Application of group theory to proper vibrations in an electric circuit

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Abstract

Group-theoretical analysis is first presented to three-dimensional behavior of an electric circuit. All the modes of proper vibration are found and assigned to each irreducible representation of symmetrical group of the circuit without solving its circuit equations. In order that an electromagnetic radiation from the outside may induce each vibration, a selection rule which is similar to that in infrared absorption must be fulfilled. The circuit may be used as a directive antenna.

1. Introduction

Group theory is used in all fields of science and technology . However, there are few examples of its application to lumped circuits as long as the author knows[1][2]. There might be no research that regards the conversion of a circuit into itself as a group yet. It is considerably strange because electric circuits have a lot of similar properties to mechanical system in which group theory is frequently used and many useful results have been achieved. Thus we present an example of electric circuit which is analogous to such a mechanical system as a molecule or a crystal lattice.

2. An Example analogous to a Mechanical System

Take the circuit in Fig.1 as an example. As is usual in circuit theory, we only take notice about the ideal property of each element and neglect its actual shape, size, or position within its relevant branch. Then the circuit can be regarded to belong to point group D_2 with four symmetry operations (E, C_{2x}, C_{2y}, C_{2z}), where E means identical operation and C_{2x}, C_{2y} and C_{2z} refer to 2-fold rotations about x-, y- , and z-axis respectively. The group has four irreducible representations as in Table 1.

We use loop analysis which is most convenient to our present purpose. State of the circuit is specified by a set of loop currents which satisfy Ohm's law and Kirchhoff's Voltage law. Though the circuit has no electric source, there can be some oscillating states which are realized as a resonance to outer electromagnetic force. According to the well-known theorem, the number of independent loop is b-n+1 where b is the number of branches and n is that of nodes. Since b=6 and n=4, there are three independent loops. They can be obtained as fundamental loops with the ordinary procedure using a tree and cotrees in graph theory[3].

If we start from the tree as drawn with solid lines in Fig.2(a), we get three fundamental loops i_α, i_β, and i_γ as in Fig.2(b) by adding each cotree to the unique tree path successively[3]. Any current flowing in the circuit is a superposition of i_α, i_β, and i_γ. The symmetry operations are written as the following matrices

which act on vector $\begin{pmatrix} i_\alpha \\ i_\beta \\ i_\gamma \end{pmatrix}$.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (1)$$

$$C_{2x} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2)$$

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$$C_{2y} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (3)$$

$$C_{2z} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (4)$$

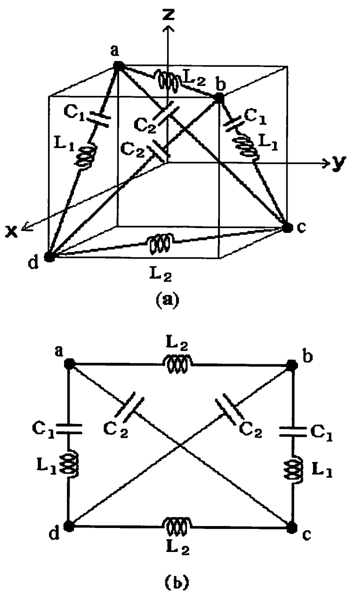

Fig. 1. A circuit belonging to point group D_2. (a) Its three-dimensional view. (b) Its circuit diagram.

Table 1. Caracters of irreducible representations for point group D_2.

	E	C_{2x}	C_{2y}	C_{2z}
A	1	1	1	1
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1

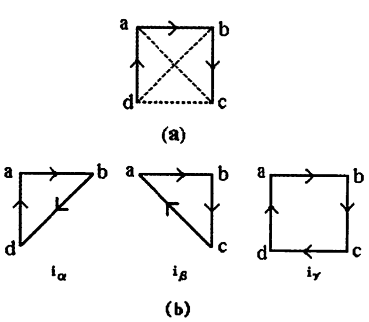

Fig. 2. (a) A tree(solid line) and cotrees(dashed lines). (b) Fundamental circuits made of the tree in (a).

