On equivalent class of descending chains of subrings

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ON EQUIVALENT CLASS OF DESCENDING CHAINS OF SUBRINGS

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Abstract

In this note we consider the generalization of vertex of finite groups to non-commutative rings. Let K be a field with characteristic p, and G a finite group. For an indecomposable KG−module V, a p−subgroup P of G is vertex of V if P satisfies the following conditions:

(a) V is P−projective

(b) For any subgroup H of G, if V is H−projective then there exist $t \in G$ such that $P^t \in H$. [1]

1. Definitions

Let Λ be a ring, Ω a class of subrings of Λ with Λ and $\mathcal{C}(\Omega)$ a class of all descending chains $B_1 \supseteq B_2 \supseteq \cdots$ of elements of Ω. For any element B of Ω, We will identify B with $B \supseteq B \supseteq \cdots$, as the element of $\mathcal{C}(\Omega)$. So we can observe that Ω is contained in $\mathcal{C}(\Omega)$.

Let $\mathfrak{B} : B_1 \supseteq B_2 \supseteq \cdots, \mathfrak{B}' : B'_1 \supseteq B'_2 \supseteq \cdots$ be elements of $\mathcal{C}(\Omega)$, we write $\mathfrak{B} \preceq \mathfrak{B}'$ if , for any B'_i, there exist B_j such that $B_j \subseteq B'_i$. If $\mathfrak{B} \preceq \mathfrak{B}'$ then there exists the map ρ of the set of all nutural numbers $\mathbb{N}$ to $\mathbb{N}$, such that if $i < j$ then $\rho(i) < \rho(j)$ and $B_{\rho(i)} \subseteq B'_i$. We remark that $i \leq \rho(i)$. If $\mathfrak{B} \preceq \mathfrak{B}'$ and $\mathfrak{B}' \preceq \mathfrak{B}$, we will write $\mathfrak{B} \sim \mathfrak{B}'$. We can easily check that the relation "$\sim$" is an equivalence relation. For verifying its transitivity, suppose that $\mathfrak{B} \sim \mathfrak{B}', \mathfrak{B}' \sim \mathfrak{B}''$, where $\mathfrak{B} : B_1 \supseteq B_2 \supseteq \cdots, \mathfrak{B}' : B'_1 \supseteq B'_2 \supseteq \cdots$ and $\mathfrak{B}'' : B''_1 \supseteq B''_2 \supseteq \cdots$. Then there exist two maps ρ, ρ' of $\mathbb{N}$ to $\mathbb{N}$ for $\mathfrak{B} \sim \mathfrak{B}', \mathfrak{B}' \sim \mathfrak{B}''$, respevtively. For any i,

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$B_{\rho(\rho'(i))} \subseteq B'_{\rho'(i)} \subseteq B''_i$. So we obtain that $\mathfrak{B} \preceq \mathfrak{B}''$. Similarly, we obtain that $\mathfrak{B}'' \preceq \mathfrak{B}$ and $\mathfrak{B} \sim \mathfrak{B}''$.

Now, we will denote the set of eqivalent classes of the reration "$\sim$" by $\tilde{C}(\Omega)$, and the equivalent class contain $\mathfrak{B}$ by $[\mathfrak{B}]$. Since distinct elements of Ω are contianed dinstinct eqivalent classes in $\tilde{C}(\Omega)$, we can observe that Ω is contained in $\tilde{C}(\Omega)$. If $\mathfrak{B}_1 \sim \mathfrak{B}'_1, \mathfrak{B}_2 \sim \mathfrak{B}'_2$ and $\mathfrak{B}_1 \preceq \mathfrak{B}_2$, then $\mathfrak{B}'_1 \preceq \mathfrak{B}'_2$. So we can difine the relation "$\leq$" in $\tilde{C}(\Omega)$ by usual way.

