STATISTICAL AND PROBABILISTIC MODELS FOR STOCK PRICES: A REVIEW

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1 Introduction

In 1973 Black, Scholes and Merton made an exciting breakthrough in the pricing of financial options based on a model which has been now called the Black-Scholes model. Among other assumptions, this model assumes that the stock prices $\{S_t,\ t \geq 0\}$ follow a geometric Brownian motion which is given by the following stochastic differential equation (SDE):

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad S_0 > 0,$$

where $\{B_t,\ t \geq 0\}$ is a standard Brownian motion, $\mu \in \mathbb{R}$ is the drift parameter and $\sigma > 0$ the diffusion parameter (volatility). Undoubtedly, the Black-Scholes model has contributed enormously to the real financial world and has had huge influence on the way that traders price and hedge options.

However, recent empirical studies show that the theoretical prices of options do not fit the market prices very well, which indicates that the Black-Scholes model has some imperfections. The imperfections are due to several aspects, but the main reason is that the geometric Brownian motion assumed for the stock prices seems to be not appropriate: It can not fit the statistical properties of many real financial time series well. In fact it has been found that the log returns do not follow a normal distribution, and volatilities change stochastically over time.

In order to improve away imperfections of the Black-Sholes model, many alternative models have been proposed for stock prices since 1980s. The ARCH model proposed by Engle (1982) has lead to the developments of a wide category of conditional heteroscedastic models and some non-linear time series models. Recently, some Lévy processes with non-Gaussian semi-heavy-tailed distributions have been proposed for modeling stock prices. Clearly, whether or not we can make rational pricing for options depends on whether or not we can find a reasonable model for our stock prices. In this note we provide a review of recent developments in modeling of stock prices. We try to compare properties of these models and point out merits and demerits of the models. By doing so we expect more flexible and realistic models

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to be developed for stock prices in future studies. Our main reference materials are Schoutens (2003), Tsay (2002), Shiryaev (1999) and references therein.

In Section 2 we summarize some significant statistical characteristics of stock prices via an empirical study of the daily log returns of IBM stock. In Section 3 we introduce some distributions that are suitable for stock prices. In Sections 5 and 6 we introduce time series models and continuous-time stochastic processes, respectively, which are relevant to, or suitable for stock prices. In Section 6 we briefly mention recent developments in pricing of options under a Lévy market. Finally, we give some concluding remarks in Section 7.

2 Statistical properties of stock prices

Let $\{S_t,\ t = 0, 1, 2, \ldots\}$ be the time series of stock prices. We can write S_t as

$$S_t = S_0 e^{X_t}, \quad t = 1, 2, \ldots$$

where $S_0 > 0$ and $X_t = \log(S_t/S_0)$ is the t-period log return (log price) of the stock. Now we can write X_t as

$$X_t = r_1 + \cdots + r_t, \quad t = 1, 2, \ldots$$

where $r_t = X_t - X_{t-1} = \log(S_t/S_{t-1})$ is the one-period log return of the stock at period t. To describe the behavior of the stock price $\{S_t\}$, it is enough to do so for its one-period log return series $\{r_t\}$. As an example, we consider the time series of daily log returns of IBM stock from July 3, 1962 to December 31, 1997 (obtained from R.S.Tsay), which is plotted in Figure 1. By a discussion of this series, we present some significant statistical properties of stock prices.

■ Distributional characteristics

For the series $\{r_t\}$ of daily log returns of IBM stock, we show the sample mean, median, standard deviation, skewness and kurtosis in Table 1. We see that the distribution has large kurtosis and negative skewness. In Figure 2 we show the histogram, box-plot, estimated density function and QQ-plot. These show that the distribution of the daily log returns of IBM stock is asymmetric with heavy tails, and is non-Gaussian.

Table 1: Descriptive statistics for daily log returns of IBM stock.

mean	median	stan.devi.	skewness	kurtosis
0.0393	0.0000	1.4808	−0.3329	18.2075

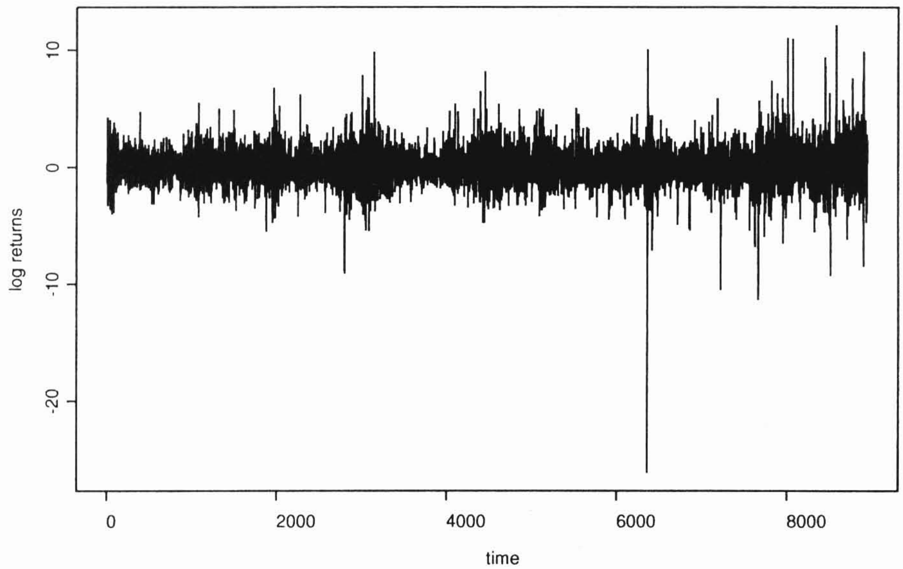

Figure 1: Time plots of daily log-returns of IBM stock from July 3, 1962 to December 31, 1997 (8938 business days).

■ Serial correlations

Figure 3 shows the sample autocorrelation functions (ACF) of the daily log returns $\{r_t\}$, squared log returns $\{r^2\}$ and absolute log returns $\{|r_t|\}$. We can observe that there are weakly serial correlations at lags 2, 5 and 20 for the daily log returns. We further observe that there are relatively large serial correlations in $\{r^2\}$ and $\{|r_t|\}$, and in particular, the slowly decaying pattern in ACF of $\{|r_t|\}$ shows some long-memory properties. Since the sample mean of the daily log returns of IBM stock is 0.0393, we can write

$$r_t = 0.0393 + a_t,$$

where $Ea_t = 0$. The above empical results show that, even if we might ignore serial correlations in $\{r_t\}$, we should not consider $\{a_t\}$ as an independent white noise.

■ Volatility clusters

Looking at Figure 1, we observe that the log returns possess a tendency that there are periods with low variation and periods with high variation. This phenomenon is called volatility-clustering effect. Also, we can see that variations seem to become large in recent years. These indicate that volatility changes stochastically over time.

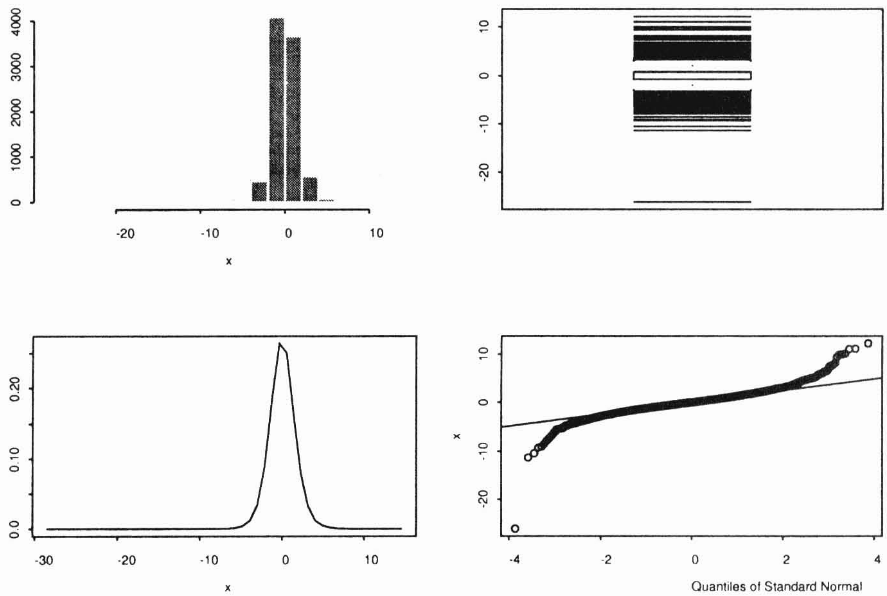

Figure 2: Distributional characteristics of IBM stock: histogram, box-plot, estimated density and QQ-plot.

3 Distributions for stock prices

So far, many empirical studies suggest that the normal distribution is not suitable for log returns of stock prices, and that distributions with heavy or semi-heavy tails should be used. In this section we introduce some distributions that are useful for log returns of stock prices.

Let $f(x)$ be a probability density function. We say that $f(x)$ has heavy right-hand tail if

$$f(x) \sim |x|^{-\alpha} L(x) \quad (x \to +\infty),$$

where $\alpha > 0$ and $L(x)$ is a slowly varying function at $+\infty$, i.e. $L(\lambda x)/L(x) \to 1$ as $x \to +\infty$ for each $\lambda > 0$. And $f(x)$ has semi-heavy right-hand tail if

$$f(x) \sim \text{const.} |x|^{\rho} e^{-\eta |x|} \quad (x \to +\infty),$$

for some $\rho \in \mathbb{R}$ and $\eta > 0$. Heavy and semi-heavy left-hand tail is defined similarly.

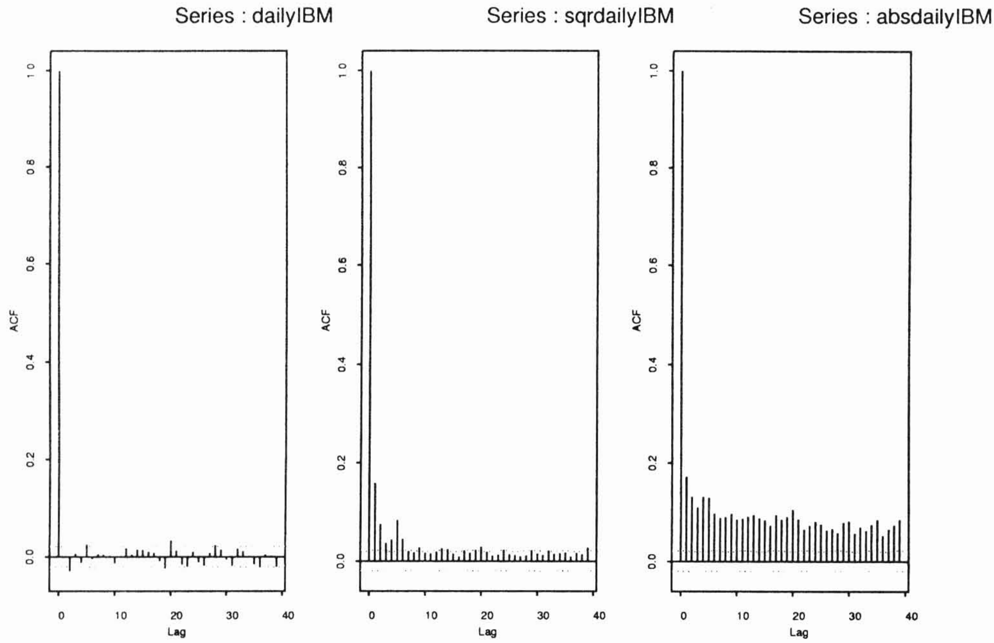

Figure 3: Sample autocorrelation functions of daily log returns of IBM stock, the absolute log returns and squared log returns (from left to right).