The set of character(the sum of diagonal elements) for each operation (3,-1,-1,-1) can reduce the matrices into irreducible representations by the orthogonal theorem in group theory[4]. If the character of the irreducible representation for each symmetry operation R is given by $\chi(R)$, and the reduction decomposes the arbitrary representation into n_j irreducible representations with character $\chi_j(R)$, then the reduction is uniquely given by

$$n_j = \frac{1}{h}\sum_R \chi_j(R)^* \chi(R), \qquad (5)$$

where R is summed over each symmetry operation and h is the number of the operations. The number of times the operation contains

A is [(1)(3)+(1)(-1)+(1)(-1)+(1)(-1)]/4=0,
B_1, [(1)(3)+(1)(-1)+(-1)(-1)+(-1)(-1)]/4=1,
B_2, [(1)(3)+(-1)(-1)+(1)(-1)+(-1)(-1)]/4=1,
and B_3, [(1)(3)+(-1)(-1)+(-1)(-1)+(1)(-1)]/4=1.

Thus the reduction yields B_1+ B_2+ B_3.

Each basis for its corresponding representation can be obtained by the following transfer projection operators[4].

$$V^{B_1} = E + C_{2x} - C_{2y} - C_{2z}, \qquad (6)$$

$$V^{B_2} = E - C_{2x} + C_{2y} - C_{2z}, \qquad (7)$$

$$V^{B_3} = E - C_{2x} - C_{2y} + C_{2z}. \qquad (8)$$

Applying these operators to i_α, for example, the following results are obtained.

$$V^{B_1}(i_\alpha) = (i_\alpha) + (-i_\alpha + i_\gamma) - (i_\beta - i_\gamma) - (-i_\beta) = 2i_\gamma, \qquad (9)$$

$$V^{B_2}(i_\alpha) = (i_\alpha) - (-i_\alpha + i_\gamma) + (i_\beta - i_\gamma) - (-i_\beta) = 2(i_\alpha + i_\beta - i_\gamma), \qquad (10)$$

$$V^{B_3}(i_\alpha) = (i_\alpha) - (-i_\alpha + i_\gamma) - (i_\beta - i_\gamma) + (-i_\beta) = 2(i_\alpha - i_\beta). \qquad (11)$$

Now we define these currents as i_1, i_2, and i_3, neglecting the meaningless coefficient 2.

$$i_1 = i_\gamma , \qquad (12)$$

$$i_2 = i_\alpha + i_\beta - i_\gamma , \qquad (13)$$

$$i_3 = i_\alpha - i_\beta . \qquad (14)$$

They are shown in Fig.3.

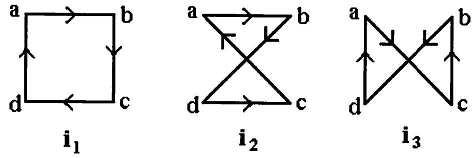

Fig. 3. Loop currents corresponding to irreducible representations.

Since they belong to irreducible representations B_1, B_2 and B_3 respectively, they are independent from each other. The circuit is regarded to be a superposition of the three independent ones in Fig.4. Then we directly obtain the following equations without using any ordinary process in loop analysis.

$$(i\omega(2L_1+2L_2)+\frac{2}{i\omega C_1})i_1=0, \quad (15)$$

$$(i\omega\cdot 2L_2+\frac{2}{i\omega C_2})i_3=0, \quad (16)$$

$$(i\omega\cdot 2L_1+\frac{2(C_1+C_2)}{i\omega C_1 C_2})i_2=0. \quad (17)$$

They give frequencies of their proper vibrations as follows.

$$\omega_1=\frac{1}{\sqrt{(L_1+L_2)C_1}}, \quad (18)$$

$$\omega_2=\frac{1}{\sqrt{L_2C_2}}, \quad (19)$$

$$\omega_3=\frac{1}{\sqrt{\frac{L_1C_1C_2}{C_1+C_2}}}. \quad (20)$$

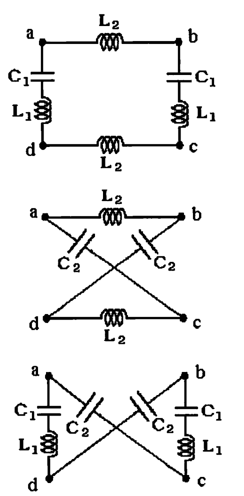

Fig. 4. Three independent circuits corresponding to irreducible representations.

