THEORETICAL RESULTS AND EMPIRICAL STUDIES ON THE FORECASTING ACCURACY OF THE HOLT'S LINEAR EXPONENTIAL SMOOTHING METHOD

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THEORETICAL RESULTS AND EMPIRICAL STUDIES ON THE FORECASTING ACCURACY OF THE HOLT'S LINEAR EXPONENTIAL SMOOTHING METHOD

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Abstract

The Holt's linear exponential smoothing method has been frequently used to forecast a time series that has a trend. In this paper, we investigate the forecasting accuracy of this method. We give theoretical results on the asymptotic prediction errors for some stochastic processes. Using real-life time series data, we show short-range forecasting performances of this method. Problems related to the range of the smoothing parameters are also discussed.

1. Introduction

Many time series exhibit a broad long run movement which is called a trend. To forecast a time series that has a trend, Holt's linear exponential smoothing method has been widely used in application fields. In this paper we investigate the forecasting accuracy of this method from theoretical and empirical aspects.

Let $\{Y_t\}$, where t is an integer, be a time series that has a trend. Suppose that we have observations $Y_1, Y_2, \cdots, Y_n$ of $\{Y_t\}$ and want to forecast the future value Y_{n+h}, where $h > 0$. Here we consider the Holt's linear exponential smoothing method. In this method, the time

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series $\{Y_t\}$ is assumed to be of the form

$$Y_t = T_t + W_t, \tag{1.1}$$

where W_t represents the noise component and T_t the trend component, which is treated locally as a linear trend. Denote by G_t be the slope of the trend at time point t. Then the forecast of the future value Y_{n+h} is obtained by extrapolating the present trend level T_n with the present slope G_n Since T_n and G_n are unknown, they should be estimated by using the observations $Y_1, Y_2, \cdots, Y_n$. Denote by $\hat{T}_t$ and $\hat{G}_t$ the estimates of T_t and G_t respectively, $t = 1, 2, \cdots, n$. In the Holt's method, the esitmates $\hat{T}_n$ and $\hat{G}_n$ are obtained recursively from the following smoothing algorithm for $t = 1, 2, \cdots, n$:

$$\begin{aligned} \hat{T}_t &= \theta_1 Y_t + (1-\theta_1)(\hat{T}_{t-1} + \hat{G}_{t-1}), \\ \hat{G}_t &= \theta_2(\hat{T}_t - \hat{T}_{t-1}) + (1-\theta_2)\hat{G}_{t-1}, \end{aligned} \tag{1.2}$$

where θ_1 and θ_2 are the smoothing parameters and their values are usually assumed $0 < \theta_1, \theta_2 \leq 1$. Denote by $\hat{Y}_t(h)$ the h-step forecast of Y_{t+h} at time t. Once $\hat{Y}_n$ and $\hat{G}_n$ are obtained, the forecast of Y_{n+h} is given by

$$\hat{Y}_n(h) = \hat{T}_n + h\hat{G}_n. \tag{1.3}$$

In order to employ the smoothing algorithm (1.2) recursively for $t = 1, 2, \cdots, n$, the initial values $\hat{T}_0$ and $\hat{G}_0$ should be given suitably. One way for getting these values is to fit a straight line $a + bt$ to all or the first part of the time series (depending on the characteristic of variation in the trend) by using least squares estimates and take $\hat{T}_0 = a$ and $\hat{G}_0 = b$ (see Abraham and Ledolter (1983)). Another way is to use backcasting (see Ledolter and Abraham (1984)). A simple way is to take $\hat{T}_2 = Y_2$ and $\hat{G}_2 = Y_2 - Y_1$ and use the smoothing algorithm recursively for $t = 3, 4, \cdots, n$ (see Granger and Newbold (1986)). Some valuable suggestions are given in Gardner (1985).

In the smoothing algorithm (1.2), the values of the smoothing parameters θ_1 and θ_2 should be chosen suitably. By using the sample, these values are usually estimated by a grid search to minimize the mean square errors (MSE) of one-step forecasts

$$Q(\theta_1, \theta_2) = \frac{1}{n} \sum_{t=1}^{n} (Y_t - \hat{Y}_{t-1}(1))^2 \tag{1.4}$$

among the range $0 < \theta_1, \theta_2 \leq 1$. Here there is a problem concerning this range. Although this range was recommended by Holt in his original work, such a range has been criticized as arbitray. In fact, McClain and Thomas (1973) showed that the Holt's method is stable over a wider range given as follows:

$$0 < \theta_1 < 2, \quad 0 < \theta_2 < (4 - 2\theta_1)/\theta_1. \tag{1.5}$$

However, it has not been clear at all whether this wider range should be used or not.

While the Holt's method has been thought to be an *ad hoc* forecasting procedure because it bases on little mathematical arguments, this method has turned out to be a popular tool among practitioners to forecast a time series that has a trend. Its popularity is due to not only the simplicity in the forecasting procedure, but also its forecasting performance. Some empirical studies have reported that there is little difference in forecasting accuracy between exponential smoothing and some mathematically sophisticated methodology such as methods based on ARIMA models among others (see Makridakis and Hibon (1979), Makridakis et al. (1984)).

Concerning statistical properties of the Holt's method, however, many aspects remain not clear. In this paper, we investigate the forecasting accuracy of the Holt's method from both treoretical and empirical viewpoints. In Section 2, we give theoretical results on asymptotic prediction errors of this method for some stochastic processes. In Section 3, we show emprical studies on the short-range forecasting performance of the Holt's method by using real life time series data. We also discuss the problem related to the range of the smoothing parameters. Then we give some concluding remarks.

2. Theoretical Results on the Asymptotic Prediction Errors

Consider the h-step forecast error of the Holt's method. Put

$$e_n(h, \boldsymbol{\theta}) = Y_{n+h} - \hat{Y}_n(h),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)'$. Suppose that $\boldsymbol{\theta}$ is estimated by minimizing the MSE of one-step forecast errors using the sample $Y_1, Y_2, \cdots, Y_n$ among the wider range Θ given by (1.5). Put

$$\hat{\boldsymbol{\theta}}_n = \arg \min_{\boldsymbol{\theta} \in \Theta} Q_n(\boldsymbol{\theta}). \tag{2.1}$$

In this section, we give asymptotic results on the h-step forecast error $e(h, \hat{\boldsymbol{\theta}}_n)$ of the Holt's method as n tends to infinity for some stochastic processes. It should be noted that the range Θ includes the usual range $(0, 1] \times (0, 1]$ and the results to be given hold for the case when $\boldsymbol{\theta}$ is estimated among the usual range.

We assume that the time series $\{Y_t\}$ satisfies

$$\nabla^2 Y_t = X_t, \tag{2.2}$$

where $\nabla = 1 - B$, B is the backward shift operator such that $BX_t = X_{t-1}$, and $\{X_t\}$ is a stationary process which can be expresses as

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{2.3}$$

where $\{\psi_j\}$ is a sequence of real numbers, $\sum_{j=0}^{\infty} |\psi_j| < \infty$, and $\{\varepsilon_t\}$ a sequence of independently identically distributed random variables such that $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$ and $E\varepsilon_t^4 = \kappa\sigma^4$, where $0 \leq \kappa < \infty$. The time series that satisfies (2.2) is a typical time series that exhibits a local linear trend.

We assume that the smoothing algorithm (1.2) is employed for $t = 1, 2, \cdots, n$, under the initial values $\hat{T}_0$ and $\hat{G}_0$ which are suitably constructed from the sample such that $E(e_0(1,\theta))^2 < \infty$ and $E(e_1(1,\theta))^2 < \infty$ for $\theta \in \Theta$.

In what follows, the one-step forecast error $e_{t-1}(1,\theta)$ is written as $e_t(\theta)$ for simplicity. We prepare some lemmas that will be needed later.