■ α-stable distributions

As a non-Gaussian distribution, the α-stable distributions can be thought to be natural generalizations of a normal distribution, since the later is a symmetric α-stable distribution with $\alpha = 2$. The distribution of a random variable X is said to be an α-stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$ such that its characteristic function is the following form

$$E\exp(i\theta X) = \begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha(1 - i\beta(\text{sign}\,\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(\text{sign}\,\theta)\ln|\theta|) + i\mu\theta\} & \text{if } \alpha = 1, \end{cases}$$

where $\theta \in \mathbb{R}$ and sign $= 1, 0, -1$ according to $\theta > 0$, $\theta = 0$ and $\theta < 0$. The parameter α is called the index of stability of the stable distribution, σ the scale parameter, β the skewness parameter and μ the shift parameter. We write this distribution as $S_\alpha(\sigma,\ \beta,\ \mu)$. When $\beta = 0$ and $\mu = 0$, the distribution $S_\alpha(\sigma, 0, 0)$ is called symmetric α-stable (SαS). An SαS rv X has a characteristic function of the simple form $E\exp(i\theta X) = e^{-\sigma^\alpha|\theta|^\alpha}$. When $\alpha = 2$, $S_2(\sigma, 0,\ \mu)$ is the normal distribution $N(\mu, 2\sigma^2)$; when $\alpha = 1$, $S_1(\sigma, 0,\ \mu)$ is a Cauchy distribution; when $\alpha = 1/2$, $S_{1/2}(\sigma, 1, 0)$ is the Lévy-Smirnov distribution. Except for these special cases, the probability densities of stable distributions are not known in explicit form. This makes it difficult to estimate parameters by using maximum likelihood estimation (MLE) method for a time series model in which the marginal distribution is α-stable.

If $X \sim S_\alpha(\sigma,\ \beta,\ \mu)$, $0 < \alpha < 2$, then X has the tail probabilities

$$\begin{cases} \lim_{\lambda\to\infty} \lambda^\alpha P\{X > \lambda\} = C_\alpha \frac{1+\beta}{2}\sigma^\alpha, \\ \lim_{\lambda\to\infty} \lambda^\alpha P\{X < -\lambda\} = C_\alpha \frac{1-\beta}{2}\sigma^\alpha, \end{cases}$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ 2/\pi & \text{if } \alpha = 1. \end{cases}$$

Thus stable distributions have heavy tails. Note that $E|X|^r = \int_0^\infty P\{|X|^r > \lambda\}d\lambda$. From the tail behavior of stable distributions, we see that if $X \sim S_\alpha(\sigma,\ \beta,\ \mu)$, $0 < \alpha < 2$, then $E|X|^\alpha = +\infty$ and $E|X|^r < +\infty$ for $0 \leq r < \alpha$. The fact that α-stable distributions with $\alpha < 2$ have infinite second moment means that many of the techniques that are valid for the Gaussian or finite-second-moment case do not apply in the α-stable case. When $\alpha \leq 1$, one also has $E|X| = +\infty$, precluding the use of expectations. These are the main source of difficulties for dealing with the α-stable distributions.

Stable distributions have been popularly used in finance literature in 1960s and 1970s (see Fama (1965); Mandelbrot and Taylor (1967)). However, presently most common opinions seem to be that using these heavy-tailed distributions with infinite variance for log returns of stock price is in conflict with econometrics and finance theories.

■ Student's t-distributions

This well-known distribution has a density given by

$$f(x;n) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \quad x \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the Gamma function and $n \in \mathbb{N}$ is the number of degrees of freedom of the distribution. A t-distribution is symmetric and heavy-tailed. Both t-distributions and α-stable distributions have Pareto-type tails, namely

$$P\{X > x\} \sim \text{const.}x^{-\beta} \quad (x \to +\infty)$$

for some $\beta > 0$. Since a t-distribution involves only one parameter, it may be not flexible enough for modeling stock prices.

■ Mixture of normal distributions

There are two kinds of mixture of normal distributions: finite-mixture and scale-mixture. For example, the following

$$r_t \sim (1-\alpha)N(\mu, \sigma_1^2) + \alpha N(\mu, \sigma_2^2)$$

is a mixture of two normal distrbutions $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, where $0 < \alpha < 1$. While a scale-mixture of normal distributions is

$$\mathcal{L}(r_t|\sigma) = N(\mu, \sigma^2),$$

where $\mathcal{L}(r_t|\sigma)$ is the conditional law of r_t given σ, and σ is itself a positive random variable following some distribution, e.g. a Gamma law.

Mixtures of normal distributions can capture the excess kurtosis. Yet it may be hard to to estimate the mixture parameter α in the finite-mixture case, and it is not easy to find a suitable distribution for σ in the scale-mixture case.

■ Infinitely divisible distributions

α-stable distributions and t-distributions metioned in the above are both typical infinitely divisible laws with heavy tails. Recent statistical studies suggest that the distributions of log returns of stock prices tend to have semi-heavy tails. So far, several semi-heavy-tailed infinitely divisible laws have been proposed in the literature. These include:

The Variance Gamma (VG) distributions;
The Normal Inverse Gaussian (NIG) distributions;
The CGMY distributions;
The Hyperbolic (HYP) distributions;
The Generalized Hyperbolic (GH) distributions.

These distributions will be introduced in Section 5, where we consider Lévy processes whose marginal laws are precisely these semi-heavy-tailed infinitely divisible laws.

4 Time series models for stock prices

Here we introduce some time series models that are useful for financial time series analysis. These include linear and nonlinear time series models. In the following we use $\{r_t\}$ to denote the one-period log returns of our stock price $\{S_t\}$, that is,

$$S_t = S_0 e^{X_t}, \quad X_t = r_1 + \cdots + r_t, \quad r_t = X_t - X_{t-1} = \log(S_t/S_{t-1}).$$

4.1 Linear time series models

A time series $\{r_t\}$ is said to be linear if it can be written as

$$r_t = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i},$$

where μ, $\psi_i \in \mathbb{R}$, $\psi_0 = 1$ are constants, and $\{a_t\}$ is a sequence of independent and identically distributed (iid) random variables with mean 0 and finite variance σ^2. Such a series $\{a_t\}$ is called white noise, which is denoted by $\{a_t\} \sim \text{iid}(0, \sigma^2)$.

The main aim of linear time series analysis is to fit a suitable model by analyzing the serial correlation structure, and then use the model to forecast the future values.

■ ARMA models

A time series $\{r_t\}$ is said to follow an autoregressive moving-average (ARMA(p,q)) model if it satisfies

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + a_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}, \quad \{a_t\} \sim \text{iid}(0, \sigma^2),$$

where ϕ_i and θ_j are constants, and $p, q \in \{0, 1, \dots\}$ are called the orders of the ARMA models. When $q = 0$, the ARMA(p, 0) model is called an AR(p) model, and when $p = 0$, the ARMA(0, q) model is called an MA(q) model.

Let B be the backward shift opeator: $Br_t = r_{t-1}$. Then an ARMA(p, q) model can be expressed as

$$\phi(B) = \phi_0 + \theta(B)a_t, \tag{4.1}$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$, which are called the AR polynomial and MA polynomial, respectively. If all of the zeroes of $\phi(z) = 0$ lie outside the unit circle, then $\{r_t\}$ is stationary, and if further all of the zeroes of $\theta(z) = 0$ lie outside the unit circle, then the above model is called a stationary and causal ARMA model. It is assumed that the AR and MA polynomials do not have common zeroes.

For a stationary time series $\{r_t\}$, we define the autocorrelation function (ACF) at lag k as

$$\rho_k = \frac{Cov(r_t,\ r_{t-k})}{Var(r_t)}, \quad k = 1, 2, \dots$$

A time series following an ARMA model has short-memory in the sense that its ACF $\{\rho_k\}$ decreases to 0 at an exponential rate. In finance literature, one often uses ARMA models to handle serial correlations. However, since the volatility of a time series following a stationary ARMA model is a constant, that is,

$$Var(r_t | r_{t-1}, \dots, r_1) = Var(a_t) = \sigma^2,$$

ARMA models are useless for describing stochastic volatility of financial time series.

■ ARIMA models

A time series $\{r_t\}$ is said to follow an autoregressive integrated moving-average (ARIMA(p, d, q)) model if

$$\phi(B)(1 - B)^d r_t = \phi_0 + \theta(B)a_t, \quad \{a_t\} \sim \text{iid}(0,\ \sigma^2), \tag{4.2}$$

where d is a non-negative integer, $\phi(B)$ is an AR polynomial of order p and $\theta(B)$ MA polynomial of order q, respectively, as given in the ARMA model (4.1). d is called the order of difference of the ARIMA model. If $d > 0$, then $\{r_t\}$ is non-stationary, but the differenced series $(1-B)^d r_t$ is stationary and follows an ARMA(p, q) model. If $p = q = 0$ and $d = 1$, then the avobe model is a random walk with drift:

$$r_t = \psi_0 + r_{t-1} + a_t, \quad t = 1, 2, \ldots$$

Although ARIMA models have extensive uses in macroeconomics and also in finance literature, it seems that log returns of many stocks/indexes are stationary.

■ ARFIMA models

In the ARIMA(p, d, q) model, if we assume that the value of the order d satisfies $d \in (-0.5,\ 0.5)$, then we obtain the so called autoregressive fractionally integrated moving-average (ARFIMA(p, d, q)) model. In this case the meaning of the operator $(1-B)^d$, $d \in (-0.5,\ 0.5)$, must be considered in the sense of binomial expansion.