Lemma 1. *For any $[\mathfrak{B}]$ in $\tilde{C}(\Omega)$, $\tilde{C}(\Omega)$ has a minimal element $[\mathfrak{B}_0]$ such that $[\mathfrak{B}_0] \leq [\mathfrak{B}]$.*

Proof. We consider a descending chain of elements of $\tilde{C}(\Omega)$

$$[\mathfrak{B}] \geq [\mathfrak{B}_1] \geq [\mathfrak{B}_2] \geq \cdots,$$

where $\mathfrak{B}_i : B_{i,1} \supseteq B_{i,2} \supseteq \cdots$. Let ρ_i be the map of $\mathbb{N}$ to $\mathbb{N}$ for $\mathfrak{B}_i \succeq \mathfrak{B}_{i+1}$. Set $\sigma_i = \rho_i \rho_{i-1} \cdots \rho_1$, and

$$\mathfrak{B}' : B_{1,1} \supseteq B_{2,\sigma_1(2)} \supseteq \cdots \supseteq B_{i,\sigma_{i-1}(i)} \supseteq \cdots.$$

For any i and j, since $j \leq \sigma_{i-1}(j)$, $B_{i,j} \supseteq B_{i,\sigma_{i-1}(j)} \supseteq B_{k,\sigma_{k-1}(k)}$, where $k = max\{i,j\}$. So $\mathfrak{B}_i \succeq \mathfrak{B}'$ and $[\mathfrak{B}']$ is a lower bound of $[\mathfrak{B}] \geq [\mathfrak{B}_1] \geq [\mathfrak{B}_2] \geq \cdots$. By Zorn's Lemma, $\tilde{C}(\Omega)$ has minimal elements $[\mathfrak{B}_0]$ such that $[\mathfrak{B}_0] \leq [\mathfrak{B}]$. □

2. Relative Projective

Let M be a left Λ−module and B a subring of Λ. M is (Λ, B)−projective if the multiplication map of $\Lambda \otimes_B M$ to M splits as a left Λ−homomorphism. For any $\mathfrak{B} : B_1 \supseteq B_2 \supseteq \cdots$ in $C(\Omega)$, we will difine that M is $(\Lambda, \mathfrak{B})$−projective if M is (Λ, B_i)−projective, for all i. If B' is a intermediate ring between B and Λ, then the multiplication map of $\Lambda \otimes_B M$ to M can be factored through $\Lambda \otimes_{B'} M$. So, if the multiplication map of $\Lambda \otimes_B M$ to M splits then the multiplication map of $\Lambda \otimes_{B'} M$ to M splits. Therefore we obtain the following lemma:

Lemma 2. *Let $\mathfrak{B}, \mathfrak{B}'$ be elements of $C(\Omega)$ such that $\mathfrak{B} \sim \mathfrak{B}'$, then M is $(\Lambda, \mathfrak{B})$−projective if and only if M is $(\Lambda, \mathfrak{B}')$−projective*

By above lemma, for any elment $[\mathfrak{B}]$ in $\tilde{C}(\Omega)$, we can define $(\Lambda, [\mathfrak{B}])-$projectivity. We will denote the set of all elements $[\mathfrak{B}]$ in $\tilde{C}(\Omega)$ such that M is $(\Lambda, [\mathfrak{B}])-$projective, by $\tilde{C}(\Omega)_M$.

Proposition 1. *If* $[\mathfrak{B}]$ *is an element in* $\tilde{C}(\Omega)_M$ *then* $\tilde{C}(\Omega)_M$ *has a minimal element* $[\mathfrak{B}_0]$, *such that* $[\mathfrak{B}_0] \leq [\mathfrak{B}]$.

Proof. Let Ω_M be the set of all elements B of Ω such that M is $(\Lambda, B)-$Projective. Then $\tilde{C}(\Omega)_M = \tilde{C}(\Omega_M)$ and $[\mathfrak{B}] \in \tilde{C}(\Omega_M)$. By Lemma 1, we can obtain the proposion . □

We denote the set of all minimal elements of $\tilde{C}(\Omega)_M$ by $mrp_\Omega(M)$. If $[\mathfrak{B}]$ is a element of $mrp_\Omega(M)$ such that $\mathfrak{B} \not\sim \cap\mathfrak{B}$, then M is not $(\Lambda, \cap\mathfrak{B})-$projective, by minimality of $[\mathfrak{B}]$.

