A PROOF OF THE EXISTENCE OF COMPLEX-VALUED SYMMETRIC α-STABLE RANDOM MEASURES

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A PROOF OF THE EXISTENCE OF COMPLEX–VALUED SYMMETRIC α-STABLE RANDOM MEASURES

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Abstract: We give a rigorous proof for the existence of complex-valued symmetric α-stable random measures.

Key words: symmetric α-stable random measures; spectral measures; stable integrals; stochastic processes

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space, $(E, \mathcal{E}, m)$ a σ-finite measure space, $M(\cdot)$ a symmetric stable random measure on $(E, \mathcal{E})$ with control measure m, $\{f_t,\ t \in T\}$ a family of measurable functions on $(E, \mathcal{E})$ satisfying suitable conditions, where T is an index set. Many real-valued non-Gaussian stable stochastic processes $\{X(t),\ t \in T\}$, for example, sub-Gaussian processes, stable Lévy motion, moving average processes, Ornstein-Uhlenbeck process and fractional stable motion, can be defined as a family of stable stochastic integrals of the form $X_t = \int_E f_t(x)M(dx)$, where M is a real-valued stable random measure. There are, however, some non-Gaussian stable stochastic processes, such as harmonizable processes, that are defined as stable stochastic integrals with respect to complex-valued stable random measures. The stable integrals can be defined in several ways. If it can be assumed that the random measure does exist, then it is

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straightforward to construct the integral as a bona fide integral. However, showing the existence of random measures is not an easy task, and this problem has seemed to be ignored unsuitably. The existence of real-valued stable random measures is discussed thoroughly by Samorodnitsky and Taqqu (1994) in their excellent book. The same authors also briefly discuss the existence of complex-valued stable random measure, but the proof is not given. In this note we present a detailed proof for this by using similar auguments and ideas as those developed by Samorodnitsky and Taqqu (1994) for the proof of existence of real-valued random measures and stable stochastic integrals.

2. Definitions and notations

A random variable X is said to have a symmetric α-stable ($S\alpha S$) distribution if there are parameters $0 < \alpha \leq 2$ and $\sigma > 0$ such that its characteristic function has the form

$$E(\exp i\theta X) = \exp\{-\sigma^\alpha|\theta|^\alpha\}, \quad \theta \in \mathbf{R}.$$

We write $X \sim S_\alpha(\sigma, 0, 0)$. The parameter α is the index of stability and σ the scale parameter.

For $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^d$, let $(\mathbf{x}, \mathbf{y}) := x_1y_1 + \cdots + x_dy_d$ denote the usual inner product in $\mathbf{R}^d$. Let $S_d := \{\mathbf{s} : \|\mathbf{s}\| = 1\}$ be the unit sphere in $\mathbf{R}^d$ and $\mathcal{B}(S_d)$ be the Borel σ-field on S_d with respect to the usual Euclidean norm. A measure on $(S_d, \mathcal{B}(S_d))$ is said to be symmetric if $\Gamma(-A) = \Gamma(A)$ for any $A \in \mathcal{B}(S_d)$. A random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is said to be a symmetric α-stable random vector in $\mathbf{R}^d$ if its characteristic function has the form

$$E\exp\{i(\boldsymbol{\theta}, \mathbf{X})\} = \exp\left\{-\int_{S_d} |(\boldsymbol{\theta}, \mathbf{s})|^\alpha \Gamma(d\mathbf{s})\right\},$$

where $0 < \alpha < 2$ and Γ is a finite symmetric measure on the measurable space $(S_d, \mathcal{B}(S_d))$. It is known that the measure Γ in this representation is unique (see Varadhan (1962)). Γ is called the spectral measure of $\mathbf{X}$. A stochastic process $\{X_t,\ t \in T\}$ is symmetric α-stable if all its finite-dimensional distributions are symmetric α-stable.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. Denote by $L^0(\Omega)$ the set of all real-valued random variables on $(\Omega, \mathcal{F}, P)$ and $L^0_{\mathbf{C}}(\Omega)$ the set of all complex-valued random variables. Let $(E, \mathcal{E}, m)$ be a measure space, where m is a σ-finite measure. Let $\mathcal{E}_0 = \{A \in \mathcal{E} :\ m(A) < \infty\}$. A set function $M : E_0 \mapsto L^0(\Omega)$ is independently scattered if $M(A_1), \cdots, M(A_k)$ are independent whenever $A_1, \ldots, A_k \in \mathcal{E}_0$ are disjoint. It is σ-additive if

$$\forall \{A_j\}_1^\infty \subset \mathcal{E}_0 : \text{ disjoint}, \bigcup_{j=1}^{\infty} A_j \in \mathcal{E}_0 \implies M(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} M(A_j) \text{ a.s.}$$