In comparison with an ARIMA model, an ARFIMA model is stationary. It can be shown that the ACF of an ARFIMA(p, d, q) model satisfies

$$\rho_k \sim \text{const.} k^{2d-1} \quad (k \to \infty); \qquad \sum_{k=1}^{\infty} |\rho_k| = +\infty \quad \text{if } d \in (0,\ 0.5).$$

This is called the long-memory property, and an ARFIMA(p, d, q) model with $d \in (0,\ 0.5)$ has long memory. In comparison with a stationary short-memory model, e.g. an ARMA model, the rate of decaying of the ACF of an ARFIMA model is rather slow.

If we look at the ACF of the absolute daily log returns of IBM stock shown in Figure 3, we find that the ACF is not large in magnitude, but it decays very slowly. This indicates that the volatility of the daily log returns of IBM stock may have long-memory. For more informations about the behaviors of ACF of absolute returns, see Ding, Granger and Engle (1993).

■ Fractional Gaussian noise

Let $\{B_H(t),\ t \geq 0\}$ be a fractional Brownian motion with Hurst self-similarity exponent $H \in (0, 1)$ (See Section 5.4), and define the time series $\{b_H(t),\ t \in \mathbb{N}\}$ by

$$b_H(t) = B_H(t) - B_H(t-1), \quad t = 1, 2, \ldots$$

Then it can be shown that the ACF of $\{b_H(t)\}$ satisfies

$$\rho_H(k) = \frac{1}{2}\left\{(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right\}, \quad k = 1, 2, \ldots$$

and

$$\rho_H(k) \sim H(2H-1)k^{2H-2} \quad (k \to \infty).$$

$\{b_H(t)\}$ is called fractional Gaussian noise. Thus, if $H = 1/2$, then $\rho_H(k) = 0$ for $k \neq 0$ and in this case $\{b_{\frac{1}{2}}(t)\}$ is a Gaussian white noise. For $H \in (0,\ 1/2)$, we have

$$\rho_H(k) < 0, \ k = 1, 2, \ldots \quad \text{and} \quad \sum_{k=0}^{\infty} |\rho_H(k)| < \infty,$$

and for $H \in (1/2,\ 1)$,

$$\rho_H(k) > 0,\ k = 1, 2, \dots \quad \text{and} \quad \sum_{k=0}^{\infty} \rho_H(k) = +\infty.$$

Thus, for $H \in (0,\ 1/2)$, $\{b_H(t)\}$ has negative serial correlations, and for $H \in (1/2,\ 1)$, $\{b_H(t)\}$ has positive serial correlations and long memory.

As was pointed out in the above, volatilities of log returns of many stocks seem to have long memory with positive serial correlations. So, a fractional Gaussian noise $b_H(t)$, with $H \in (1/2,\ 1)$, might be a useful model for volatility $\{\log \sigma_t\}$. Here we note that, in some studies it is observed that $\{\log \frac{\sigma_t}{\sigma_{t-1}}\}$ appears to have negative serial correlations (Shiryaev (1999), P.234). The reason may be that, in fact $\{\log \sigma_t\}$ is a fractional Gaussian noise with $H \in (1/2,\ 1)$, and over-differencing $\{\log \sigma_t\}$ to $\{\log \sigma_t - \log \sigma_{t-1}\}$ makes $\{\log \frac{\sigma_t}{\sigma_{t-1}}\}$ a fractional Gaussian noise with $H \in (0,\ 1/2)$.

4.2 Conditional heteroscedastic time series models

Let $\{r_t = \log(S_t/S_{t-1})\}$ be the log returns of a stock with prices $\{S_t\}$. We write

$$r_t = \mu_t + a_t,$$

where

$$\mu_t = E(r_t|\mathcal{F}_{t-1}), \quad \mathcal{F}_{t-1} = \sigma(r_1, \dots, r_{t-1}),$$

where $\sigma(r_1, \dots, r_{t-1})$ denotes the σ-field generated by $r_1, \dots, r_{t-1}$. So μ_t is the conditional mean of r_t given $r_1, \dots, r_{t-1}$. Obviously, μ_t is the one-step ahead forecast of r_t, and a_t is the forecast error. It is easy to see that $\{a_t\}$ is a martingale difference. Note that the volatility of $\{r_t\}$ is given by

$$\sigma^2 = Var(r_t|\mathcal{F}_{t-1}) = E((r_t - \mu_t)^2|\mathcal{F}_{t-1}) = E(a_t^2|\mathcal{F}_{t-1}).$$

In 1982, Engle proposed the ARCH model. Since then, many conditional heteroscedastic models have been introduced. Basically, these models assume that $\{r_t\}$ can be written formally as a linear time series model, for example, a stationary ARMA$(p,\ q)$ model:

$$r_t = \mu_t + a_t = (\phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}) + a_t.$$

However, here $\{a_t\}$ is assumed to be uncorrelated, but not independent. So $\{r_t\}$ is in general not a linear time series. Conditional heteroscedastic time series models are concerned with behavior of $\{a_t\}$ by specifying the dynamic structure of volatility $\{\sigma_t^2\}$. The manner under which σ_t^2 evoles over time distinguishes one model from another. These models are reviewed in this section.

■ ARCH models

The autoregressive conditional hereroscedastic (ARCH) model assumes that, the mean-corrected log return $a_t = r_t - \mu_t$ is serially uncorrelated, but dependent. The dependence of $\{a_t\}$ is described by a dynamic structure of its volatility $\{\sigma^2\}$. Precisely, an ARCH(m) model assumes that

$$\begin{aligned} a_t &= \sigma_t \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{iid}(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \end{aligned}$$

where $\alpha_0 > 0$ and $\alpha_i > 0$. In addition, the coefficients need to satisfy some other regularity conditions if $\{a_t\}$ is required to be second-order or fourth-order stationary. The distribution of $\{\varepsilon_t\}$ is often assumed to be a standard normal distribution $N(0,1)$ or a standardized Student's t-distribution with degree n (often $n = 3 \sim 6$ is prespecified).

If we assume that $\{r_t\}$ follows in the form an ARMA(p, q) model with $\{a_t\}$ following an ARCH(m) model, we say that $\{r_t\}$ follows an ARMA(p, q)–ARCH(m) model.

Let $\eta_t = a_t^2 - \sigma_t^2$. Since $\sigma_t^2 = E(\sigma_t^2|\mathcal{F}_{t-1})$, $\{\eta_t\}$ is a martingale difference. The ARCH model can written as

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + \eta_t,$$

which is formally an AR(m) model for $\{a_t^2\}$. However, $\{\eta_t\}$ is not an iid white noise.

Here we mention some properties of ARCH models. It is easy to see that $\{a_t\}$ is serially uncorrelated. However, since $\{a_t^2\}$ follows a form of an AR(m) model, it is serially correlated. As we have observed in the daily log returns of IBM stock, this phenomenon has been often observed in finance indexes.

From the volatility equation in the ARCH model, we can see that the volatility σ_t^2 becomes large when the past shocks $\{a_{t-i}^2\}_{i=1}^m$ are large, and so the present shock a_t tends to take a large value in modulus. The similar is for the case of small values. So ARCH models are suitable for describing volatility-clusters phenomenon. However, ARCH models have some weaknesses. The most serious one is that they allow positive and negative shocks to have the same impects on volatility, which is in conflict with finance theories.

■ GAHCH models

Generalized ARCH (GARCH) models were proposed by Bollerslev (1986) in the same spirit of ARMA models. As before, let $a_t = r_t - \mu_t$ be the mean-corrected log return. Then $\{a_t\}$ follows a GARCH(p, q) model if

$$\begin{aligned} a_t &= \sigma_t \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{iid}(0,1), \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \end{aligned}$$

where α_0, α_i and β_j are nonnegative, and $\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1$.

A GARCH(p, q) model can be put in the form of ARMA model. Let $\eta_t = a_t^2 - \sigma_t^2$. Then $\{\eta_t\}$ is a martingale difference, and we have

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}. \tag{4.3}$$

So $\{a_t^2\}$ follows formally an ARMA model. As a generalization of ARCH model, GARCH models possess similar properties as those of ARCH models.

When the AR polynomial of the GARCH model (4.3) has a unit root, we obtain the so-called integrated GARCH (IGARCH) model, which is similar to an ARIMA model. For example, an IGARCH$(1,1)$ model is

$$a_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + (1 - \beta_1) a_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

A key feature of IGACH models is that the impact of shock η_{t-i}, $i > 0$, on a_t^2 is persistent.

In some cases, the returns of a stock may depend on its volatility. For example,

$$r_t = \mu + c\sigma_t^2 + a_t,$$

where $\{a_t\}$ follows a GARCH model and c is the risk premium parameter. Such models are called GARCH-M models.

ARCH and GARCH models have very extensive applications in modeling of log returns of stocks, and also other financial indexes.

■ EGARCH models

This model aims at overcoming some weaknesses of GARCH models by allowing positive and negative shocks to have asymmetric effects on volatility. Nelson (1991) proposed the following exponential GARCH (EGARCH) model:

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \quad \{\varepsilon_t\} \sim \mathrm{iid}(0,1), \\ \log \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \left[\theta \varepsilon_{t-i} + \gamma(|\varepsilon_{t-i}| - E|\varepsilon_{t-i}|)\right] + \sum_{j=1}^{q} \beta_j \log \sigma_{t-j}^2, \end{aligned}$$

where the coefficients satisfy some conditions. Note that if $\{\varepsilon_t\} \sim \mathrm{iid}N(0,1)$, then $E|\varepsilon_t| = \sqrt{\frac{2}{\pi}}$.

■ CHARMA models

Tsay (1987) proposed the following coditional heteroscedastic ARMA (CHARMA) model:

$$r_t = \mu_t + a_t, \quad a_t = \delta_{1t} a_{t-1} + \cdots + \delta_{mt} a_{t-m} + \eta_t,$$

where $\{\eta_t\} \sim \mathrm{iid}N(0, \sigma_\eta^2)$, $\{\delta_t\} = \{(\delta_{1t}, \ldots, \delta_{mt})'\}$ is a sequence of iid random vectors with mean 0 and nonnegative definite covariance matrix Ω, and $\{\delta_t\}$ is independent of $\{\eta_t\}$. It can be shown that, under such a model, the volatility $\sigma_t^2 = E(a_t^2|\mathcal{F}_{t-1})$ satisfies

$$\sigma_t^2 = \sigma_\eta^2 + (a_{t-1}, \dots, a_{t-m})\Omega(a_{t-1}, \dots, a_{t-m})'.$$

One of the main features of CHARMA models is that it allows the presence of cross-products of the lagged shocks of a_t in the volatility equation. Tsay (1987) argued that the cross-product terms might be useful in some applications.