Lemma 1 *The one-step forecast error $\{e_t(\theta)\}$ satisfies*

$$\nabla^2 Y_t = e_t(\theta) - g_1(\theta)e_{t-1}(\theta) - g_2(\theta)e_{t-2}(\theta) \tag{2.4}$$

for $t \geq 3$, where $g_1(\theta) = 2 - \theta_1 - \theta_1\theta_2$ and $g_2(\theta) = -(1-\theta_1)$.

Proof. See Roberts (1982). □

Lemma 2 *$\{e_t(\theta)\}$ has the following representation which is unique in the mean square sense:*

$$e_t(\theta) = \sum_{j=0}^{t-3} \phi_j(\theta) X_{t-j} + \sum_{j=1}^{2} \alpha_j^{(t)}(\theta) e_j(\theta) \tag{2.5}$$

for $t \geq 3$, where

$$\phi_j(\theta) = \begin{cases} 1 & \text{if } j = 0, \\ \sum_{k=1}^{j} g_k(\theta)\phi_{j-k}(\theta) & \text{if } 1 \leq j \leq 2, \\ \sum_{k=1}^{2} g_k(\theta)\phi_{j-k}(\theta) & \text{if } j \geq 3 \end{cases} \tag{2.6}$$

and

$$\alpha_j^{(t)}(\theta) = \begin{cases} \delta(j,t) & \text{if } t = 1, 2, \\ \sum_{k=1}^{2} g_k(\theta)\alpha_j^{(t-k)}(\theta) & \text{if } t \geq 3 \end{cases} \tag{2.7}$$

for $j = 1, 2$, where $\delta(j,t) = 1$ if $t = j$ and $\delta(j,t) = 0$ if $t \neq j$.

Proof. It is easy to show that the results hold for $t = 3$. Then Lemma 2 can be shown by induction. We omit the details. □

Lemma 3 *For any $\theta \in \Theta$, there exist positive constants M_1, M_2 and r, $0 < r < 1$, such that*

$$|\phi_j(\theta)| \le M_1 j^2 r^j \quad \textit{for} \quad j = 1, 2, \cdots$$

and

$$|\alpha_j^{(t)}(\theta)| \le M_2 t^2 r^t \quad \textit{for} \quad j = 1, 2;\ t = 1, 2, \cdots.$$

Proof. It follows from (2.6) and (2.7) that $\phi_j(\theta)$ and $\alpha_j^{(t)}(\theta)$ satisfy the following difference equations:

$$[1 - g_1(\theta)B - g_2(\theta)B^2]\phi_j(\theta) = 0,$$

$$[1 - g_1(\theta)B - g_2(\theta)B^2]\alpha_j^{(t)}(\theta) = 0, \quad j = 1, 2.$$

It is easy to show that, for any $\theta \in \Theta$,

$$1 - g_1(\theta)z - g_2(\theta)z^2 \ne 0 \quad \text{for} \quad |z| \le 1.$$

Then Lemma 3 follows from Corollary 3.6.1 in Brockwell and Davis (1991). The details are omitted. □

Let $\gamma(h) = E(X_{t+h}X_t)$. We will use '$Z_n \to Z$ a.s.' to denote that $\{Z_n\}$ converges to Z almost surely as n tends to infinity.

Lemma 4 *We have*

$$Q_n(\theta) \to \bar{Q}(\theta) \ \textit{a.s.} \quad \textit{uniformly in } \theta \in \Theta,$$

where

$$\bar{Q}(\theta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \phi_i(\theta)\phi_j(\theta)\gamma(i-j).$$

Proof. Lemma 4 can be shown by using the similar arguments as in Lemma 3 in Chen (1994). The details are omitted. □

Lemma 5 *The h-step forecast error $e_t(h, \theta)$ satisfies*

$$e_t(h, \theta) = \sum_{j=0}^{h-1} v_j(\theta) e_{t+h-j}(\theta),$$

where

$$v_j(\theta) = \begin{cases} 1, & j = 0, \\ \theta_1 + j\theta_1\theta_2, & j = 1, 2, \cdots. \end{cases}$$

Proof. From (1.2), we have

$$\hat{T}_t - \hat{T}_{t-1} + \hat{G}_t - \hat{G}_{t-1} = (\theta_1 + \theta_1\theta_2)e_t(\theta) + \hat{G}_{t-1}.$$

Now consider $e_t(2, \theta)$. We have

$$\begin{aligned} e_t(2,\theta) &= Y_{t+2} - \hat{T}_t - 2\hat{G}_t \\ &= Y_{t+2} - (\hat{T}_{t+1} + \hat{G}_{t+1}) + \hat{T}_{t+1} + \hat{G}_{t+1} - \hat{T}_t - \hat{G}_t - \hat{G}_t \\ &= e_{t+2}(\theta) + (\theta_1 + \theta_1\theta_2)e_{t+1}(\theta). \end{aligned}$$

Hence Lemma 5 holds for $h = 2$. The proof of Lemma 5 is completed by induction. □

Using the above lemmas, now we obtain the following results. We use '$Z_n \Rightarrow F$' to denote that $\{Z_n\}$ converges in law to the distribution F as n tends to infinity. Denote by $N(0, V)$ the normal distribution with mean 0 and variance V.

THEOREM 1 *Suppose that $\bar{Q}(\theta)$ has a minimal point θ_0 in Θ and θ_0 exists uniquely. Then the followings hold:*

(i) $\hat{\theta}_n \rightarrow \theta_0$ *a.s.*

(ii) *If $\theta_0 \in Int(\Theta)$ and, in (2.3), $\{\varepsilon_t\}$ is Gaussian and $\{\psi_j\}$ satisfies $\sum_{j=0}^{\infty} j|\psi_j| < \infty$, then*

$$e_n(h, \hat{\theta}_n) \Rightarrow N(0, \bar{Q}(h, \theta_0)), \quad h = 1, 2, \cdots,$$

where

$$\bar{Q}(h, \theta_0) = \sum_{i=0}^{h-1}\sum_{j=0}^{h-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} v_i(\theta_0)v_j(\theta_0)\phi_k(\theta_0)\phi_l(\theta_0)\gamma(i - j + k - l).$$

Proof. The results can be shown by using the similar arguments as in Theorems 1 and 2 in Chen (1994). □

REMARK. Under suitable conditions, Theorem 1 can be used to give an approximate h-step prediction interval for the Holt's method by considering the sample version of a suitably truncated form of $\bar{Q}(h, \theta_0)$.

3. Empirical Studies

In this section, we show short-range forecasting performance of the Holt's method using real time series data. We also discuss the problem related to the range of the smoothing parameters by showing whether the forecasting accuracy is improved or not when the wider range Θ is used.

Our emprical studies have been performed as follows. For a time series data $Y_1, Y_2, \cdots, Y_n$, we calculated the one-step forecasts $\hat{Y}_{m-1}(1)$, $m = n_0 + 1, \cdots, n$, by taking the first $m-1$ observations $Y_1, \cdots, Y_{m-1}$ as a sample and shifting m from $n_0 + 1$ to n. The smoothing algorithm (1.2) was used for $t = 3, 4, \cdots, m-1$, under the initial values $\hat{T}_2 = Y_2$ and $\hat{G}_2 = Y_2 - Y_1$. The smoothing parameters were estimated among the usual range $(0, 1] \times (0, 1]$ and the wider range Θ. Then the forecasting accuracies of the one-step forecasts corresponding to these two ranges were compared. The following three summary statistics can be used as a benchmark for evaluating the forecasting accuracy:

$$
\begin{aligned}
RMSE &= \sqrt{\frac{1}{n-n_0} \sum_{m=n_0+1}^{n} [e_m(\hat{\theta}_{m-1})]^2}, \\
MAE &= \frac{1}{n-n_0} \sum_{m=n_0+1}^{n} |e_m(\hat{\theta}_{m-1})|, \\
MAPE &= \frac{1}{n-n_0} \sum_{m=n_0+1}^{n} \left| \frac{e_m(\hat{\theta}_{m-1})}{Y_m} \right| \times 100\%.
\end{aligned}
$$

Our empirical studies were performed for the following data and the results are as follows:

(i) Case 1: *Quarterly Iowa Nonfarm Income*

This data was collected from Abraham and Ledolter (1983) (see Series 1, pp.419), which is the quarterly Iowa nonfarm income for 1948–1979. Figure 1-1 is a plot of the data. The time series shows a smoothly increasing trend.