We have got all the results only with group-theoretical method. Such a procedure is very similar to one obtaining normal vibration of a molecule or a crystal lattice. We may say i_1, i_2, and i_3 are normal coordinates of the system. Figure 4 is not a self-evident conclusion. There might not be another method of easily explaining why another loop, for instance, the loop of triangle a-b-d does not cause the vibration of normal mode. Similarity goes further to a selection rule of infrared absorption. The situation is shown in Fig.5. When an alternating magnetic field is applied along x-, y- , or z-axis in Fig.1, current i_1, i_2 or i_3 will be induced respectively. Especially if the frequency coincides with the corresponding one, a large current will flow by resonance. Therefore an electromagnetic wave from the outside will be absorbed if its frequency is appropriate. If the wave is polarized, the absorption will depend on the angle between the wave and the circuit. Such a circuit behaves as an antenna that can receive the three frequencies separately by changing its direction. The selection rule is as follows: The absorption of an electromagnetic wave is allowed if a loop of current transforms as the same irreducible representation of one or more of the Cartesian coordinates[4]. In the present example, it is satisfied by all three coordinate x, y and z, for they belong to irreducible representations B_1, B_2 and B_3 respectively.

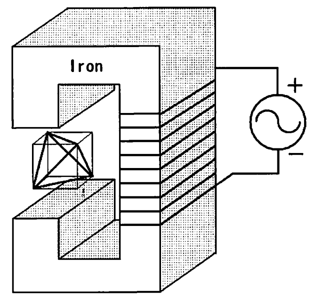

Fig. 5. An equipment to detect resonant electromagnetic oscillations.

3. An Example Peculiar to Networks or Graphs

The above example was analyzed in almost complete imitation of a mechanical system. The symmetry of circuits, however, exceed that of a mechanical system considerably. The symmetrical essence of circuits lie in their connecting structures and not in the spatial configurations. A true symmetrical operation on a circuit is an exchange of nodes which keep the equivalence of the circuit.

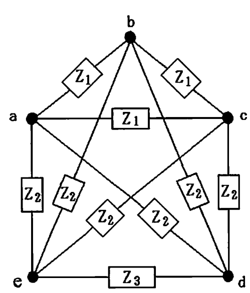

Fig.6. A circuit with 12 symmetry operations.

For example, the circuit in Fig. 6 has no spatial symmetry except a reflection in a vertical plane that contain node b and the center of branch de, but it keeps equivalence for any interchange between node a, b and c, and independently between d and e. The circuit is equivalent under the following 12 permutations which form a group.

$$\begin{aligned} C_1 &= (a)(b)(c)(d)(e), \\ C_2 &= (abc)(de), \\ C_3 &= (acb)(d)(e), \\ C_4 &= (a)(b)(c)(de), \\ C_5 &= (abc)(d)(e), \\ C_6 &= (acb)(de), \\ C_7 &= (ab)(c)(d)(e), \\ C_8 &= (a)(bc)(d)(e), \\ C_9 &= (ac)(b)(d)(e), \\ C_{10} &= (ab)(c)(de), \\ C_{11} &= (a)(bc)(de), \\ C_{12} &= (ac)(b)(de). \end{aligned} \quad (21)$$

Let us call the group G_{12} for the present. Since it is isomorphic or equivalent to point group C_{6v}, its irreducible representations Γ_1 to Γ_6 are readily given with their characters as in Table 2.

Table 2. Characters of irreducible representations for group G_{12}.