■ RCA models

Nicholls and Quinn (1982) introduced the random coefficient autoregressive (RCA) model to account for variability among different subjects and to obtain a better description of the conditional mean equation of the time series by allowing the parameters to evolve with time. This model is as follows:

$$r_t = \phi_0 + \sum_{i=1}^{p}(\phi_i + \delta_{it})r_{t-i} + a_t,$$

where ϕ_is are constants, $\{\delta_t\} = \{(\delta_{1t}, \dots, \delta_{pt})'\}$ is a sequence of iid random vectors with mean 0 and nonnegative definite covariance matrix Ω_δ, $\{a_t\} \sim \text{iid}(0, \sigma_a^2)$, and $\{\delta_t\}$ is independent of $\{a_t\}$. Under this model, the conditional mean and variance (volatility) of $\{r_t\}$ are given by

$$\begin{aligned} \mu_t &= E(r_t|\mathcal{F}_{t-1}) = \phi_0 + \sum_{i=1}^{p}\phi_i r_{t-i}, \\ \sigma_t^2 &= \sigma_a^2 + (r_{t-1}, \dots, r_{t-p})\Omega_\delta(r_{t-1}, \dots, r_{t-p})'. \end{aligned}$$

Note that σ_t^2 is a quadratic form of $r_{t-1}, \dots, r_{t-p}$.

■ SV models

A stochastic volatility (SV) model is of the following form:

$$\begin{aligned} &r_t = \mu_t + a_t,\ a_t = \sigma_t\varepsilon_t,\ \{\varepsilon_t\} \sim \text{iid}N(0,1), \\ &(1 - \alpha_1 B - \cdots - \alpha_m B^m)\log\sigma_t^2 = \alpha_0 + \nu_t, \quad \{\nu_t\} \sim \text{iid}N(0, \sigma_\nu^2), \end{aligned}$$

where $\{\varepsilon_t\}$ is independent of $\{\nu_t\}$, and the AR polynomial is assumed to be stationary. Properties of SV models can be found in Taylor (1994), Shiryaev (1999) and the references therein.

■ Long-memory SV models

These models can be written in the following form:

$$\begin{aligned} &r_t = \mu_t + a_t,\ a_t = \sigma_t\varepsilon_t,\ \{\varepsilon_t\} \sim \text{iid}N(0,1), \\ &\sigma_t^2 = \sigma^2 e^{u_t},\ \phi(B)(1-B)^d u_t = \theta(B)\eta_t,\ \{\eta_t\} \sim \text{iid}N(0, \sigma_\eta^2), \end{aligned}$$

where the AR and MA polynomials are assumed to satisfy usual conditions of an ARFIMA model, $d \in (0, 0.5)$, $\sigma > 0$, and $\{\varepsilon_t\}$ is independent of $\{\eta_t\}$.

Applications of long-memory SV models in finance literature can be found in Bollerslev and Jubinski (1999) and Ray and Tsay (2000).

4.3 Nonlinear time series models with nonlinear conditional mean

The conditional heteroscedastic time series models introduced in the above section are essentially nonlinear in the conditional variance σ_t^2, but linear in the conditional mean μ_t. There are also some nonlinear time series models that let the conditional mean μ_t evole over time according to some simple parametric nonlinear structures. Combined with other time series models, these models may be useful for modeling stock prices. These models are reviewed in the following.

■ Bilinear models

These models were introduced by Granger and Anderson (1978), which can be written as follows:

$$r_t = c + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j a_{t-j} + \sum_{i=1}^{m}\sum_{j=1}^{s} \beta_{ij} r_{t-i} a_{t-j} + a_t,$$

where $\{a_t\} \sim \text{iid}(0, \sigma_a^2)$, p, q, m, s are nonnegative integers. Statistical properties and applications of these models can be found in Subba Rao and Gabr (1984).

■ SETAR-GARCH models

A time series $\{x_t\}$ is said to follow a k-regime self-exciting threshold AR (SETAR) model with threshold variable x_{t-d} if the jth regime satisfies

$$x_t = \phi_0^{(j)} + \phi_1^{(j)} x_{t-1} + \cdots + \phi_p^{(j)} x_{t-p} + a_t^{(j)}, \qquad \text{if } \gamma_{j-1} \le x_{t-d} < \gamma_j,$$

for $j = 1, 2, \ldots, k$, where k and d are positive integers, the coefficients $\phi_i^{(j)}$s are real constants, the thresholds γ_js are real constants such that $-\infty = \gamma_0 < \gamma_1 < \cdots < \gamma_{k-1} < \gamma_k = +\infty$, and $\{a_t^{(j)}\} \sim \text{iid}(0, \sigma_j^2)$ are mutually independent for different j. d is called the delay parameter. A SETAR model is a piecewise linear AR model in the threshold space.

In the finance literature, SETAR models are useful for modeling of valatility. Let $\{r_t\}$ be the log returns of a stock. For example, the following model are often used:

$$r_t = \mu_t + a_t, \quad a_t = \sigma_t \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{iid}(0,1),$$

$$\{\sigma_t^2\} \sim \begin{cases} \text{GARCH}(p_1, q_1) & \text{if } a_{t-1} \le 0, \\ \text{GARCH}(p_2, q_2) & \text{if } a_{t-1} > 0. \end{cases}$$

Such a model is called SEATR-GARCH model. In comparison with GARCH models, SETAR-GARCH models are able to capture the asymmetric effects of positive and negative shocks a_{t-1}. Some other forms of threshold volatility models have also been proposed in the literature (See Rabemananjara and Zakoian (1993); Zakoian (1994)).

■ Markov switching models

Hamilton (1989) introduced the following Markov switching AR model, where the transition is driven by a hidden two-state Markov chain $\{s_t\}$:

$$x_t = \begin{cases} c_1 + \sum_{i=1}^{p_1} \phi_{1,i} x_{t-i} + a_{1t} & \text{if } s_t = 1, \\ c_2 + \sum_{i=1}^{p_2} \phi_{2,i} x_{t-i} + a_{2t} & \text{if } s_t = 2, \end{cases}$$

where $\{a_{it}\} \sim \text{iid}(0, \sigma_i^2)$ are assumed to be independent for $i = 1, 2$, and $\{s_t\}$ is a Markov chain with two states $\{1, 2\}$, and transition probabilities given by

$$P(s_t = 2 | s_{t-1} = 1) = w_1, \quad P(s_t = 1 | s_{t-1} = 2) = w_2.$$

Combined with GARCH models, Markov switching models are useful in modeling of financial indexes. For example, we can consider the following model:

$$r_t = \begin{cases} \mu_1 + \beta_1 \sigma_t + \sigma_t \varepsilon_t, & \{\sigma_t^2\} \sim \text{GARCH}(p_1, q_1) & \text{if } s_t = 1, \\ \mu_2 + \beta_2 \sigma_t + \sigma_t \varepsilon_t, & \{\sigma_t^2\} \sim \text{GARCH}(p_2, q_2) & \text{if } s_t = 2, \end{cases}$$

where $a_t = \sigma_t \varepsilon_t$, $\{\varepsilon_t\} \sim \text{iid}N(0, 1)$ and $\{s_t\}$ is a hidden Markov chain with two states $\{1, 2\}$ as described in the above. This is a Markov switching model with GARCH-M type dynamics. Markov switching models with other type of conditional heteroscedastic structures can also be considered. We note that statistical inferences of Markov switching models are very involved. Recent developments in Markov Chain Monte Carlo (MCMC) method enable us to make statistical inferences for these models (see Tsay (2002)).

5 Continuous-time stochastic processes for stock prices

In the following we introduce some continuous-time stochastic processes that are suitable for, or relevant to stock prices. These include some diffusion processes, Lévy processes, stochastically time-changed Lévy processes and self-similar processes.

Let $\{S_t,\ t \geq 0\}$ denote the stock price. We write

$$S_t = S_0 e^{X_t}, \quad S_0 > 0.$$

So $\{X_t,\ t \geq 0\}$ is the log price of the stock. In the following, the models are specified ether for $\{S_t\}$ or for $\{X_t\}$.

5.1 Diffusion models

■ Brownian motion with drift

This is given by

$$S_t = S_0 + \mu t + \sigma B_t,$$

where $\{B_t,\ t \geq 0\}$ is a standard Brownian motion, $\mu \in \mathbb{R}$ is the drift parameter and $\sigma > 0$ the diffusion (volatility) parameter, which are assumed to be constants. Writing in the form of SDE, it becomes

$$dS_t = \mu dt + \sigma dB_t.$$

This model was originally developed by Bachelier (1990) for modeling stock prices. Note that the discrete-time version of a Browinan motion with drift is a random walk with drift. These models had been widely accepted and used as models of stock prices until 1950s. However, this model has many defficiencies. For example, under such a model, the stock price $\{S_t\}$ may take negative values, which can not take place in practice .

■ Geometric Brownian motion

Samuelson (1965) proposed a model for log stock prices as follows:

$$\log \frac{S_t}{S_0} = (\mu - \frac{\sigma^2}{2})t + \sigma B_t,$$

or in another form

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t},$$

where $\{B_t\}$ is a standard Brownian motion. It can be written in the form of SDE as

$$dS_t = S_t(\mu dt + \sigma dB_t).$$

This diffusion model is called a geometric Brownian motion.

Let $r_t = \log(S_t/S_{t-1})$ be the log return of $\{S_t\}$. Then we have

$$r_t = (\mu - \frac{\sigma^2}{2}) + \sigma\varepsilon_t =: m + \sigma\varepsilon_t,$$

where $\{\varepsilon_t\} \sim \text{iid}N(0, 1)$. Under a geometric Brownian motion, we have

$$r_t \sim \text{iid}N(m, \sigma^2).$$

However, as was discussed in Section 2, many empirical studies show that, in general, the time series $\{r_t\}$ is neither independent nor Gaussian. Common opinions now seem to be that geometric Brownian motions are not suitable for describing behaviors of stock prices.

■ Diffusion models with time-varying drift and volatility

Another imperfection of geometric Brownian motions is that volatility is assumed to be unchanged, which is in conflict with empirical evidences. To overcome these weaknesses of geometric Brownian motions, a natural idea is to let the drift and volatility evolve with time. This leads to the following model:

$$dS_t = S_t(\mu(t)dt + \sigma(t)dB_t),$$

where $\{B_t\}$ is a standard Brownian motion, and $\mu(t)$, $\sigma(t)$ are deterministic fucntions of time satisfying certain conditions. In this case, the solution of the SDE is given by

$$S_t = S_0 e^{X_t},$$

where

$$X_t = \int_0^t \left(\mu(s) - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s) dB_s.$$

It may be more reasonable to make the function $\sigma(t)$ change stochastically over time. For example, Dupire (1993, 1994) introduced the following model:

$$dS_t = S_t(\mu(t)dt + \sigma(S_t, t)dB_t).$$

In financial applications, these models would be useless unless the functional forms of $\mu(t)$ and $\sigma(S_t, t)$ could be specified suitably, which is not easy at all. In fact, different forms of the functions correspond to different models for stock price.