There are 128 observations in the data. We calculated the one-step forecasts of $Y_{41}, \cdots, Y_{128}$ by the Holt's method based on the usual range and the wider range. Estimated values of the smoothing parameters among the wider range coincide with those among the usual range. So the forecasts based on these two ranges are the same with each other. Figure 1-2 gives the plot of the one-step forecasts, where 'Observed' is the plot of the observations, 'Holt' the plot of the one-step forecasts by the Holt's method. $RMSE$, MAE and $MAPE$ of these forecasts are given in Table 1. Estimated values for the smoothing parameters θ_1 and θ_2 are plotted in Figures 1-3 and 1-4 respectively. For this data, the accuracy of one-step forecasts by the Holt's method is very good.

(ii) Case 2: *Annual Populations of Okinawa*

Figure 2-1 gives a plot of the annual populations of Okinawa from 1946 to 1995. There is a missing value in the data which corresponds to the year 1951. We treated this missing value by taking the average of the values of 1950 and 1952. It seems that there are some structural changes in the trend, possibly in 1971 for example.

We calculated the one-step forecasts of $Y_{16}, \cdots, Y_{50}$, that is, the forecasts of 1961–1995, by the Holt's method using the usual range and the wider range of the smoothing parameters. A plot of these forecasts is given in Figure 2-2, where 'Holt (usual)' is the plot of the one-step forecasts using the usual range and 'Holt (wider)' the one-step forecasts using the wider range Θ. $RMSE$, MAE and $MAPE$ are listed in Table 2. Estimated values of θ_1 and θ_2 in these forecasts are plotted in Figures 2-3 and 2-4 respectively. It can be seen that, when the wider range is used, estimated values of the smoothing parameter θ_1 go slightly beyond the usual range. While both the one-step forecasts using the usual range and the wider range perform very well, the latter appears to be slightly worse than the former.

(iii) Case 3: *University of Iowa Student Enrollments*

This data was from Abraham and Ledolter (1983) (see Table 3.11, pp.116) and was used by these authors to show the forecasting performance of the Brown's double exponential smoothing method. Figure 3-1 gives a plot of the data—the annual student enrollments (fall and spring semesters combined) at the University of Iowa, from 1951/52 to 1979/80. It appears that the trend of the time series changes fiercely over the time.

There are 29 observations in the data. We calculated the one-step forecasts of $Y_{11}, \cdots, Y_{29}$ by the Holt's method using the usual as well as the wider ranges of the smoothing parameters. The forecasts are plotted in Figure 3-2. $RMSE$, MAE and $MAPE$ are listed in Table 3. The overall forecasting performance for this data set is satisfactory, though it is somewhat worse than that for those two data sets given in the above. The estimated values of θ_1 and θ_2 among the usual range and the wider range Θ are plotted in Figures 3-3 and 3-4 respectively. In the case when the wider range is used, the estimated values of the smoothing parameters, especially θ_2, largely go beyond the usual range. In this case, there is a tendency to overshoot sudden changes in the trend, as can be seen in Figure 3-2. Table 3 shows that the forecasting accuracy of the Holt's method using the wider range gets worse than that using the usual range.

(iv) Case 4: *Yearly total farm loan demand in the U.S.A.*

This data was from Nazem (1988), where it was uased to show applications of ARIMA models in forecasting (see §14.1). Figure 4-1 gives a plot of the data: yearly total farm loan demand in the U.S.A.

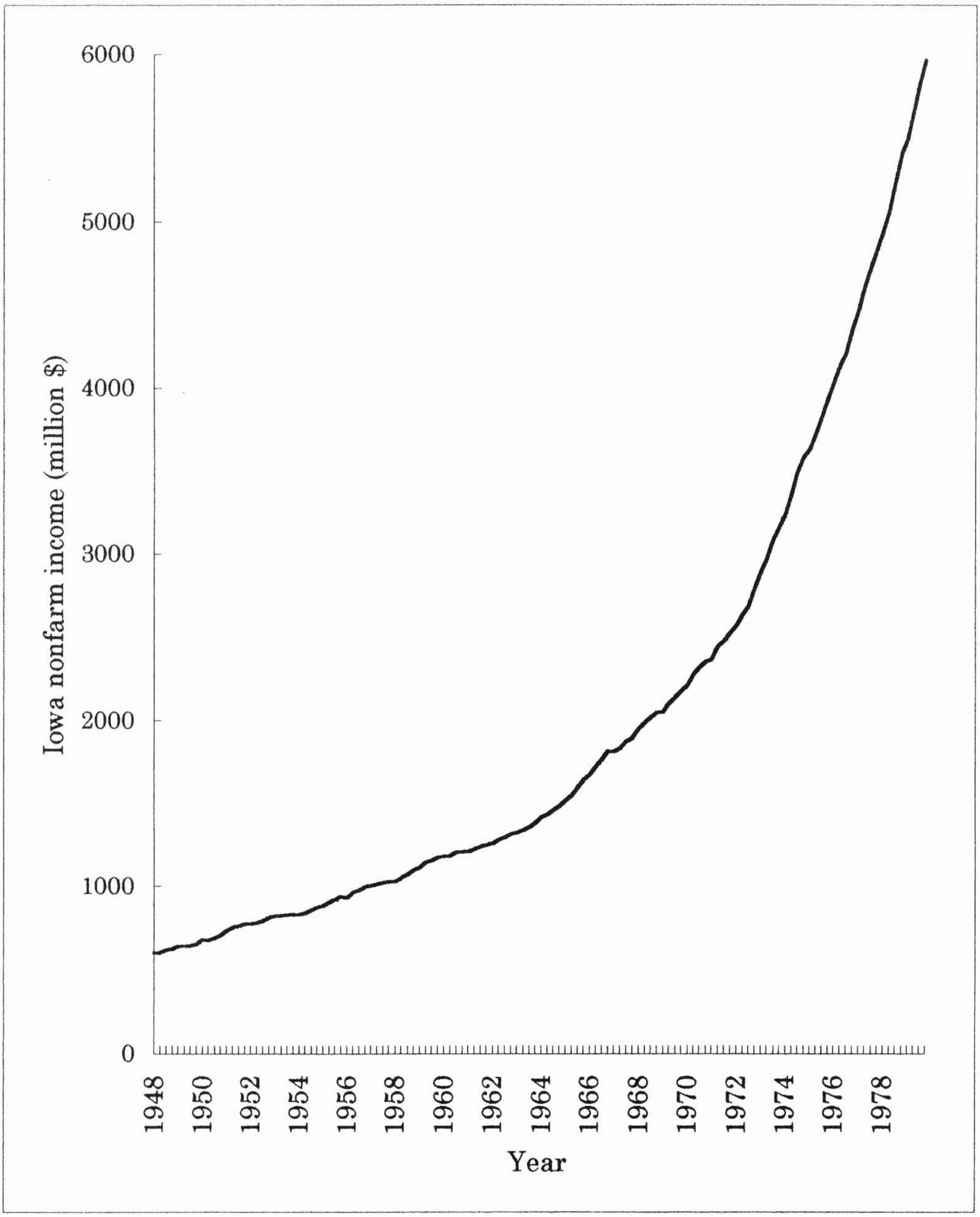

Figure 1-1. Quarterly Iowa nonfarm income (in millions of dollars), first quarter 1948 to fourth quarter 1979.