A real-valued symmetric α-stable random measure M on $(E, \mathcal{E})$ is an independently scattered σ-additive set function $M : E_0 \mapsto L^0(\Omega)$ such that $M(A) \sim S_\alpha((m(A)^{1/\alpha}, 0, 0))$ for each $A \in \mathcal{E}_0$, and such that $\{M(A),\ A \in \mathcal{E}_0\}$ is a symmetric α-stable stochastic process. The measure m is called the control measure of M.

Let $(S_2, \mathcal{B}(S_2))$ be the unit sphere in $\mathbf{R}^2$ equipped with the Borel σ-field. Let $(E, \mathcal{E})$ be a measurable space. Let k be a measure on the product space $(E \times S_2, \mathcal{E} \otimes \mathcal{B}(S_2))$ satisfying the following condition: for every $A \in \mathcal{E}$ such that $k(A \times S_2) < \infty$, $k(A \times \cdot)$ is a finite symmetric measure on $(S_2, \mathcal{B}(S_2))$. Let

$$\mathcal{E}_0 = \{A \in \mathcal{E} :\ k(A \times S_2) < \infty\}.$$

A complex-valued $S\alpha S$ random measure M on $(E, \mathcal{E})$ is an independently scattered σ-additive complex-valued set function

$$M :\ \mathcal{E}_0 \mapsto L^0_{\mathbf{C}}(\Omega)$$

such that, for every $A \in \mathcal{E}_0$, $M^{(1)}(A) := \mathrm{Re} M(A)$ and $M^{(2)}(A) := \mathrm{Im} M(A)$ are $S\alpha S$ in $\mathbf{R}^2$ with spectral measure $k(A \times \cdot)$, and such that $\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ is an $\mathbf{R}^2$-valued $S\alpha S$ process. The latter fact means that any finite-dimensional distributions of

$$(M^{(1)}(A_1), M^{(2)}(A_1), \ldots, M^{(1)}(A_k), M^{(2)}(A_k)),\ \forall A_j \in \mathcal{E}_0,\ \forall k \in \mathbf{N},$$

are $S\alpha S$ in $\mathbf{R}^{2k}$. The measure k is called the circular control measure of M.

3. Proof of the existence of complex-valued $S\alpha S$ random measures

Let M be the complex-valued $S\alpha S$ random measure with circular control measure k, as was defined in the previous section. To prove the existence of M, we use the same ideas as those in Samorodnitsky and Taqqu (1994). This will be accomplished by specifying the finite-dimensional distributions of M, showing that they are consistent and then applying Kolmogorov's existence theorem to conclude that M is a well-defined stochastic process, and after this, showing that M possesses the other properties that a complex-valued $S\alpha S$ random measure must have.

At first we consider what the finite dimensional distributions of M should be.

Proposition 1 *If M is a complex-valued $S\alpha S$ random measure on $(E, \mathcal{E})$ with circular control measure k, then for $A_1, \ldots, A_d \in \mathcal{E}_0$, the random vector $(M^{(1)}(A_1), M^{(2)}(A_1), \ldots, M^{(1)}(A_d), M^{(2)}(A_d))$ has the characteristic fuction*

$$\begin{aligned}
&\phi_{A_1,\ldots,A_d}(\theta_1^{(1)}, \theta_1^{(2)}, \ldots, \theta_d^{(1)}, \theta_d^{(2)}) \\
&\quad := E\exp\Big\{i\Big(\sum_{j=1}^{d}(\theta_j^{(1)}M^{(1)}(A_j) + \theta_j^{(2)}M^{(2)}(A_j))\Big)\Big\} \\
&\quad = \exp\Big\{-\int_E\int_{S_2}\Big|\sum_{j=1}^{d}(s_1\theta_j^{(1)} + s_2\theta_j^{(2)})1_{A_j}(x)\Big|^\alpha k(dx, d\mathrm{s})\Big\},
\end{aligned}$$

where $\theta_j^{(1)}$, $\theta_j^{(2)} \in \mathbf{R}$, $\mathrm{s} = (s_1, s_2) \in S_2$ and $1_{A_j}(x)$ is the indicator function.