Another way for improving away those weaknesses of geometric Brownian motions is to make the volatility change stochastically according to a diffusion model, and this leads to the following model (see Hull and White (1988)):

$$\begin{aligned} dS_t &= S_t(\mu(t, S_t, \sigma_t)dt + \sigma_t dB_t^{(1)}), \\ d\nu_t &= a(t, \nu_t)dt + b(t, \nu_t)dB_t^{(2)}, \end{aligned}$$

where $\nu_t = \log \sigma_t^2$, $\{B_t^{(1)}\}$ and $\{B_t^{(2)}\}$ are two independent standard Brownian motions, and the drift and diffusion terms in the equations are assumed to satisfy certain measurability and integrability conditions. Under such a model, we arrive at an incomplete financial market.

5.2 Lévy processes for stock prices

The non-Gaussian characteristics of the distributions of log returns of stocks leads to considerations of using non-Gaussian processes for modeling of stock prices. Since Lévy processes are the most natural generalization of Brownian motions, and their marginal distributions involve a wide class of infinitely divisible laws that are non-Gaussian, Lévy processes are thought to be more flexible for modeling stock prices. Recently, many Lévy processes with distributions that seem to match the empirical distributions of log returns have been proposed. These are introduced in this section.

At first, we introduce some basic results on Lévy processes. Assume that we have a filtered probability space $(\Omega, \{\mathcal{F}_t, t \geq 0\}, \mathcal{F}, P)$ satisfying the usual conditions. Let $X = \{X_t, t \geq 0\}$ be a $\mathbb{R}$-valued stochastic process adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, with $X_0 = 0$ (a.s.). Then X is a Lévy process if

(i) X has independent increments; that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$; and

(ii) X has stationary increments; that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$; and

(iii) X is continuous in probability; that is, $X_s \xrightarrow{P} X_t$ as $s \to t$ for every $t \geq 0$.

Every Lévy process has a unique càdlàg modification which is itself a Lévy process.

It can be shown that marginal distributions of Lévy processes are infinitely divisible. Inversely, for each infinitely divisible distribution μ there exists a Lévy process X such that μ is the distribution of X_1.

Let $\phi_t(u) = Ee^{iuX_t}$ be the characteristic function of X_t. It can be shown that, there exists a continuous function $\psi(u)$ with $\psi(0) = 0$ such that

$$\phi_t(u) = e^{t\psi(u)}$$

for every $t > 0$ and $u \in \mathbb{R}$. Note that $\phi_1(u) = e^{\psi(u)}$. Thus marginal distributions of a Lévy process are completely determined by the marginal distribution of X_1. The function $\psi(u)$ is called the characteristic exponent of the Lévy process $X = \{X_t,\ t \geq 0\}$, which satisfies the following Lévy-Khintchine formula:

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}})\nu(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ such that

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty.$$

Therefore, a Lévy process is completely determined by the triplet $[\gamma,\ \sigma^2,\ \nu(dx)]$, which is often called Lévy triplet. The measure ν is called the Lévy measure, and if $\nu(dx) = u(x)dx$ for some function $u(x)$, then $u(x)$ is called the Lévy density. The Lévy density must have zero mass at the origin, but does not need to be integrable.

The Lévy-Khintchine formula shows that, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian motion part and a pure jump part. If X has no Brownian motion part ($\sigma^2 = 0$), we say that the Lévy process is of pure jump. The Lévy measure $\nu(dx)$ dictates how the jumps occur: jumps of sizes in a set A occur according to a Poisson process with intensity $\int_A \nu(dx)$.

5.2.1 Lévy models with constant volatility

Now we introduce some Lévy processes that are proposed recently for modeling stock prices. Let $\{S_t\}$ be the stock price. Write

$$S_t = S_0 e^{X_t}.$$

Recall that in the case of a geometric Brownian motion, we have

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t,\ X_0 = 0,$$

which is a Brownian motion with drift. In the following Lévy models the log stock price $\{X_t\}$ is assumed to be some Lévy processes whose marginal distributions are some special infinitely divisible laws having semi-heavy tails and excess kurtosis.

■ VG Lévy processes

$\{X_t,\ t \geq 0\}$ is a Variance Gamma (VG) Lévy process if it is a Lévy process such that the distribution of X_1 is a VG distribution $\mathrm{VG}(\sigma,\ \nu,\ \theta)$, whose characteristic function (ch.f) $\phi_{\mathrm{VG}}(u;\ \sigma,\ \nu,\ \theta) = Ee^{iuX_1}$ is

$$\phi_{\mathrm{VG}}(u;\ \sigma,\ \nu,\ \theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu}.$$

This distribution is infinitely divisible. However, its density is not known explicitly. Its Lévy triplet is $[\gamma,\ 0,\ \nu_{\mathrm{VG}}(dx)]$, where

$$\gamma = \frac{-C(G(e^{-M} - 1) - M(e^{-G} - 1))}{MG},$$

$$\nu_{\mathrm{VG}}(dx) = \begin{cases} Ce^{Gx}|x|^{-1}dx, & x < 0, \\ Ce^{-Mx}x^{-1}dx, & x > 0, \end{cases}$$

where

$$\begin{aligned} C &= 1/\nu > 0, \\ G &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} > 0, \\ M &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} > 0. \end{aligned}$$

A VG Lévy process has no Browian motion part.

If the distribution $\mathrm{VG}(\sigma,\ \nu,\ \theta)$ is reparameterized in terms of C, G and M, then the ch.f of $\mathrm{VG}(C,\ G, M)$ is

$$\phi_{\mathrm{VG}}(u;\ C,\ G,\ M) = \left(\frac{GM}{GM + (M - C)iu + u^2}\right)^C.$$

A VG Lévy process is a Gamma time-changed Brownian motion with drift. Here we introduce the Gamma distributions and Gamma Lévy processes. The Gamma distribution $\mathrm{Gamma}(a,\ b)$ has a density of the form

$$f_{\mathrm{Gamma}}(x;\ a,\ b) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}1_{\{x>0\}},$$

where $a,\ b > 0$. A Gamma distribution $\mathrm{Gamma}(a,\ b)$ is infinitely divisible with Lévy triplet $[\gamma,\ 0,\ \nu(dx)]$ given by

$$\gamma = a(1 - e^{-b})/b, \quad \nu(dx) = ae^{-bx}x^{-1}1_{\{x>0\}}dx.$$

A Gamma distribution is also self-decomposable, where the self-decomposability is defined as follows: A distribution with ch.f $\phi(u)$ is called to be self-decomposable if

$$\phi(u) = \phi(cu)\phi_c(u), \quad \forall u \in \mathbb{R}, \ \forall c \in (0,1),$$

for some family of characteristic functions $\{\phi_c(u) : \ c \in (0,\ 1)\}$. In this case we also say that the corresponding distribution belongs to Lévy's class L, which is a subclass of infinitely divisible distributions. For details of self-decomposability, see Sato (1999).

A Gamma Lévy process $\{G_t,\ t \geq 0\}$ is a Lévy process whose marginal distributions are Gamma laws. Let $X^{(\mathrm{VG})} = \{X_t^{(\mathrm{VG})},\ t \geq 0\}$ be a VG Lévy process such that $X_1^{(\mathrm{VG})} \sim \mathrm{VG}(\sigma,\ \nu,\ \theta)$. Also let $\{G_t,\ t \geq 0\}$ be a Gamma Lévy process such that $G_1 \sim \mathrm{Gamma}(a,\ b)$ with $a = b = 1/\nu$. Then it can be shown that, there exists a stadard Brownian motion such that

$$X_t^{(\mathrm{VG})} = \theta G_t + \sigma B_{G_t}.$$

VG Lévy processes were introduced and studied by Madan and Seneta (1987, 1990). Applications of VG Lévy processes in finance can be found in Madan and Miline (1991) and Madan, Carr and Chang (1998).

■ NIG Lévy processes

$\{X_t,\ t \geq 0\}$ is a Normal Inverse Gaussian (NIG) Lévy process if it is a Lévy process such that the distribution of X_1 is a NIG distribution $\mathrm{NIG}(\alpha,\ \beta,\ \delta)$, where $\alpha > 0,\ -\alpha < \beta < \alpha$ and $\delta > 0$, whose characteristic function (ch.f) $\phi_{\mathrm{NIG}}(u;\ \alpha,\ \beta,\ \delta) = Ee^{iuX_1}$ is

$$\phi_{\mathrm{NIG}}(u;\ \alpha,\ \beta,\ \delta) = \exp\left(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2})\right).$$

This distribution is infinitely divisible. Its Lévy triplet is $[\gamma,\ 0,\ \nu_{\mathrm{NIG}}(dx)]$, where

$$\gamma = \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) dx,$$

where $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $K_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ (See Schoutens (2003); Abramowitz and Stegun (1968)). The Lévy measure is given by

$$\nu_{\mathrm{NIG}}(dx) = \frac{\delta\alpha}{\pi} \frac{e^{\beta x} K_1(\alpha|x|)}{|x|} dx.$$

A NIG Lévy process has no Brownian motion part.

A NIG distribution $\mathrm{NIG}(\alpha,\ \beta,\ \delta)$ is infinitely divisible. Its density is given by

$$f_{\mathrm{NIG}}(x;\ \alpha,\ \beta,\ \delta) = \frac{\delta\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta x\right) \frac{K_1(\alpha\sqrt{\delta^2 + x^2})}{\sqrt{\delta^2 + x^2}}.$$

This distribution has semi-heavy tails:

$$f_{\mathrm{NIG}}(x;\ \alpha,\ \beta,\ \delta) \sim \text{const.}|x|^{-3/2}e^{(\mp\alpha+\beta)x} \quad \text{as } x \to \pm\infty.$$

A NIG Lévy process is an Inverse Gaussian time-changed Brownian motion. A Inverse Gaussian (IG) distribution IG$(a,\ b)$ has a ch.f of the form

$$\phi_{\mathrm{IG}}(u;\ a,\ b) = \exp\left(-a(\sqrt{b^2 - 2iu} - b)\right),$$

where $a,\ b > 0$. This distribution is self-decomposable with Lévy triplet $[\gamma,\ 0,\ \nu_{\mathrm{IG}}(dx)]$ given by

$$\gamma = \frac{a}{b}(2N(b) - 1),$$

where $N(\cdot)$ is the distribution function of $N(0,\ 1)$, and

$$\nu_{\mathrm{IG}}(dx) = (2\pi)^{-1/2}ax^{-3/2}\exp(-\frac{1}{2}b^2x)1_{\{x>0\}}dx.$$

An IG Lévy process $\{I_t,\ t \geq 0\}$ is a Lévy process such that I_1 follows an IG$(a,\ b)$ law.