	C_1	C_2,C_6	C_3,C_5	C_4	C_7,C_8,C_9	C_{10},C_{11},C_{12}
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	-1	1
Γ_5	2	1	-1	-2	0	0
Γ_6	2	-1	-1	2	0	0

If we choose fundamental circuits as in Fig. 7, each operation on them can be represented in a six-dimensional matrix such as

$$C_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

An analysis based on (5) clarifies that any currents are decomposed into $\Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_6$. Their bases are constituted as follows.

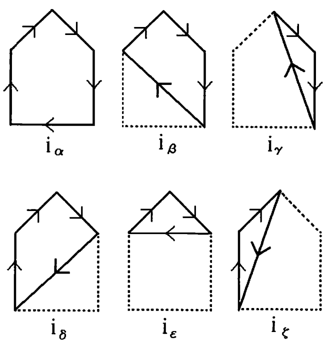

Fig. 7. Fundamental circuits.

$$\Gamma_2: \quad i_1 = i_\varepsilon. \quad (23)$$

$$\Gamma_3: \quad i_2 = -3i_\alpha + i_\beta + i_\gamma + i_\delta + i_\varsigma. \quad (24)$$

$$\Gamma_5: \quad \begin{cases} i_3 = -i_\beta + i_\gamma + i_\varsigma, \\ i_4 = i_\beta + i_\gamma - 2i_\delta + i_\varsigma. \end{cases} \quad (25)$$

$$\Gamma_6: \quad \begin{cases} i_5 = i_\beta - 2i_\gamma - i_\delta + 2i_\varsigma, \\ i_6 = 2i_\beta - i_\gamma + i_\delta - 2i_\varepsilon + i_\varsigma. \end{cases} \quad (26)$$

The resultant currents are shown in Fig. 8. Application of Kirchhoff's Voltage law in each graph gives the condition for a resonant or proper vibration as follows.

$$\begin{aligned} &Z_1 = 0 \quad (\text{for } i_1), \\ &2Z_2 + 3Z_3 = 0 \quad (\text{for } i_2), \\ &Z_2 = 0 \quad (\text{for } i_3 \text{ and } i_4), \\ &3Z_2 + 2Z_3 = 0 \quad (\text{for } i_5 \text{ and } i_6). \end{aligned} \quad (27)$$

If we assume that

$$\begin{aligned} Z_1 &= i(\omega L_1 - \frac{1}{\omega C_1}), \\ Z_2 &= i(\omega L_2 - \frac{1}{\omega C_2}), \\ Z_3 &= i(\omega L_3 - \frac{1}{\omega C_3}). \end{aligned} \quad (28)$$

then the resonant frequencies are given as follows.

$$\begin{aligned} \omega_{\Gamma_2} &= \frac{1}{\sqrt{L_1 C_1}}, \\ \omega_{\Gamma_3} &= \sqrt{\frac{2C_2 + 3C_3}{6C_2 C_3 (2L_2 + 3L_3)}}, \\ \omega_{\Gamma_5} &= \frac{1}{\sqrt{L_2 C_2}}, \\ \omega_{\Gamma_6} &= \sqrt{\frac{3C_2 + 2C_3}{6C_2 C_3 (3L_2 + 2L_3)}}. \end{aligned} \quad (29)$$

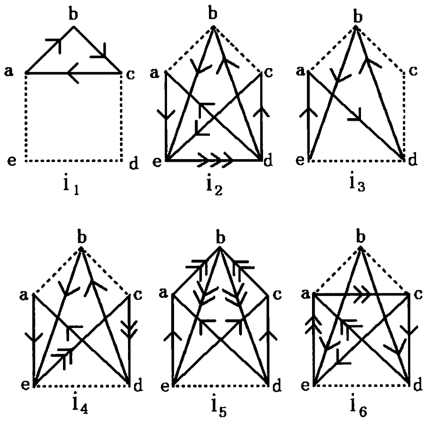

Fig. 8. Independent currents belonging to the irreducible representations.

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