PROOF: We have

$$E \exp\Big\{i(\theta_j^{(1)} M^{(1)}(A_j) + \theta_j^{(2)} M^{(2)}(A_j))\Big\}$$
$$= \exp\Big\{ - \int_{S_2} \big|\theta_j^{(1)} s_1 + \theta_j^{(2)} s_2\big|^\alpha k(A_j, ds)\Big\}$$
$$= \exp\Big\{ - \int_{S_2} \big|\theta_j^{(1)} s_1 + \theta_j^{(2)} s_2\big|^\alpha \int_E 1_{A_j}(x) k(dx, ds)\Big\}$$
$$= \exp\Big\{ - \int_E \int_{S_2} \big|(\theta_j^{(1)} s_1 + \theta_j^{(2)} s_2) 1_{A_j}(x)\big|^\alpha k(dx, ds)\Big\}.$$

Suppose firstly that $A_1, \ldots, A_d \in \mathcal{E}_0$ are disjoint. Since the random vectors

$$(M^{(1)}(A_1), M^{(2)}(A_1)), \ldots, (M^{(1)}(A_d), M^{(2)}(A_d))$$

are independent, we have

$$E \exp\Big\{i \sum_{j=1}^{d} (\theta_j^{(1)} M^{(1)}(A_j) + \theta_j^{(2)} M^{(2)}(A_j))\Big\}$$
$$= \exp\Big\{ - \int_E \int_{S_2} \sum_{j=1}^{d} \big|(s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}) 1_{A_j}(x)\big|^\alpha k(dx, ds)\Big\}$$
$$= \exp\Big\{ - \int_E \int_{S_2} \Big|\sum_{j=1}^{d} (s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}) 1_{A_j}(x)\Big|^\alpha k(dx, ds)\Big\}. \quad (3.1)$$

When $A_1, \ldots, A_d \in \mathcal{E}_0$ are not disjoint, we can decompose the sets $A_1, \ldots, A_d$ into disjoint subsets $B_1, \ldots, B_m \in \mathcal{E}_0$ such that, for each A_j, there exist $B_{k_1}, \ldots, B_{k_j} \subset \{B_1, \ldots, B_m\}$ such that $A_j = B_{k_1} \cup \cdots \cup B_{k_j}$. Let

$$1_{B_k \subset A_j} := \begin{cases} 1 & \text{if } B_k \subset A_j, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\sum_{j=1}^{d} (\theta_j^{(1)} M^{(1)}(A_j) + \theta_j^{(2)} M^{(2)}(A_j))$$
$$= \sum_{j=1}^{d} (\theta_j^{(1)} \sum_{k=1}^{m} M^{(1)}(B_k) 1_{B_k \subset A_j} + \theta_j^{(2)} \sum_{k=1}^{m} M^{(2)}(B_k) 1_{B_k \subset A_j})$$
$$= \sum_{k=1}^{m} \Big[(\sum_{j=1}^{d} \theta_j^{(1)} 1_{B_k \subset A_j}) M^{(1)}(B_k) + (\sum_{j=1}^{d} \theta_j^{(2)} 1_{B_k \subset A_j}) M^{(2)}(B_k)\Big].$$