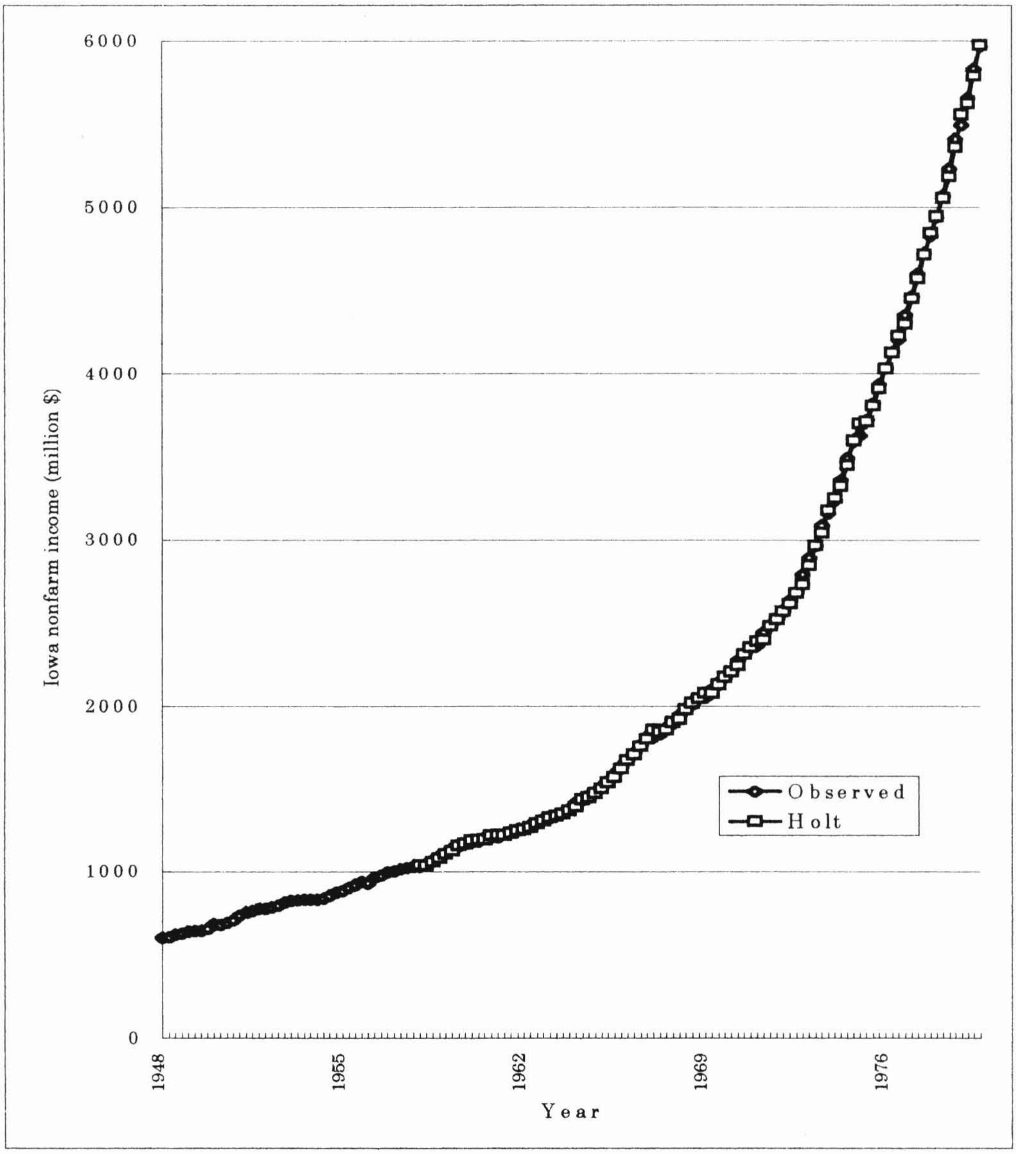

Figure 1-2. One-step forecasts of quarterly Iowa nonfarm income, first quarter 1958 to fourth quarter 1979.

Table 1. RMSE, MAE and MAPE of one-step forecasts in Figure 1-1.

	RMSE	MAE	MAPE (%)
Holt's method	2.42	16.14	0.67

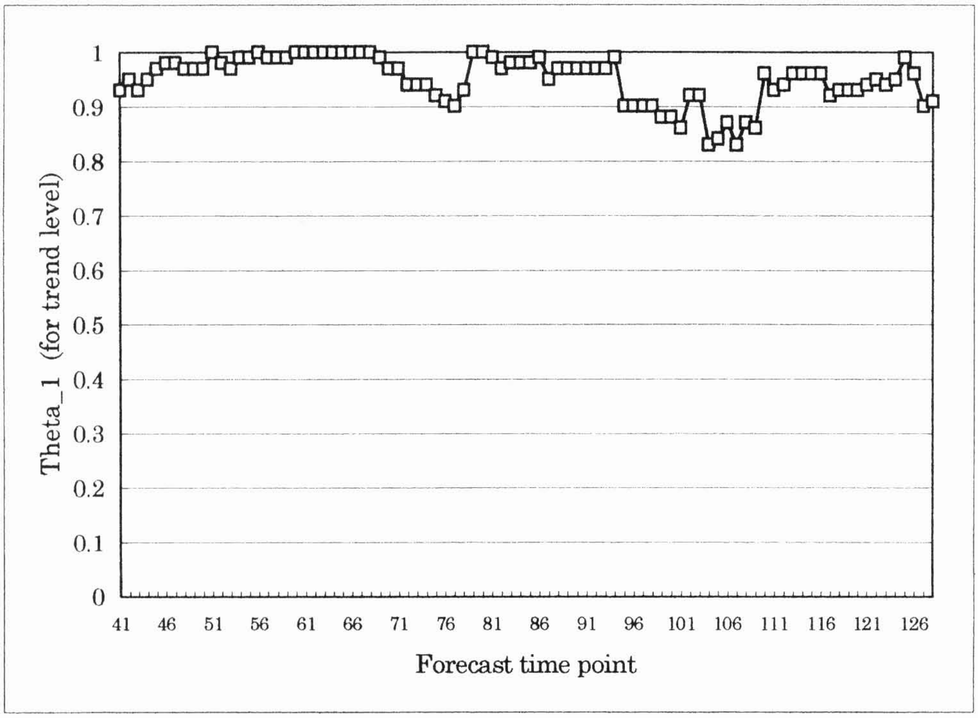

Figure 1-3. Estimated values of θ_1 for forecasts in Figure 1-2.