Let $X^{(\mathrm{NIG})} = \{X_t^{(\mathrm{NIG})},\ t \geq 0\}$ be a NIG Lévy process such that $X_1^{(\mathrm{NIG})} \sim$ NIG$(\alpha,\ \beta,\ \delta)$. Then it can be shown that

$$X_t^{(\mathrm{NIG})} = \beta\delta^2 I_t + \delta B_{I_t},$$

where $\{B_t\}$ is a standard Brownian motion, and $\{I_t\}$ is an IG Lévy process such that $I_1 \sim$ IG$(a,\ b)$ with $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$.

The NIG distributions and Lévy processes were introduced by Barndorff-Nielsen (1995). Their applications in finance can be found in Barndorff-Nielsen (1995, 1997, 1998); Barndorff-Nielsen and Shephard (2001).

■ HYP/GH Lévy processes

The GH Lévy process is generated by the Generalized Hyperbolic (GH) distribution GH$(\alpha,\ \beta,\ \delta,\ \nu)$, whose ch.f is

$$\phi_{\mathrm{GH}}(u;\ \alpha,\ \beta,\ \delta,\ \nu) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\nu/2} \frac{K_\nu(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{K_\nu(\delta\sqrt{\alpha^2 - \beta^2})},$$

where $K_\nu(\cdot)$ is the modified Bessel function of the third kind with index ν. Its density is given by

$$f_{\mathrm{GH}}(x;\ \alpha,\ \beta,\ \delta,\ \nu) = a(\alpha,\ \beta,\ \delta,\ \nu)(\delta^2 + x^2)^{(\nu-1/2)/2}K_{\nu-1/2}(\alpha\sqrt{\delta^2 + x^2})e^{\beta x},$$

where

$$a(\alpha,\ \beta,\ \delta,\ \nu) = \frac{(\alpha^2 - \beta^2)^{\nu/2}}{\sqrt{2\pi}\alpha^{\nu-1/2}\delta^\nu K_\nu(\delta\sqrt{\alpha^2 - \beta^2})},$$

where $\delta > 0$, $|\beta| < \alpha$ and $\nu \in \mathbb{R}$. The Generalized Hyperbolic (GH) distribution $\mathrm{GH}(\alpha,\ \beta,\ \delta,\ \nu)$ is infinitely divisible. Its Lévy measure is rather involved. The $\mathrm{GH}(\alpha,\ \beta,\ \delta,\ \nu)$ has semi-heavy tails:

$$f_{\mathrm{GH}}(x;\ \alpha,\ \beta,\ \delta,\ \nu) \sim \text{const.}|x|^{\nu-1}e^{(\mp\alpha+\beta)x} \quad \text{as } x \to \pm\infty.$$

When $\nu = 1$, the $\mathrm{GH}(\alpha,\ \beta,\ \delta,\ 1)$ distribution is called a Hyperbolic distribution $\mathrm{HYP}(\alpha,\ \beta,\ \delta)$, which was introduced by Eberlein and Keller (1995). The GH distribution was introduced by Eberlein and Prause (1998). Also, the GH distributions include the VG and NIG distributions as special cases. The $\mathrm{VG}(\sigma,\ v,\ \theta)$ law can be obtained from the $\mathrm{GH}(\alpha,\ \beta,\ \delta,\ \nu)$ law by letting $\nu = \sigma^2/v$, $\alpha = \sqrt{(2/v) + (\theta^2/\sigma^4)}$, $\beta = \theta/\sigma^2$ and $\delta \to 0$. For the NIG laws, it holds that

$$\mathrm{NIG}(\alpha,\ \beta,\ \delta) = \mathrm{GH}(\alpha,\ \beta,\ \delta,\ -1/2).$$

Financial applications of GH Lévy models can be found in Eberlein and Keller (1995), Eberlein and Prause (1998), Eberlein, Keller and Prause (1998) and Eberlein and Hammerstein (2002).

■ Meixner Lévy processes

The Meixner Lévy process was introduced by Schoutens and Teugels (1998). This Lévy process is generated by the Meixner distribution $\mathrm{Meixner}(\alpha,\ \beta,\ \delta)$, whose ch.f is

$$\phi_{\mathrm{Meixner}}(u;\ \alpha,\ \beta,\ \delta) = \left(\frac{\cos(\beta/2)}{\cosh((\alpha u - i\beta)/2)}\right)^{2\delta}.$$

This distribution is infinitely divisible and has semi-heavy tails. Its Lévy triplet is $[\gamma,\ 0,\ \nu(dx)]$, where

$$\gamma = \alpha\delta\tan(\beta/2) - 2\delta\int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}dx, \quad \nu(dx) = \delta\frac{\exp(\beta x/\alpha)}{x\sinh(\pi x/\alpha)}dx.$$

The Meixner Lévy process has no Brownian motion part.

Financial applications of the Meixner Lévy processes can be found in Schoutens (2000, 2001, 2002).

■ CGMY Lévy processes

The CGMY Lévy process was introduced by Carr, Geman, Madan and Yor (2002, 2003). It is generated by the CGMY distribution $\mathrm{CGMY}(C,\ G,\ M,\ Y)$ with ch.f given by

$$\phi_{\mathrm{CGMY}}(u;\ C,\ G,\ M,\ Y) = \exp(C\Gamma(-Y)((M-iu)^Y - M^Y + (G+iu)^Y - G^Y)),$$

where $C,\ G,\ M > 0$ and $Y \in (-\infty,\ 2)$. It is infinitely divisible with Lévy triplet $[\gamma,\ 0,\ \nu_{\mathrm{CGMY}}(dx)]$, where

$$\gamma = C\left(\int_0^1 e^{-Mx}x^{-Y}dx - \int_{-1}^0 e^{Gx}|x|^{-Y}dx\right),$$

$$\nu_{\mathrm{CGMY}}(dx) = \begin{cases} Ce^{Gx}(-x)^{-1-Y}dx, & x < 0, \\ Ce^{-Mx}x^{-1-Y}dx, & x > 0. \end{cases}$$

The CGMY process has no Brownian motion part.

The VG distribution/process is a special case of CGMY: it holds that

$$\mathrm{VG}(C,\ G,\ M) = \mathrm{CGMY}(C,\ G,\ M,\ 0).$$

Financial applications of CGMY processes can be found in Carr et al. (2002, 2003).

★ Comments

The above Lévy models together with the Black-Scholes model were compared by Schoutens (2003), by fitting them into S&P500 European call options. As a result, all the Lévy models perform significantly better than the Black-Scholes model. Also, the CGMY Lévy model seems to perform some what better than the other Lévy models.

5.2.2 Lévy models with stochastic volatility

Most recent studies on Lévy processes in finance literature are concerned with using Lévy models with stochastic volatility. There are essentially two methods to introduce volatility into Lévy models. One way is to make the volatility parameter in a geometric Brownian motion to evolve according to an Ornstein-Uhlenbeck (OU) process driven by a Lévy process. Another way is to make random changes in time of the models according to some nonnegative Lévy processes. The former way leads to some BNS-SV models, which were introduced by Barndorff-Nielsen and Shephard (2001), while the latter way leads to some stochastically time-changed Lévy Models, which were introduced by Carr et al. (2003). These models are introduced in this section.

Here we introduce the OU processes that will be needed in the Lévy models. An OU process $\{y_t,\ t \geq 0\}$ driven by a Brownian motion satisfies

$$dy_t = -\lambda y_t dt + \sigma dB_t,\ t \geq 0,$$

where $\lambda > 0$ and $\{B_t\}$ is a standard Brownian motion. Recently, Barndorff-Nielsen and Shephard (2001, 2003) introduced the following OU processes driven by a Lévy process:

$$dy_t = -\lambda y_t dt + dz_{\lambda t},\ y_0 > 0,\ \lambda > 0,\ t \geq 0,$$

where $\{z_t\}$ is a subordinator, that is, a Lévy process with no Brownian motion part, nonnegative drift and positive increments. $\{z_t\}$ is called the background driving Lévy process (BDLP). We can see that the above OU process is strictly positive and $y_t \geq y_0 e^{-\lambda t}$.

Barndorff-Nielsen and Shephard (2001) show that, if the initial value y_0 is chosen according to a self-decomposable law D on $\mathbb{R}_+$, then the OU process $\{y_t\}$ is strictly stationary and the marginal distribution is precisely D. They call such a process $\{y_t\}$ as D-OU process and show that the solution of $\{y_t\}$ is given by

$$y_t = y_0 e^{-\lambda t} + e^{-\lambda t} \int_0^{\lambda t} e^s dz_s.$$

On the other hand, if the BDLP $\{z_t\}$ has marginal laws such that $z_1 \sim \tilde{D}$, then $\{y_t\}$ is called an OU-$\tilde{D}$ process. In finance literature, suitable self-decomposable laws include the Gamma distribution and the inverse Gaussian (IG) distribution. Correspondingly, the Gamma-OU and IG-OU processes are often used to handle volatility effects.

Barndorff-Nielsen and Shephard (2001) also introduced the integrated OU (intOU) processes of $\{y_t\}$:

$$Y_t = \int_0^t y_s ds.$$

The intOU process $\{Y_t,\ t \geq 0\}$ has continuous sample paths when $\lambda > 0$. Such an intOU process serves as a process according to which time is changed randomly.

■ BNS-SV models

Let $\{S_t,\ t \geq 0\}$ be the stock-price process. Let $Z_t := \log S_t$ be the log stock price. Under the Black-Sholes model, we have

$$dS_t = S_t(\mu dt + \sigma dB_t),\ S_0 > 0,$$

where $\{B_t\}$ is a standard Brownian motion. By Itô's formula, $\{Z_t\}$ satisfies

$$dZ_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t, \quad Z_0 = \log S_0.$$

In order to handle stochastically changing volatility, Barndorff-Nielsen and Shephard (2001) assume that the volatility evolve according to an OU process driven by a Lévy process and propose the following Lévy model:

$$\begin{aligned} dZ_t &:= d\log S_t = (\mu - \tfrac{1}{2}\sigma_t^2)dt + \sigma_t dB_t + \rho dz_{\lambda t}, \quad Z_0 = \log S_0, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dz_{\lambda t}, \end{aligned}$$

where $\mu \in \mathbb{R}$, $\rho \leq 0$, $\{B_t\}$ is a standard Brownian motion, $\{z_t\}$ is a subordinator, and $\{B_t\}$ is independent of $\{z_t\}$. In financial applications, the volatility process $\{\sigma_t^2\}$ is often taken as a Gamma-OU or IG-OU process.