Using this and (3.1), we have

$$
\begin{aligned}
&E\exp\Big\{i\sum_{j=1}^{d}(\theta_j^{(1)}M^{(1)}(A_j)+\theta_j^{(2)}M^{(2)}(A_j))\Big\}\\
&\quad = E\exp\Big\{i\sum_{k=1}^{m}\Big[(\sum_{j=1}^{d}\theta_j^{(1)}1_{B_k\subset A_j})M^{(1)}(B_k)+(\sum_{j=1}^{d}\theta_j^{(2)}1_{B_k\subset A_j})M^{(2)}(B_k)\Big]\Big\}\\
&\quad = \exp\Big\{-\int_E\int_{S_2}\Big|\sum_{k=1}^{m}\Big((\sum_{j=1}^{d}\theta_j^{(1)}1_{B_k\subset A_j})s_1+(\sum_{j=1}^{d}\theta_j^{(2)}1_{B_k\subset A_j})s_2\Big)1_{B_k}(x)\Big|^{\alpha}\\
&\qquad\qquad k(dx,d\mathbf{s})\Big\}\\
&\quad = \exp\Big\{-\int_E\int_{S_2}\Big|\sum_{k=1}^{m}\sum_{j=1}^{d}(\theta_j^{(1)}s_1+\theta_j^{(2)}s_2)1_{B_k\subset A_j}1_{B_k}(x)\Big|^{\alpha}k(dx,d\mathbf{s})\Big\}\\
&\quad = \exp\Big\{-\int_E\int_{S_2}\Big|\sum_{j=1}^{d}(\theta_j^{(1)}s_1+\theta_j^{(2)}s_2)\sum_{k=1}^{m}1_{B_k\subset A_j}1_{B_k}(x)\Big|^{\alpha}k(dx,d\mathbf{s})\Big\}\\
&\quad = \exp\Big\{-\int_E\int_{S_2}\Big|\sum_{j=1}^{d}(s_1\theta_j^{(1)}+s_2\theta_j^{(2)})1_{A_j}(x)\Big|^{\alpha}k(dx,d\mathbf{s})\Big\}.
\end{aligned}
$$

This completes the proof. ∎

Proposition 2 *The characteristic function $\phi_{A_1,\ldots,A_d}$ given in Proposition 1 is the characteristic function of a $S\alpha S$ distribution in $\mathbf{R}^{2d}$.*

Proof: Let $E_+ := \{x\in E:\ \sum_{j=1}^{d}1_{A_j}(x)>0\}$. Define $\mathbf{g}: E_+\times S_2\mapsto S_{2d}$ by

$$
\mathbf{g}(x,\mathbf{s})=(g_1^{(1)}(x,\mathbf{s}),g_1^{(2)}(x,\mathbf{s}),\ldots,g_d^{(1)}(x,\mathbf{s}),g_d^{(2)}(x,\mathbf{s})),
$$

where

$$
\begin{aligned}
g_j^{(1)}(x,\mathbf{s}) &= \frac{1_{A_j}(x)s_1}{(\sum_{j=1}^{d}1_{A_j}(x))^{1/2}},\\
g_j^{(2)}(x,\mathbf{s}) &= \frac{1_{A_j}(x)s_2}{(\sum_{j=1}^{d}1_{A_j}(x))^{1/2}}.
\end{aligned}
$$

We have

$$
\phi_{A_1,\ldots,A_d}(\theta_1^{(1)},\theta_1^{(2)},\ldots,\theta_d^{(1)},\theta_d^{(2)})
$$

$$= \exp\Big\{ - \int_{E_+} \int_{S_2} \Big| \sum_{j=1}^{d} (s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}) 1_{A_j}(x) \Big|^{\alpha} k(dx, d\mathbf{s}) \Big\}$$

$$= \exp\Big\{ - \int_{E_+} \int_{S_2} \Big| \sum_{j=1}^{d} \theta_j^{(1)} g_j^{(1)}(x, \mathbf{s}) + \sum_{j=1}^{d} \theta_j^{(2)} g_j^{(2)}(x, \mathbf{s}) \Big|^{\alpha}$$

$$\Big| \sum_{j=1}^{d} 1_{A_j}(x) \Big|^{\alpha/2} k(dx, d\mathbf{s}) \Big\} \quad (3.2)$$

$$= \exp\Big\{ - \int_{S_{2d}} \Big| \sum_{j=1}^{d} \theta_j^{(1)} s_j^{(1)} + \sum_{j=1}^{d} \theta_j^{(2)} s_j^{(2)} \Big|^{\alpha} \Gamma(d\mathbf{s}_{2d}) \Big\},$$

where $\mathbf{s}_{2d} := (s_1^{(1)}, s_1^{(2)}, \ldots, s_d^{(1)}, s_d^{(2)}) \in S_{2d}$ and for each $A \in \mathcal{B}(S_{2d})$,

$$\Gamma(A) := \int_{\mathbf{g}^{-1}(A)} \Big| \sum_{j=1}^{d} 1_{A_j}(x) \Big|^{\alpha/2} k(dx, d\mathbf{s}).$$