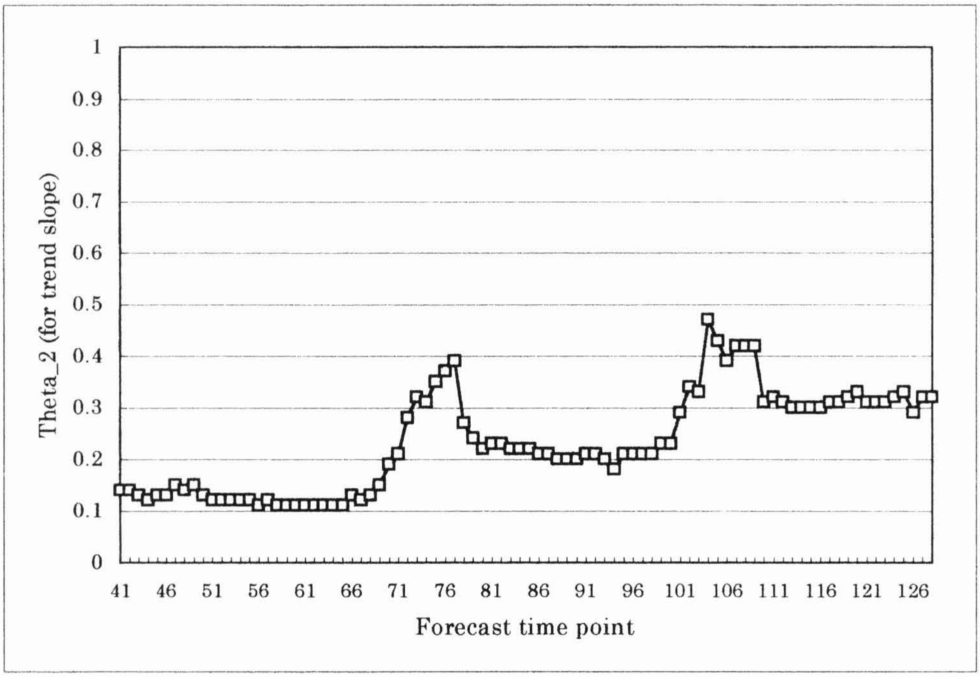

Figure 1-4. Estimated values of θ_2 for forecasts in Figure 1-2.

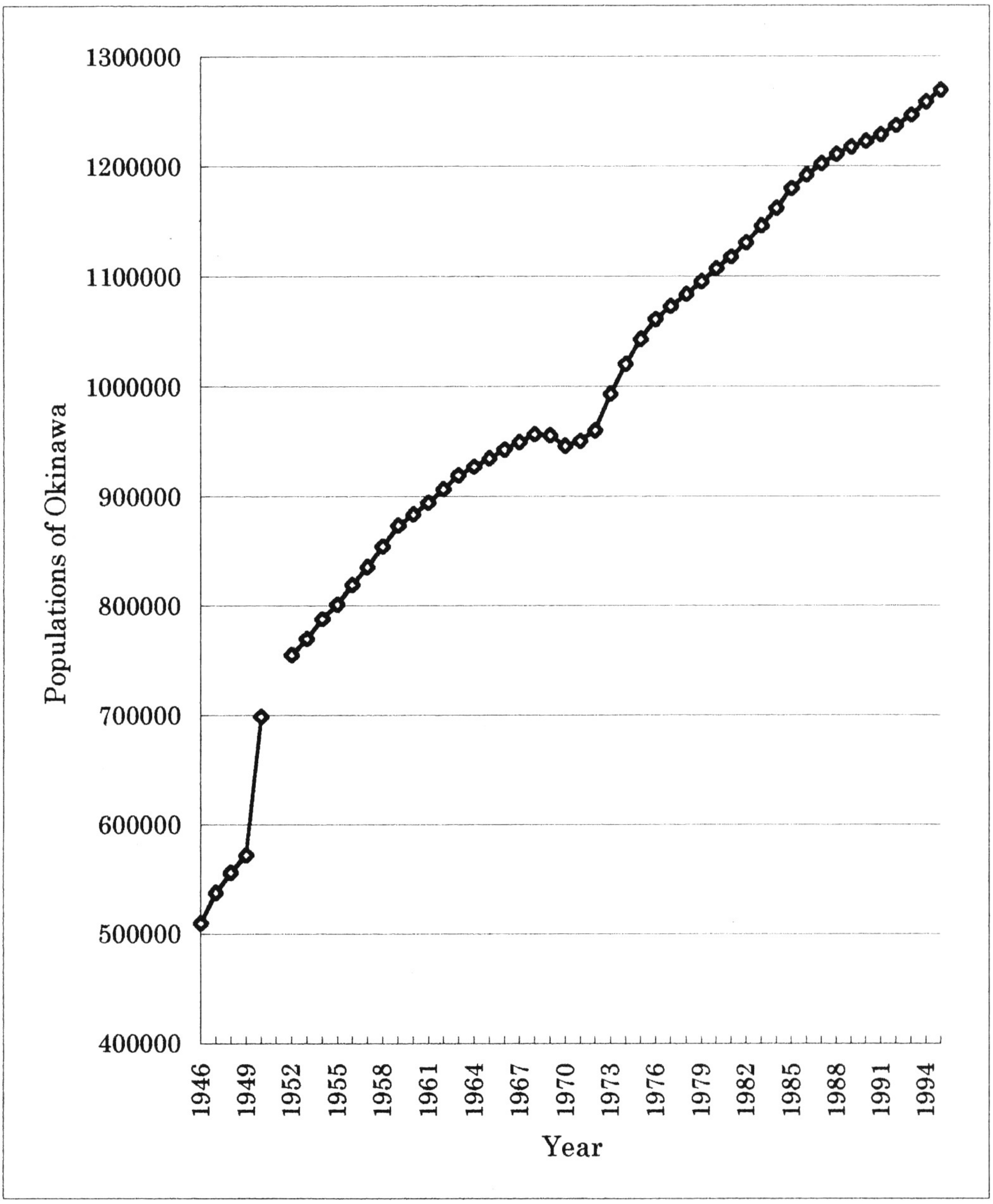

Figure 2-1. Populations of Okinawa, 1946 to 1995.

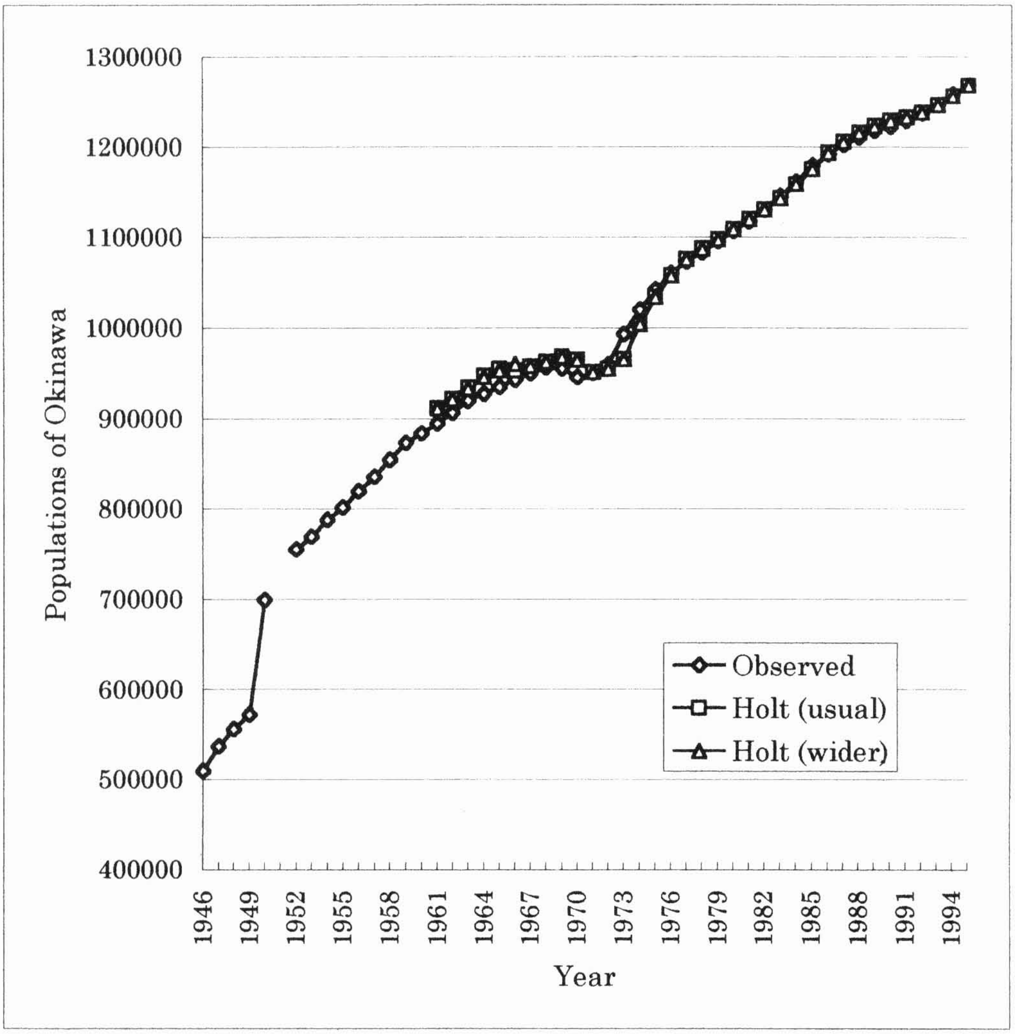

Figure 2-2. One-step forecasts of populations of Okinawa, 1961 to 1995.