The above model is arbitrage-free, but incomplete. This means that there exists more than one equivalent martingale measure. For more details about the BNS-SV models, see Barndorff-Nielsen and Shephard (2001, 2003); Barndorff-Nielsen, Nicolato and Shephard (2002).

■ Stochastic time changes in Lévy models

A Lévy model with stochastic time changes is as follows:

$$S_t = S_0 e^{X_{Y_t}},$$

where $\{X_t\}$ is some Lévy processes that are suitable for stock price, for example, the VG, NIG, HYP/GH, Meixner or CGMY Lévy processes discussed in the previous section, and $\{Y_t\}$ is a positive process used for random time change. In financial applications, candidates for $\{Y_t\}$ are Gamma-OU and IG-OU processes. Carr et al. (2003) use as the rate of time change the Cox-Ingersoll-Ross (CIR) process — the classical example of a mean-reverting positive process $\{y_t\}$, which is given by

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dB_t.$$

For details on stochastic time changes in Lévy models, see Carr, et al. (2003).

★ Comments

Schoutens (2003) also compared the behaviours between Lévy models with constant volatility and those with stochastic volatility, by fitting them into S&P500 European call options. All the Lévy models with stochastic volatility perform significantly better than the Lévy models with constant volatility. Also, the Lévy models with stochastic time changes seem to perform better than the BNS-SV models.

5.3 Self-similar processes

Recent empirical studies in finance show that the log stock price processes $X = \{X_t\} := \{\log \frac{S_t}{S_0}\}$ possess many characteristics that are typical for self-similar processes. In this section we provide a review of such processes.

At first, we introduce some relevant results on self-similar processes. An $\mathbb{R}$-valued process $\{X_t\}$ is said to be self-similar if for any $a > 0$, there exists $b > 0$ such that

$$\{X_{at}\} \stackrel{d}{=} \{bX_t\},$$

where '$\stackrel{d}{=}$' means equality of all finite dimensional distribuions of two processes:

$$(X_{at_1}, \ldots, X_{at_m}) \stackrel{d}{=} b(X_{t_1}, \ldots, X_{t_m})$$

for any choice of $t_i > 0$ and $m \in \mathbb{N}$. It can be shown that, if $\{X_t\}$ is self-similar, nontrivial and stochastically continuous at $t = 0$, then for a and b in the above equation, there exists a unique $H \geq 0$ such that $b = a^H$. H is called the exponent of self-similarity of the process $\{X_t\}$. In this case, $\{X_t\}$ is written as H-selfsimilar (H-ss, for short).

Self-similar processes are closely related to strictly stationary processes through a nonlinear time change, referred to as the Lamperti transformation. Precisely, if $\{Y_t\}$ is a strictly stationary process, and define

$$X_t = t^H Y_{\log t}, \ t > 0; \ X_0 = 0,$$

for some $H > 0$. Then $\{X_t\}$ is H-selfsimilar. Conversely, if $\{X_t\}$ is H-ss and we define

$$Y_t = e^{-tH} X_{e^t}, \ t \in \mathbb{R},$$

then $\{Y_t\}$ is strictly stationary.

It is known that, self-similar processes can have independent and stationary increments, or independent and non-stationary increments, or dependent and stationary increments. In the following, some typical self-similar processes will be introduced.

■ α-stable Lévy motions

$\{X_t, \ t \geq 0\}$ is said to be an α-stable Lévy motion if it is a Lévy process such that the law of X_1 is a strictly α-stable distribution. It can be shown that an α-stable Lévy motion is H-selfsimilar with

$$H = \frac{1}{\alpha} \in [\frac{1}{2}, \ \infty).$$

Conversely, if $\{X_t\}$ is H-selfsimilar with $H \in [1/2, \ \infty)$ and has stationary and independent increments, then it is an α-stable Lévy motion with $\alpha = 1/H$ (see Embrechts and Maejima (2002); Samorodnitsky and Taqqu (1994)). Therefore α-stable Lévy motions are the only selfsimilar processes with stationary and independent increments.

α-stable Lévy motions are natural generalizations of Brownian motions ($\alpha = 2$). Some works suggest that α-stable Lévy motions should be useful for modeling of financial indexes, e.g. Shiryaev (1999). However, these processes do not have finite second-order moments, which is said to contradict finance and econometric theories. Further studies on these processes should be made.

■ Self-similar processes with non-stationary increments

There are many selfsimilar processes that have independent but non-stationary increments. However, statistical properties of log stock prices often appear to have dependent and stationary increments. So, self-similar processes with independent and non-stationary increments may not be suitable for modeling stock prices.

■ Fractional Brownian motion

Let $0 < H \leq 1$. An $\mathbb{R}$-valued Gaussian process $\{B_H(t), \ t \geq 0\}$ is called a fractional Brownian motion if $E[B_H(t)] = 0$ and

$$E[B_H(s)B_H(s)] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] E[B_H^2(1)].$$

It is easy to show that a fractional Brownian motion $\{B_H(t)\}$ is H-selfsimilar with stationary increments. If $H = 1/2$, it is a Brownian motion; if $H \neq 1/2$, the increments are not independent. Furthermore, it can be shown that a fractional Brownian motion is not a semimartingale unless $H = 1/2$.

Fractional Brownian motions can be thought in some sense to be a generalization of Brownian motions. This suggests that it may be reasonable to replace the Brownian motion $\{B_{1/2}(t)\}$ in the Black-Scholes model with the fractional Brownian motion $\{B_H(t)\}$ to get the following stock-price model:

$$dS(t) = S(t)(\mu dt + \sigma dB_H(t)).$$

However, since a fractional Brownian motion $\{B_H(t)\}$ is in general not a semimartingale, the stochastic integral $\int f(s,\ \omega)dB_H(s)$ can not be defined as an Itô integral, and it must be defined suitably in some other ways (see Embrechts and Maejima (2002)). Fractional Brownian motions are Gaussian, where as log stock prices have in general non-Gaussian distributions. Furthermore, fractional Brownian motions have special dependence structure, which may be not flexible enough to fit that of log stock prices very well. So fractional Brownian motions might be not suitable for modeling of stock price.

■ Self-similar process with stationary and dependent increments

Here we consider some aspects and examples of general selfsimilar processes with stationary but dependent increments. Let $\{X_t,\ t \geq 0\}$ be a H-selfsimilar processes with stationary increments. It can be shown that (see Embrechts and Maejima (2002); Samorodnitsky and Taqqu (1994)) the followings hold:

If $E|X_1|^\gamma < \infty$ for some $\gamma \in (0,\ 1)$, then $0 < H < 1/\gamma$;
If $E|X_1|^\gamma < \infty$ for some $\gamma \geq 1$, then $0 < H \leq 1$;
If $H > 1$, then $E|X_1|^{1/H} = +\infty$;
If $E|X_1| < \infty$ and $0 < H < 1$, then $EX_t = 0$ for all $t \geq 0$;
If $H > 0$, then $X_0 = 0$ (a.s.);
If $E|X_1| < \infty$ and $H = 1$, then $X_t = tX_1$ (a.s.).

Now we consider the dependence structure of H-selfsimilar processes $\{X_t\}$ with stationary increments (H-sssi, for short) and finite variances $EX_t^2 < \infty$. Let $0 < H < 1$, and define the time series $\{r_t\}$ by

$$r_t = X_t - X_{t-1}, \quad t = 1, 2, \ldots$$

Then it can be shown that the ACF $\{\rho_k\}$ of $\{r_t\}$ satisfies

$$\rho_k \sim H(2H-1)k^{2H-2} \quad (k \to \infty).$$

Thus, if $H = 1/2$, $\{r_t\}$ is serially uncorrelated; If $H \in (0,\ 1/2)$, then $\{r_t\}$ is negatively correlated and $\sum_{k=1}^{\infty} |\rho_k| < \infty$; If $H \in (1/2,\ 1)$, then $\{r_t\}$ is positively correlated and $\sum_{k=1}^{\infty} \rho_k = \infty$, which means that $\{r_t\}$ has long memory.

○ α-stable processes

Fractional Brownian motions $\{B_H(t)\}$ with $H \neq 1/2$ are selfsimilar processes with dependent increments. Other examples of such processes include α-stable stochastic processes which are not Lévy processes. A stochastic process $\{X_t\}$ is said to be α-stable if all its finite-dimensional distributions are α-stable. It is strictly stable or symmetric stable if all its finite-dimensional distributions are, respectively, strictly stable or symmetric stable. It can be shown that if $\{X_t\}$ is an α-stable H-sssi process with $0 < \alpha < 2$, then it is strictly stable.

In the following we give some examples of α-stable H-sssi processes.

◊ Linear fractional stable motions

A linear fractional stable motion (LFSM) $\{L_{\alpha,H}(a,b;t),\ t \in \mathbb{R}\}$ is an extension of a fractional Brownian motion, which is given by

$$L_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} f_{\alpha,H}(a,b;t,x)M(dx),$$

where

$$f_{\alpha,H}(a,b;t,x) = a\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) + b\left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right),$$

where $x_+ := \max(x,\ 0)$, $x_- := \max(-x,\ 0)$, $0^s := 0$ for $s \le 0$, a, b are real constants, $|a| + |b| > 0$, $0 < \alpha \le 2$, $0 < H < 1$, $H \ne 1/\alpha$ and $M(dx)$ is an α-stable random measure on $\mathbb{R}$ with Lebesgue control measure and skewness intensity $\beta(x)$, $x \in \mathbb{R}$, satisfying certain conditions. Here the integral $\int f(x)M(dx)$ is defined in the sense of stable integrals (see Samorodnitsky and Taqqu (1994)). In the special case when $a = b = 1$, the LFSM is called a well-balanced LFSM. If $\alpha = 2$, the LFSM reduces to fractional Brownian motion. It can be shown that $\{L_{\alpha,H}(a,b;t)\}$ is H-sssi, and $L_{\alpha,H}(a,b;t) \sim S_\alpha(\sigma_t,\ \beta_t,\ 0)$, where σ_t and β_t are certain functions of a, b, α, H and t.

A fractional stable noise is defined by

$$r_t = L_{\alpha,H}(a,b;t) - L_{\alpha,H}(a,b;t-1), \quad t \in \mathbb{Z}.$$

It is a strictly stationary time series with α-stable distributions, and possesses long memory properties in a certain sense (see Samorodnitsky and Taqqu (1994)).