For an element λ of Λ and an unit μ of Λ, we will denote $\mu^{-1}\lambda\mu$ by λ^μ. For a subring B of Λ, we will denote the subring of all elements b^μ where b is element of B, by B^μ. And for an element $\mathfrak{B} : B_1 \supseteq B_2 \supseteq \cdots$ in $C(\Omega)$, we will denote $B_1^\mu \supseteq B_2^\mu \supseteq \cdots$ by $\mathfrak{B}^\mu$. Let G be a group of units of Λ. We will say that G acts on Ω if B^μ is in Ω, for any B in Ω and any μ in G, and similarly we will define that the action of G on $mrp_\Omega(M)$.

Proposition 2. *Let* G *be a group of units of* Λ. *If* G *acts on* Ω *then* G *acts on* $mrp_\Omega(M)$.

Proof. Let B be in Ω, φ the map of $\Lambda \otimes_B M$ to $\Lambda \otimes_{B^\mu} M$ defined by $\lambda \otimes m \mapsto \lambda\mu \otimes \mu^{-1}m$, and ψ the map of $\Lambda \otimes_{B^\mu} M$ to $\Lambda \otimes_B M$ defined by $\lambda \otimes m \mapsto \lambda\mu^{-1} \otimes \mu m$, where $\lambda \in \Lambda, m \in M$, and $\mu \in G$. For any $b \in B$,

$$\lambda b\mu \otimes \mu^{-1}m = \lambda\mu(\mu^{-1}b\mu) \otimes \mu^{-1}m = \lambda\mu \otimes (\mu^{-1}b\mu)\mu^{-1}m = \lambda\mu \otimes \mu^{-1}bm$$

in $\Lambda \otimes_{B^\mu} M$, and

$$\lambda(\mu^{-1}b\mu)\mu^{-1} \otimes \mu m = \lambda\mu^{-1}b \otimes \mu m = \lambda\mu^{-1} \otimes b\mu m = \lambda\mu^{-1} \otimes \mu(\mu^{-1}b\mu)m$$

in $\Lambda \otimes_B M$. So φ and ψ are well-defined left $\Lambda-$homomorphism. Since $\varphi\psi$ and $\psi\varphi$ are the identity maps, $\Lambda \otimes_B M$ is isomorphic to $\Lambda \otimes_{B^\mu} M$ as left $\Lambda-$modules. Therefore M is $(\Lambda, B)-$projective if and only if M is $(\Lambda, B^\mu)-$projective.

Let $[\mathfrak{B}]$ be in $mrp_\Omega(M)$. By the above sequence, M is $(\Lambda, \mathfrak{B}^\mu)$−projective. By Proposition 1, there exist $[\mathfrak{B}_0]$ in $mrp_\Omega(M)$ such that $[\mathfrak{B}_0] \leq [\mathfrak{B}^\mu]$. Let ρ be the map of $\mathbb{N}$ to $\mathbb{N}$ for $[\mathfrak{B}_0] \leq [\mathfrak{B}^\mu]$. Since $B_{0,\rho(i)} \subseteq B_i^\mu$, $B_{0,\rho(i)}^{\mu^{-1}} \subseteq B_i$. So $[\mathfrak{B}_0^{\mu^{-1}}] \leq [\mathfrak{B}]$. Since M is $(\Lambda, \mathfrak{B}_0)$−projective, M is $(\Lambda, \mathfrak{B}_0^{\mu^{-1}})$−projective, and $[\mathfrak{B}_0^{\mu^{-1}}] \in \tilde{\mathcal{C}}(\Omega)_M$. For minimality of $[\mathfrak{B}]$, we obtain that $[\mathfrak{B}] = [\mathfrak{B}_0^{\mu^{-1}}]$ and $[\mathfrak{B}^\mu] = [\mathfrak{B}_0] \in mrp_\Omega(M)$. □

Let K be a field with characteristic p, G a finite group, and V an indecomposable KG−module. Set $\Omega = \{KH | H \text{ is subgroup of } G\}$. Since Ω is finite set,

$$\mathcal{C}(\Omega) = \tilde{\mathcal{C}}(\Omega) = \Omega,$$

and

$$mrp_\Omega(M) = \{KH | H = \mathrm{vx}(V)^t, \ t \in G\},$$

where $\mathrm{vx}(V)$ is a vertex of V.

References

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