Table 2. RMSE, MAE and MAPE of one-step forecasts in Figure 2-2.

	RMSE	MAE	MAPE (%)
Holt's method (usual)	1727.61	7477.74	0.754
Holt's method (wider)	1745.44	7511.27	0.760

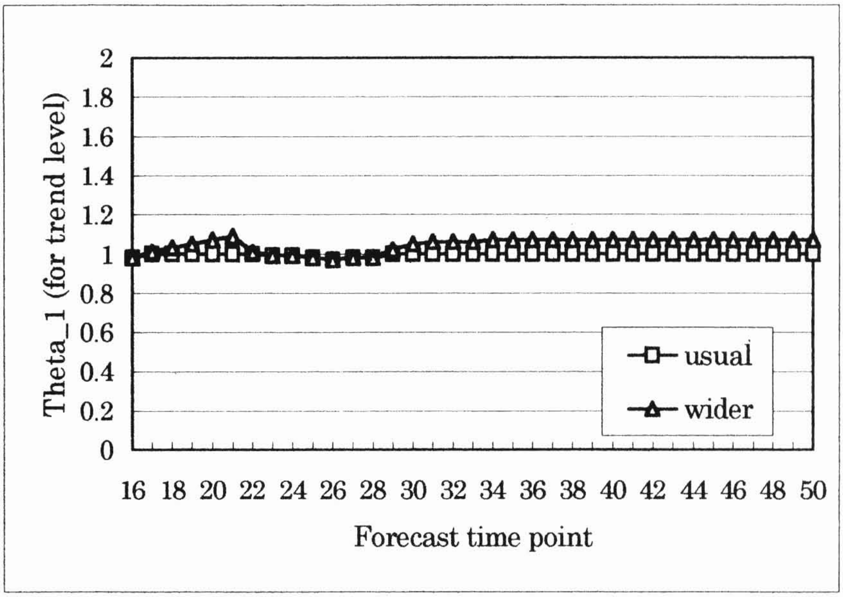

Figure 2-3. Estimated values of θ_1 for forecasts in Figure 2-2 using the usual range and the wider range.

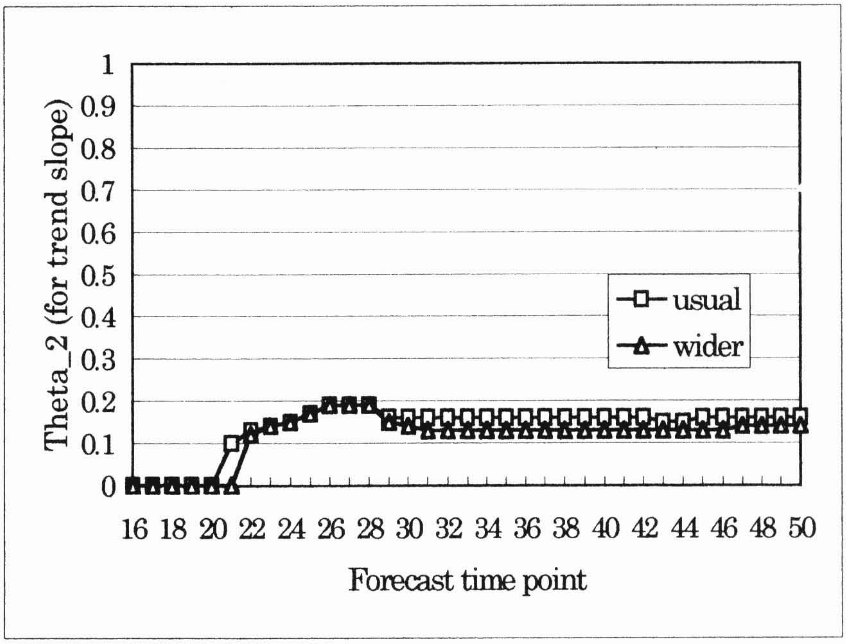

Figure 2-4. Estimated values of θ_2 for forecasts in Figure 2-2 using the usual range and the wider range.

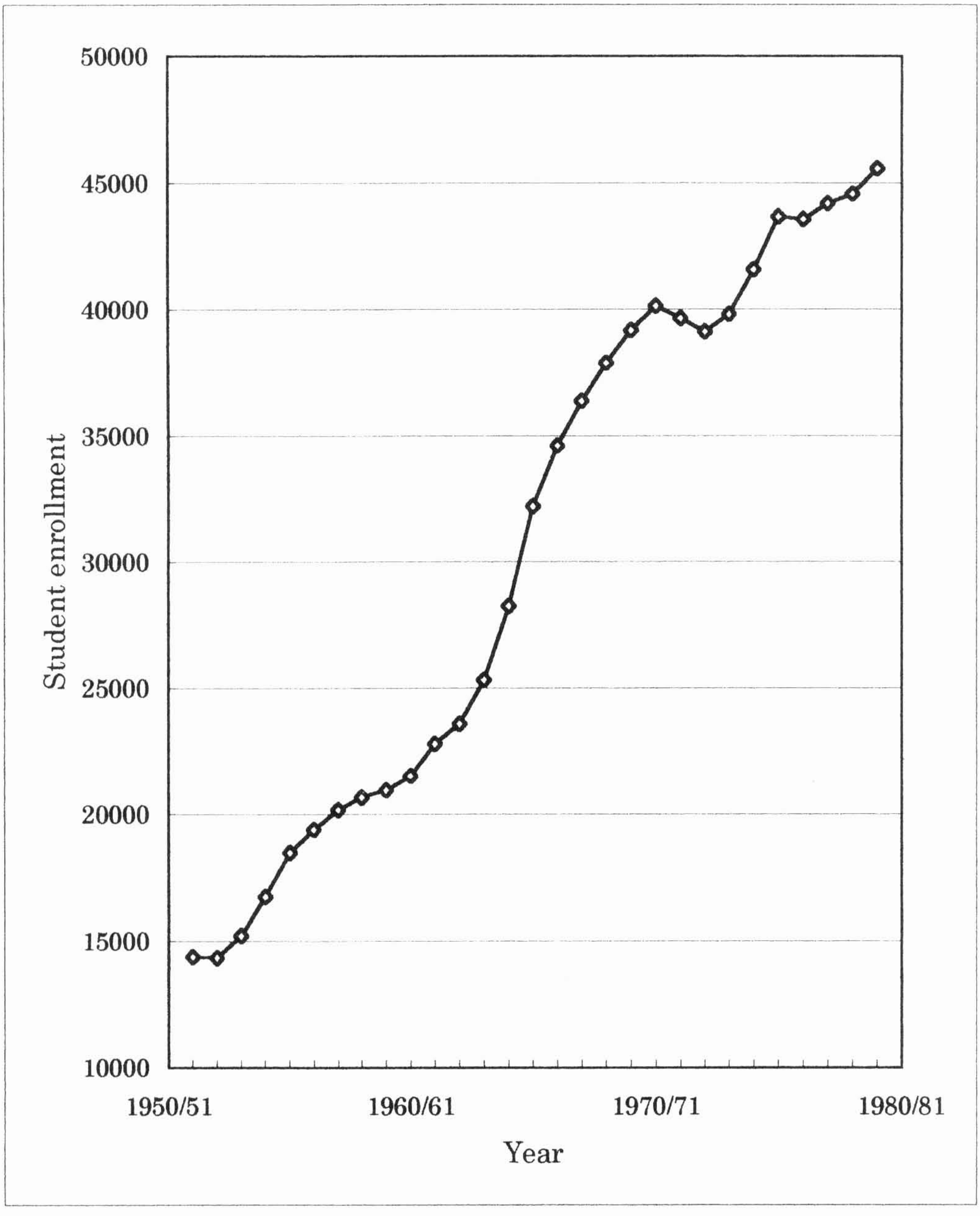

Figure 3-1. University of Iowa student enrollment, 1951/52 to 1979/80.