Other important α-stable and H-sssi processes include log-fractional stable motions and real harmonizable fractional stable motions (see Samorodnitsky and Taqqu (1994)) that may be useful for modeling log stock prices.

◊ Subordinated processes

A subordinated process $\{X_t\}$ is an α-stable process with representation

$$X_t = \int_\Omega Y(t,x)M(dx),$$

where $Y = \{Y(t,x),\ t \in T,\ x \in \Omega\}$ is a stochastic process defined on $(\Omega,\ \mathcal{F},\ P)$ and M is an α-stable random measure with control measure P.

If M is $S\alpha S$, $0 < \alpha < 2$, and Y is Gaussian, then X is the $S\alpha S$ sub-Gaussian processes; If Y is $S\alpha' S$ with $0 < \alpha < \alpha' < 2$, then X is the $S\alpha S$ sub-stable processes.

It can be shown that if $Y(t,x)$ is H-ssi with finite α-order moments and if M is a $S\alpha S$ random measure, then the subordinated processes $\{X_t\}$ is $S\alpha S$ and H-sssi.

○ Selfsimilar processes with finite variances

Except for the fractional Brownian motions, the selfsimilar processes metioned in the above have infinite variances. However, there are also some non-Gaussian selfsimilar processes with finite variances, which may be useful in modeling of stock price. Some details can be found in Embrechts and Maejima (2002) and the references therein.

★ R/S-analysis of self-similar processes

Now we briefly mention the R/S analysis for selfsimilar processes. Let $\{S_t,\ t \geq 0\}$ be our stock price. As before, write

$$S_t = S_0 e^{X_t}.$$

Here, if we assume that the log stock price process $\{X_t,\ t \geq 0\}$, $X_0 = 0$, is some selfsimilar process with exponent of similarity H, then it is important to estimate the value of H. Our previous discussions show that, if $H \in (1/2,\ 1)$, then the time series of log returns $r_t = X_t - X_{t-1}$ has long memory. In order to estimate the exponent H, Hurst (1951) proposed the so-called R/S-statistic given as follows:

$$\begin{aligned} R_n &:= \max_{1 \leq t \leq n} \left(X_t - \frac{t}{n} X_n \right) - \min_{1 \leq t \leq n} \left(X_t - \frac{t}{n} X_n \right), \\ S_n &:= \sqrt{\frac{1}{n} \sum_{t=1}^{n} (r_t - \bar{r}_n)^2}, \end{aligned}$$

where $\bar{r}_n = \frac{1}{n} \sum_{t=1}^{n} r_t$ — the sample mean of $\{r_t\}$, and S_n is just the sample standard deviation of $\{r_t\}$. The ratio $Q_n := R_n / S_n$ is called R/S-statistic.

We metion some statistical properties of Q_n here (see Embrechts and Maejima (2002); Beran (1994)):

(1) If $\{X_t\}$ is $\frac{1}{2}$-sssi such that $\{r_t\}$ is iid with finite variances, then

$$n^{-1/2} Q_n \Rightarrow \sup_{0 \leq t \leq 1} B_t^0 - \inf_{0 \leq t \leq 1} B_t^0,$$

where $\{B_t^0,\ 0 \leq t \leq 1\}$ is a Brownian bridge, and '$\Rightarrow$' denotes convergence in distribution.

(2) If $\{X_t\}$ is $\frac{1}{2}$-sssi such that $\{r_t\}$ is iid and is in the domain of attraction of an α-stable distribution with $0 < \alpha \leq 2$, then

$$n^{-1/2} Q_n \Rightarrow \xi \quad (n \to \infty),$$

where ξ is a non-degenerate random variable.

(3) If $\{X_t\}$ is $\frac{1}{2}$-sssi such that $\{r_t\}$ is strictly stationary, $\{r_t^2\}$ is ergodic and $n^{-\frac{1}{2}} \sum_{j=1}^{[nt]} r_j$ converges to a Brownian motion, then the conclusion of (2) also holds in this case.

(4) If $\{X_t\}$ is H-sssi such that $\{r_t\}$ is strictly stationary, $\{r_t^2\}$ is ergodic and $n^{-H} \sum_{j=1}^{[nt]} r_j$ converges to a fractional Brownian motion, then

$$n^{-H} Q_n \Rightarrow \xi \quad (n \to \infty),$$

where ξ is a non-degenerate random variable.

The above results show that, for a H-sssi process $\{X_t\}$, its R/S-statistic Q_n satisfies essentially the following equation:

$$\log Q_n = a + H \log n + \zeta_n$$

for sufficiently large n, where $a \in \mathbb{R}$ and ζ_n is some non-degenerate rv. Using this simple regression model, we can estimate H by using least squares (LS) method or least absolute deviations (LAD) method.

Many empirical studies show that the Hurst exponents H of selfsimilarity of log prices of many stocks and financial indexes are often larger than 1/2. For example, it was shown in Peters (1991) that,

H(S&P500)=0.78; H(IBM)=0.72;
H(Apple Computer)=0.75; H(Consolidated Edison)=0.68.

These indicate that, stock price processes are possibly selfsimilar processes with stationary increments that have long memory.

6 Pricing of European options

In most cases, the probability density functions of Lévy processes are either difficult to be known in an exact form, or much involved. The main tool for analyzing Lévy processes is the characteristic functions. Here we introduce some recent results on pricing formulas of European options through using characteristic functions (see Bakshi and Madan (2000); Carr and Madan (1998)). Also, Nualart and Schoutens (2001) and Raible (2000) derive a partial differential integral equation (PDIE) for the price of European call option.

Let $B = \{B_t,\ t \geq 0\}$ be a riskless asset (the bank account) and $S = \{S_t,\ t \geq 0\}$ the price of a stock/index. Assume that $B_t = e^{rt}$, where $r > 0$ is the interest rate. For a European call option of $\{S_t\}$ with strike price K and expiry time T, the pay-off function is given by $\max\{S_T - K,\ 0\}$. Then the arbitrage-free price of $C(K, T; t)$ of the European call option at time $t \in (0, T]$ is given by

$$C(K, T; t) = E_Q \left[e^{-r(T-t)} \max\{S_T - K,\ 0\} | \mathcal{F}_t \right],$$

where the expectation E_Q is taken with respect to an equivalent martingale measure Q, $\{\mathcal{F}_t,\ t \geq 0\}$ is the natural filtration of $\{S_t\}$, and $e^{-r(T-t)}$ is the discounting factor. If we know the density function $f_Q(s,T)$ of S_T under Q, then we can price European call options, for example, at time 0, by

$$\begin{aligned} C(K,T;0) &= e^{-rT}\int_0^\infty f_Q(s,T)\max\{s-K,\ 0\}ds \\ &= e^{-rT}\int_K^\infty sf_Q(s,T)ds - Ke^{-rT}\Pi_2, \end{aligned}$$

where $\Pi_2 = Q\{S_T > K\}$.

Bakshi and Madan (2000) obtain a formula for a European call option through the characteristic function. Let $\phi(u)$ be the ch.f of $\log S_T$:

$$\phi(u) = E_Q e^{iu\log S_T}.$$

The price of European call option at time 0 is given by

$$C(K,T;0) = S_0\Pi_1 - Ke^{-rT}\Pi_2,$$

where

$$\begin{aligned} \Pi_1 &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\log K}\phi(u-i)}{iu\phi(-i)}\right)du, \\ \Pi_2 &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\log K}\phi(u)}{iu}\right)du. \end{aligned}$$

Using this pricing formula, pricing of a European call option can be easily undertaken for a Lévy-market model.

Let $S_t = S_0 e^{X_t}$. Assume that $\{X_t\}$ is a Lévy process with Lévy triplet $[\gamma,\ 0,\ \nu^Q(dx)]$ under the equivalent martingale measure Q. Denote the price of European call option at time t by $V(t,\ X_t)$. Nualart and Schoutens (2001) and Raible (2000) show that, the price $V(t,\ X_t)$ can be obtained from the following PDIE:

$$\begin{aligned} rV(t,\ x) &= \gamma\frac{\partial}{\partial x}V(t,\ x)+\frac{\partial}{\partial t}V(t,\ x) \\ &\quad + \int_{-\infty}^\infty\left[V(t,\ x+y)-V(t,\ x)-y\frac{\partial}{\partial x}V(t,\ x)\right]\nu^Q(dy), \\ V(T,\ x) &= F(x), \end{aligned}$$

where $F(X_T) := \max(S_0e^{X_T}-K,\ 0)$. This is an analogue of the Black-Sholes PDE.

7 Concluding remarks

In this review we have introduced a wide class of statistical and probabilistic models that are suitable for, or relevant to modeling of stock prices, including time series models, Brownian-motion-based diffusion models and Lévy-market models. Here we give a summarization, and also, some opinions and suggestions.

- The semi-heavy-tailed infinitely divisible distributions such as the VG, NIG, CGMY, HYP/GH and Meixner distributions seem to fit stock prices well. However, other distributions such as α-stable distributions, t-distributions and mixture of normal distributions should not be abandoned, on which more studies are needed.

- Conditional heteroscedastic time series models such as GARCH, EGACH and SV models are suitable for modeling stock prices. But these nonlinear models should be used in combination with linear time series models such as ARMA models to handle serial correlations, leading to models of ARMA-GARCH or ARMA-EGARCH models, and so on.

- Log stock prices do not have independent increments. Thus continuous-time stochastic processes having independent increments are not very suitable for modeling log stock prices; so Brownian motions and also Lévy processes are not very satisfactory, though Lévy processes can largely improve upon Brownian motions.

- Lévy processes (including Brownian motions) with stochastic time changes seem to fit log stock prices well. Also, BNS-SV models seem to perform well. The reason may be that making stochastic time changes in the Lévy models or introducing stochastic volatility into the BNS-SV models can simultaneously introduce flexible serial correlation structures into the increments to make them dependent.

- Log stock prices possess many of the typical properties of self-similar processes. α-stable processes such as linear fractional stable motions and subordinated processes, should be useful for modeling log stock prices. In particular, self-similar processes with stationary and dependent increments having long memory and finite variances are expected to model log stock prices well, and more self-similar processes having such properties need to be developed.

- So far, there have been few studies on comparisons of behaviors between discrete and continuous-time models for stock prices, and more statistical studies on this aspect are needed.

- In some studies, unknown parameters in the models used for pricing of options are estimated by minimizing the root-mean-square error between the market's and the model's prices of options. However, it is possible that the market prices of options are mis-priced, and in this case if we estimate model parameters in the above way, it will lead to a mis-specified model for the stock prices. From the statistical point of view, parameters should be estimated by using maximum likelihood method or some other appropriate methods, and pricing of options should be undertaken under the model fitted in this way. Further studies on this aspect are needed.

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