Now let us show that Γ is a symmetric measure on S_{2d}. For each $A \in \mathcal{B}(S_2)$ such that $\mathbf{g}^{-1}(A) := U \times V \in \mathcal{E} \times \mathcal{B}(S_2)$, which generates the σ-filed $\mathcal{E} \otimes \mathcal{B}(S_2)$, we have $\mathbf{g}^{-1}(-A) := U \times (-V)$ by the definition of $\mathbf{g}$. Therefore,

$$\begin{aligned}
\Gamma(-A) &= \int_{\mathbf{g}^{-1}(-A)} \Big| \sum_{j=1}^{d} 1_{A_j}(x) \Big|^{\alpha/2} k(dx, d\mathbf{s}) \\
&= \int_{U \times (-V)} \Big| \sum_{j=1}^{d} 1_{A_j}(x) \Big|^{\alpha/2} k(dx, d\mathbf{s}) \\
&= \int_{U \times V} \Big| \sum_{j=1}^{d} 1_{A_j}(x) \Big|^{\alpha/2} k(dx, d\mathbf{s}) \\
&= \Gamma(A),
\end{aligned}$$

because $k(U \times \cdot)$ is a symmetric measure on $(S_2, \mathcal{B}(S_2))$. This completes the proof. ∎

REMARK 1. In the above proof of Proposition 2, we used the condition that $k(A \times \cdot)$ is a symmetric measure on S_2. In fact, Proposition 2 holds without this condition (note that this condition is used only in the proof of Claim 1 to be given later), and the proof is given as follows. From (3.2) we have

$$
\begin{aligned}
&\phi_{A_1,\dots,A_d}(\theta_1^{(1)},\theta_1^{(2)},\dots,\theta_d^{(1)},\theta_d^{(2)})\\
&= \exp\Big\{-\int_{E_+}\int_{S_2}\Big|\sum_{j=1}^d (s_1\theta_j^{(1)}+s_2\theta_j^{(2)})1_{A_j}(x)\Big|^\alpha k(dx,d\mathrm{s})\Big\}\\
&= \exp\Big\{-\int_{E_+}\int_{S_2}\Big(\Big|\sum_{j=1}^d \theta_j^{(1)}g_j^{(1)}(x,\mathrm{s})+\sum_{j=1}^d \theta_j^{(2)}g_j^{(2)}(x,\mathrm{s})\Big|^\alpha\\
&\qquad +\Big|-\sum_{j=1}^d \theta_j^{(1)}g_j^{(1)}(x,\mathrm{s})-\sum_{j=1}^d \theta_j^{(2)}g_j^{(2)}(x,\mathrm{s})\Big|^\alpha\Big)\\
&\qquad\qquad \frac{1}{2}\Big|\sum_{j=1}^d 1_{A_j}(x)\Big|^{\alpha/2} k(dx,d\mathrm{s})\Big\}\\
&=: \exp\Big\{-\int_{S_{2d}}\Big|\sum_{j=1}^d \theta_j^{(1)}s_j^{(1)}+\sum_{j=1}^d \theta_j^{(2)}s_j^{(2)}\Big|^\alpha \tilde{\Gamma}(d\mathrm{s}_{2d})\Big\},
\end{aligned}
$$

where for $A\in\mathcal{B}(S^{2d})$,

$$
\tilde{\Gamma}(A) := \frac{1}{2}\Big(\int_{\mathbf{g}^{-1}(A)}+\int_{\mathbf{g}^{-1}(-A)}\Big)\Big|\sum_{j=1}^d 1_{A_j}(x)\Big|^{\alpha/2} k(dx,ds)
$$