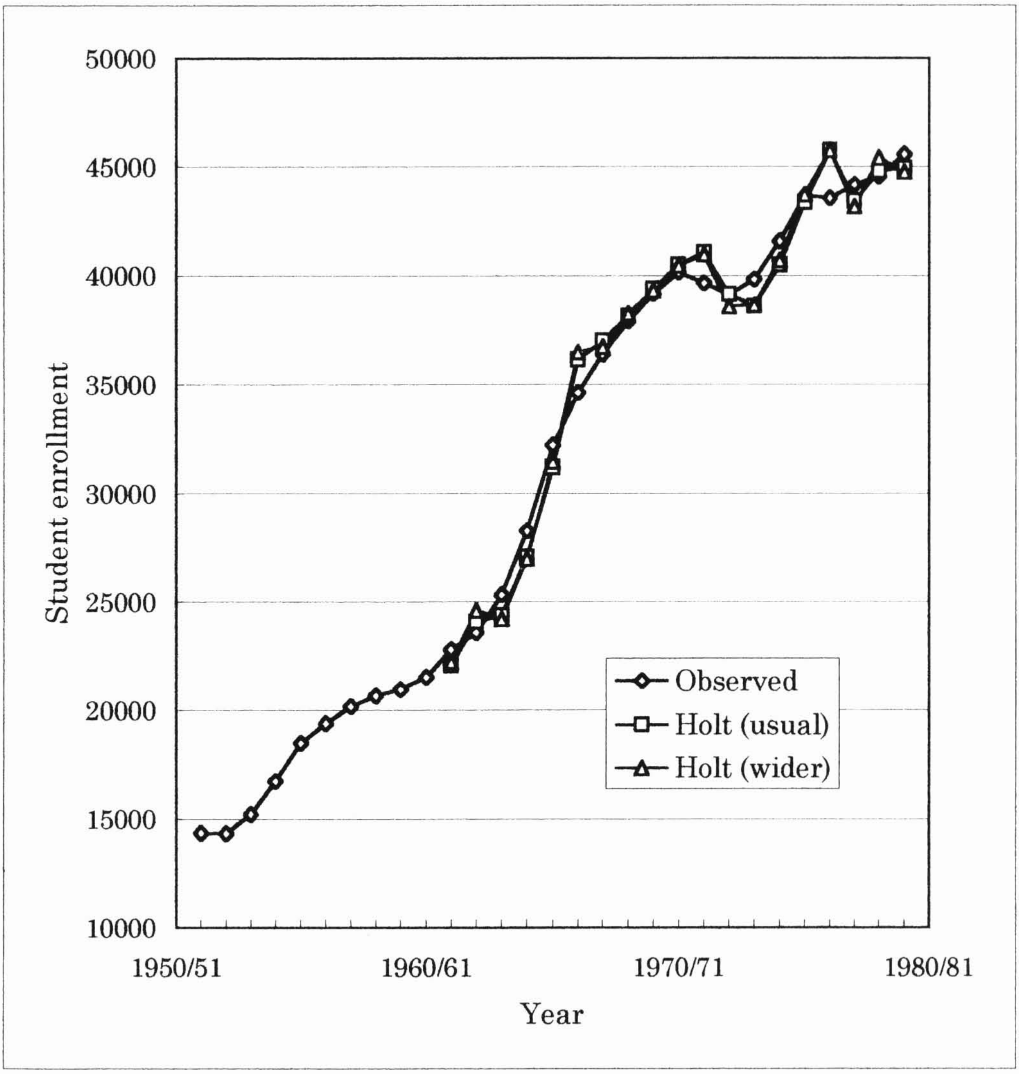

Figure 3-2. One-step forecasts for University of Iowa student enrollment, 1961/62 to 1979/80.

Table 3. RMSE, MAE and MAPE of one-step forecasts in Figure 3-2.

	RMSE	MAE	MAPE (%)
Holt's method (usual)	221.83	801.71	2.28
Holt's method (wider)	237.60	878.94	2.51

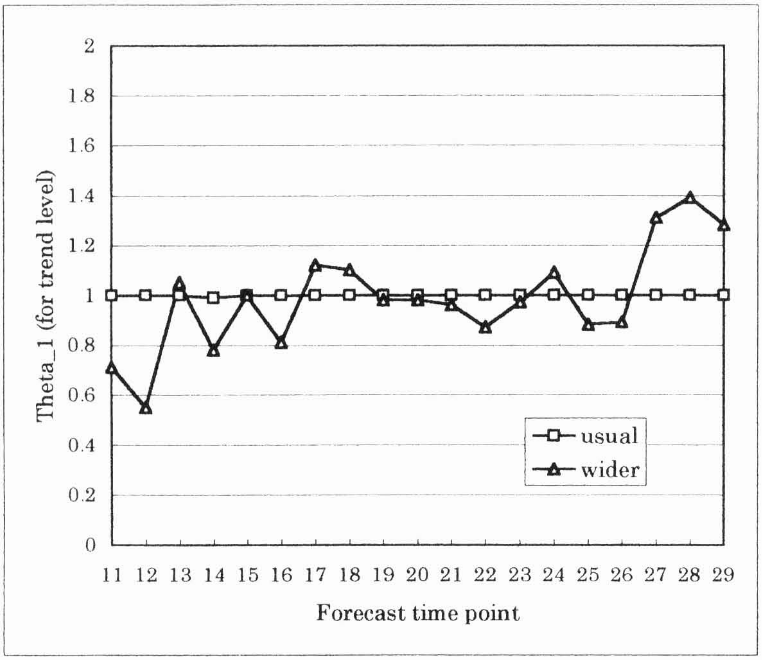

Figure 3-3. Estimated values of θ_1 for forecasts in Figure 3-2 using the usual and the wider range.

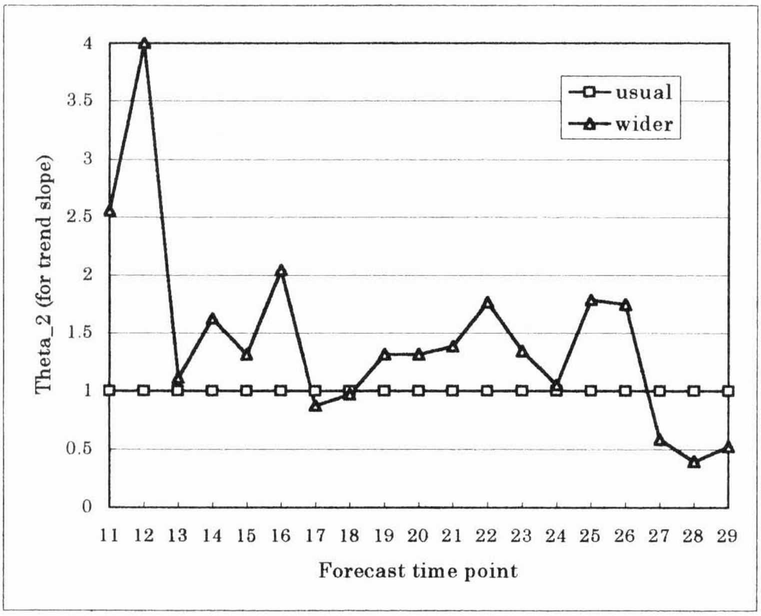

Figure 3-4. Estimated values of θ_2 for forecasts in Figure 3-2 using the usual and the wider range.

