Similarly there is the converse .

Because $\boldsymbol{p}$ is onto, we can ascertain that $\left\{f_i'' \in Ann\eta'' ; 1 \le i \le r'\right\}$ is a basis of $Ann\ \eta''$ if $\left\{f_i' \in Ann\eta' ; 1 \le i \le r'\right\}$ is a basis of $Ann\ \eta'$.

And its converse is similar. Therefore, we can introduce the result of this Lemma. □

Lemma 3.2

Let p be a projection from M to M/G.

If $\boldsymbol{g}'' = p^* \boldsymbol{g}'$, then $\dim \boldsymbol{g}'' = \dim \boldsymbol{g}'$ and $\operatorname{rank} \boldsymbol{g}'' = \operatorname{rank} \boldsymbol{g}'$.

Proof. Since p is a projection, if $\left\{f_i' ; 1 \le i \le k'\right\}$ is a basis of $\boldsymbol{g}'$, then $\left\{f_i'' ; 1 \le i \le k'\right\}$ is a basis of $\boldsymbol{g}''$, and $\dim \boldsymbol{g}'' = \dim \boldsymbol{g}'$.

For the rank of the Lie algebra , let $\operatorname{rank} \boldsymbol{g}' = m$.

Suppose that $\operatorname{rank} \boldsymbol{g}'' = \min\left\{\dim Ann\ \eta'' ; \eta'' \in g''\right\} < m$, then there is a covector η'' such that $\dim Ann\ \eta'' < m$, and so , there is a covector η' such that $\dim Ann\ \eta' < m$ from Lemma 3.1. Therefore ,

$\operatorname{rank} \boldsymbol{g}' = \min\left\{\dim Ann\ \eta' ;\ \eta' \in g'\right\} < m$. It is contradiction. □

Lemma 3.3

Suppose that $\boldsymbol{g}'' = p^* \boldsymbol{g}'$, and $\eta'' = p^* \eta'$, then

$(Ann\xi) \oplus (Ann\eta'') = Ann(\xi \oplus \eta'')$, where $\xi \in \boldsymbol{g}^*$, $\eta'' \in (\boldsymbol{g}'')^*$.

Proof. Let $f + f'' \in (Ann\xi) \oplus (Ann\eta'')$, $\forall g \in \boldsymbol{g}$, $g'' \in \boldsymbol{g}''$.

By the use of Proposition 2.1,

$$(ad_{f+f''}{}^*(\xi \oplus \eta''))(g+g'')$$
$$= (\xi \oplus \eta'')(\{f,g\} + \{f,g''\} + \{f'',g\} + \{f'',g''\})$$
$$= \xi\{f,g\} + \eta''\{f'',g''\} = ad_f{}^*\xi(g) + ad_{f''}{}^*\eta''(g'') = 0$$

$\therefore\ ad_{f+f''}{}^*(\xi \oplus \eta'') = 0$, i.e., $f + f'' \in Ann(\xi \oplus \eta'')$

Similarly, there is the converse. □

Proposition 3.4

Let M^{2n} be a symplectic manifold with the form ω. Let *sgrd F* be a Hamiltonian field on M such that all elements of the Lie algebra $\boldsymbol{g}$ are integrals of *sgrad F*. Suppose that $\boldsymbol{g}'$ is the linear space of functions on *M/G* such that their restriction onto Q_ξ constitutes the finite dimensional Lie algebra of integrals of the vector field *E(F)*. Suppose that dim $\boldsymbol{g}$ + rank $\boldsymbol{g}$ $< 2n$ and the reduced system *E(F)* on Q_ξ is completely integrable in the noncommutative sense on Q_ξ. Then *sgrad F* is completely integrable in the noncommutative sense on M^{2n}.

Proof. We constructed the Lie algebra $\boldsymbol{g} \oplus \boldsymbol{g}''$ by Proposition 2.1, and we could get { F, $\boldsymbol{g} \oplus \boldsymbol{g}''$ } = 0. Therefore it is sufficient that we prove dim($\boldsymbol{g} \oplus \boldsymbol{g}''$) + rank($\boldsymbol{g} \oplus \boldsymbol{g}''$) = $2n$.
Let dim $\boldsymbol{g}' = k'$, rank $\boldsymbol{g}' = r'$, then dim $\boldsymbol{g}'' = k'$, rank $\boldsymbol{g}'' = r'$ from Lemma 3.2. Let $\xi \oplus \eta''$ be the covector of $\boldsymbol{g} \oplus \boldsymbol{g}''$. From Lemma 3.3, rank($\boldsymbol{g} \oplus \boldsymbol{g}''$) = min{dim($Ann(\xi \oplus \eta'')$)} = $r + r'$.
By the assumption of that dim $\boldsymbol{g}'$ + rank $\boldsymbol{g}'$ = dim $Q_\xi = 2n - k - r$, we can get $k' + r' = 2n - k - r$. Therefore dim ($\boldsymbol{g} \oplus \boldsymbol{g}''$) + rank($\boldsymbol{g} \oplus \boldsymbol{g}''$) = $(k + k') + (r + r') = 2n$. Then the new Lie algebra $\boldsymbol{g} \oplus \boldsymbol{g}''$ is complete integration of Hamiltonian system *sgrad F* in the non-commutative sense. □

For the positive integer number r , if Q_ξ is the certain non empty symplectic manifold in Proposition 3.4 and $2n\text{-}k\text{-}r \geqq 2$, then $k < 2(n\text{-}1)$.

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