Clearly, $\tilde{\Gamma}$ is symmetric. ∎

Now let $\{\mu_{A_1,\dots,A_d}:\ A_1,\dots,A_d\in\mathcal{E}_0,\ d\in\mathbf{N}\}$ be the probability distributions corresponding to the characteristic functions

$$
\{\phi_{A_1,\dots,A_d}(\theta_1^{(1)},\theta_1^{(2)},\dots,\theta_d^{(1)},\theta_d^{(2)}):\ A_1,\dots,A_d\in\mathcal{E}_0,\ d\in\mathbf{N}\},
$$

which are specified in Proposition 1. It is easy to see that the family of finite dimensional distributions possesses the consistency: for any permutation $(\pi(1),\dots,\pi(d))$ of $(1,\dots,d)$, it holds that

$$
\begin{aligned}
&\phi_{A_{\pi(1)},\dots,A_{\pi(d)}}(\theta_{\pi(1)}^{(1)},\theta_{\pi(1)}^{(2)},\dots,\theta_{\pi(d)}^{(1)},\theta_{\pi(d)}^{(2)})\\
&= \phi_{A_1,\dots,A_d}(\theta_1^{(1)},\theta_1^{(2)},\dots,\theta_d^{(1)},\theta_d^{(2)}),
\end{aligned}
$$

and for each $n\le d$ it holds that

$$
\begin{aligned}
&\phi_{A_1,\dots,A_n}(\theta_1^{(1)},\theta_1^{(2)},\dots,\theta_n^{(1)},\theta_n^{(2)})\\
&= \phi_{A_1,\dots,A_d}(\theta_1^{(1)},\theta_1^{(2)},\dots,\theta_n^{(1)},\theta_n^{(2)},0,\dots,0).
\end{aligned}
$$

By Kolmogorov's existence theorem, there is an $\mathbf{R}^2$-valued stochastic process $\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ whose finite dimensional distributions are $S\alpha S$ as given in Propositions 1 and 2. Now we show that $\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ satisfies the other properties required by a complex-valued $S\alpha S$ random measure.

Claim 1. *For each $A \in \mathcal{E}_0$, $(M^{(1)}(A), M^{(2)}(A))$ is $S\alpha S$ in $\mathbf{R}^2$ with spectral measure $k(A \times \cdot)$.*

Proof: We have

$$\begin{aligned}
&E \exp\left\{i(\theta^{(1)} M^{(1)}(A) + \theta^{(2)} M^{(2)}(A))\right\} \\
&\quad = \exp\left\{ - \int_E \int_{S_2} \left|(\theta^{(1)} s_1 + \theta^{(2)} s_2)\right|^\alpha 1_A(x) k(dx, d\mathbf{s})\right\} \\
&\quad = \exp\left\{ - \int_{S_2} \left|\theta^{(1)} s_1 + \theta^{(2)} s_2\right|^\alpha \int_E 1_A(x) k(dx, d\mathbf{s})\right\} \\
&\quad = \exp\left\{ - \int_{S_2} \left|\theta^{(1)} s_1 + \theta^{(2)} s_2\right|^\alpha k(A, d\mathbf{s})\right\}.
\end{aligned}$$

Since, by assumption, $k(A \times \cdot)$ is a symmetric measure on S_2, it must be the spectral measure of $(M^{(1)}(A), M^{(2)}(A))$ by unicity of the spectral measure. ∎

Claim 2. $\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ *is independently scattered.*

Proof: For any disjoint sets $A_1, \ldots, A_d \in \mathcal{E}_0$, we have

$$\begin{aligned}
&E \exp\left\{i \sum_{j=1}^{d} (\theta_j^{(1)} M^{(1)}(A_j) + \theta_j^{(2)} M^{(2)}(A_j))\right\} \\
&\quad = \exp\left\{ - \int_E \int_{S_2} \left|\sum_{j=1}^{d} (s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}) 1_{A_j}(x)\right|^\alpha k(dx, d\mathbf{s})\right\} \\
&\quad = \exp\left\{ - \int_E \int_{S_2} \sum_{j=1}^{d} \left|s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}\right|^\alpha 1_{A_j}(x) k(dx, d\mathbf{s})\right\} \\
&\quad = \prod_{j=1}^{d} \exp\left\{ - \int_E \int_{S_2} \left|s_1 \theta_j^{(1)} + s_2 \theta_j^{(2)}\right|^\alpha 1_{A_j}(x) k(dx, d\mathbf{s})\right\} \\
&\quad = \prod_{j=1}^{d} E \exp\left\{(\theta_j^{(1)} M^{(1)}(A_j) + \theta_j^{(2)} M^{(2)}(A_j))\right\}. \quad ∎
\end{aligned}$$

CLAIM 3. *$\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ is finitely additive.*

PROOF: It is enough to show this for $M^{(1)}$. For any disjoint sets $A_1, \ldots, A_d \in \mathcal{E}_0$, by Proposition 1 we have

$$
\begin{aligned}
&E \exp\Big\{i\theta\Big(M^{(1)}(\bigcup_{j=1}^{d} A_j) - \sum_{j=1}^{d} M^{(1)}(A_j)\Big)\Big\} \\
&= E \exp\Big\{i\Big(\theta M^{(1)}(\bigcup_{j=1}^{d} A_j) - \theta M^{(1)}(A_1) - \cdots - \theta M^{(1)}(A_d)\Big)\Big\} \\
&= \exp\Big\{-\int_E \int_{S_2} \Big|\theta s_1\Big(1_{\bigcup_{j=1}^{d} A_j}(x) - 1_{A_1}(x) - \cdots - 1_{A_d}(x)\Big)\Big|^{\alpha} k(dx, ds)\Big\} \\
&= 1.
\end{aligned}
$$

This means that $M^{(1)}(\bigcup_{j=1}^{d} A_j) - \sum_{j=1}^{d} M^{(1)}(A_j) = 0$ a.s. ∎

CLAIM 4. *$\{(M^{(1)}(A), M^{(2)}(A)),\ A \in \mathcal{E}_0\}$ is σ-additive.*

PROOF: Since this is equivalent to that both $M^{(1)}$ and $M^{(2)}$ are σ-additive, it is enough to show this for $M^{(1)}$. Take any disjoint sequence $\{A_j\}_{j=1}^{\infty} \subset \mathcal{E}_0$ such that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{E}_0$. Then

$$
+\infty > k((\bigcup_{j=1}^{\infty} A_j) \times S_2) = k(\bigcup_{j=1}^{\infty} (A_j \times S_2)) = \sum_{j=1}^{\infty} k(A_j \times S_2).
$$

We have

$$
M^{(1)}(\bigcup_{j=1}^{\infty} A_j) - \sum_{j=1}^{n} M^{(1)}(A_j) = M^{(1)}(\bigcup_{j=n+1}^{\infty} A_j) \sim S_\alpha(\sigma_n, 0, 0),
$$

where by Propositions 1 and 2, we have

$$
\begin{aligned}
\sigma_n^{\alpha} &= \int_E \int_{S_2} |s_1|^{\alpha} 1_{\bigcup_{j=n+1}^{\infty} A_j}(x) k(dx, ds) \\
&= \int_{S_2} |s_1|^{\alpha} \int_E 1_{\bigcup_{j=n+1}^{\infty} A_j}(x) k(dx, ds) \\
&\le k\Big(\Big(\bigcup_{j=n+1}^{\infty} A_j\Big) \times S_2\Big) \\
&= \sum_{j=n+1}^{\infty} k(A_j \times S_2) \downarrow 0
\end{aligned}
$$

as $n \to \infty$. This fact, by Property 2.8.3 in Samorodnitsky and Taqqu (1994), means that

$$M^{(1)}(\bigcup_{j=1}^{\infty} A_j) - \sum_{j=1}^{n} M^{(1)}(A_j) \ \to \ 0 \quad \text{in probability}$$

as $n \to \infty$, which in turn means that

$$M^{(1)}(\bigcup_{j=1}^{\infty} A_j) - \sum_{j=1}^{n} M^{(1)}(A_j) \ \to \ 0 \quad \text{a.s.}$$

as $n \to \infty$, because $\sum_{j=1}^{n} M^{(1)}(A_j)$ has independent summands. ∎

Now combining all the above arguments completes the proof of the existence of complex-valued $S\alpha S$ random measures.

REMARK 2. The arguments given here can be easily extended to show the existence of an $\mathbf{R}^n$-valued symmetric α-stable random measure for any $n \in \mathbf{N}$.

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