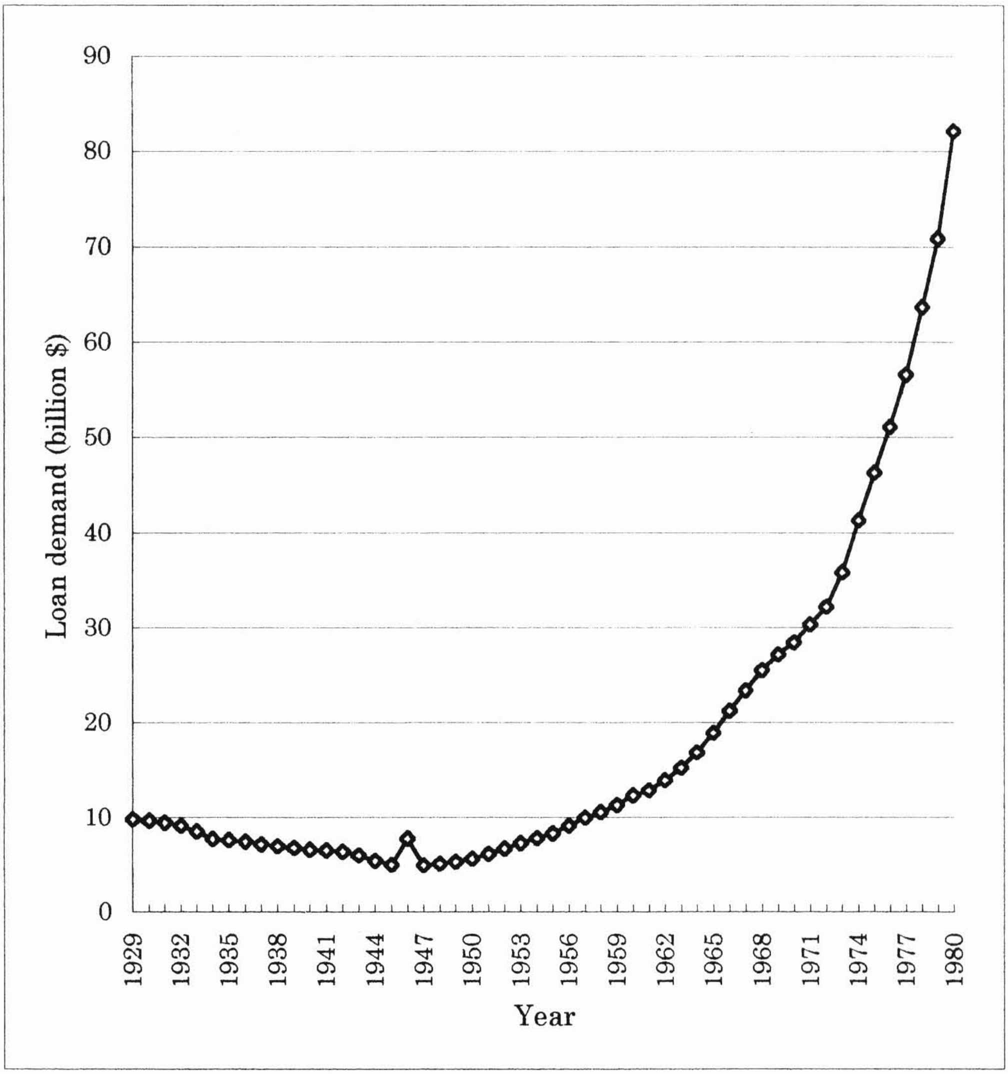

Figure 4-1. Yearly total farm loan demand in the U.S.A., 1929-1980.

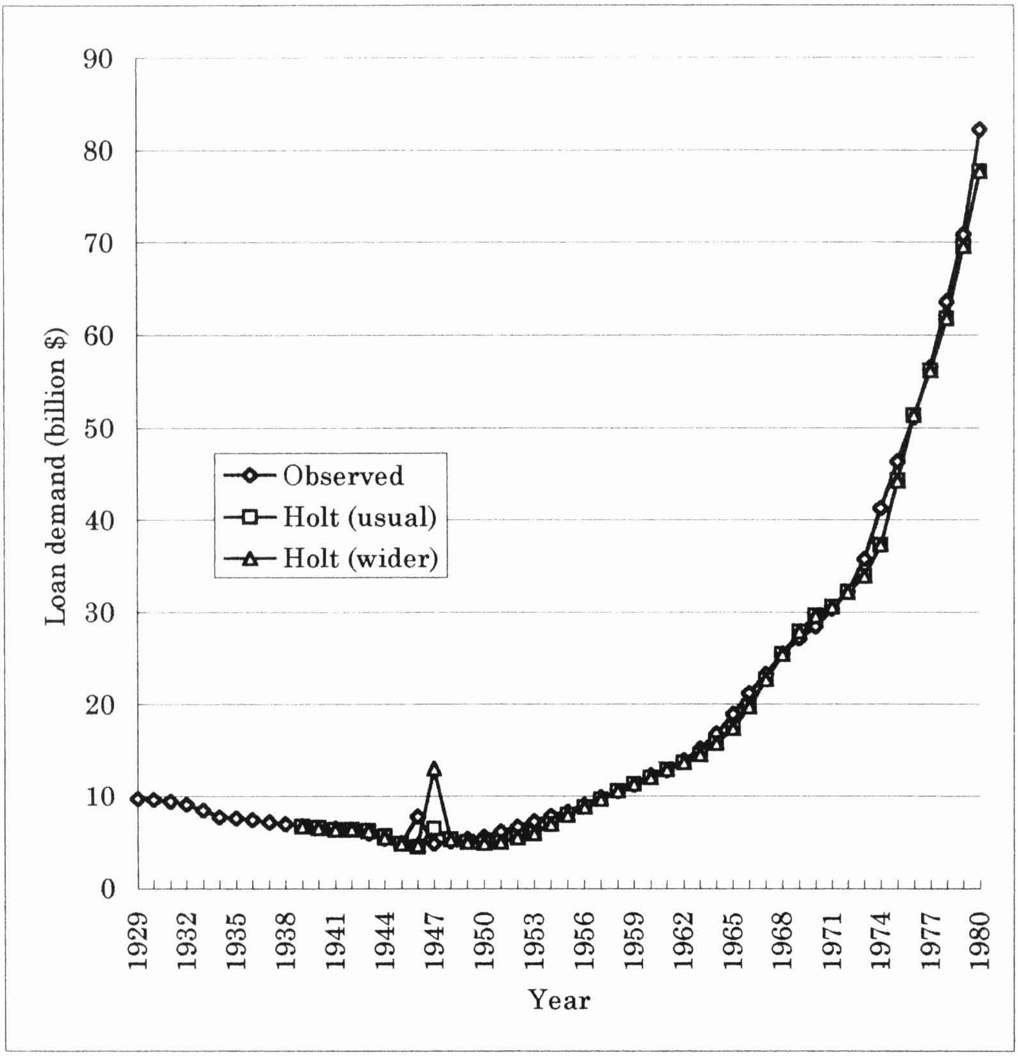

Figure 4-2. one-step forecasts of yearly total farm loan demand in the U.S.A., 1939 to 1980.

Table 4. RMSE, MAE and MAPE of one-step forecasts in Figure 4-2.

	RMSE	MAE	MAPE (%)
Holt (usual)	0.207	0.869	6.120
Holt (wider)	0.279	1.018	9.205

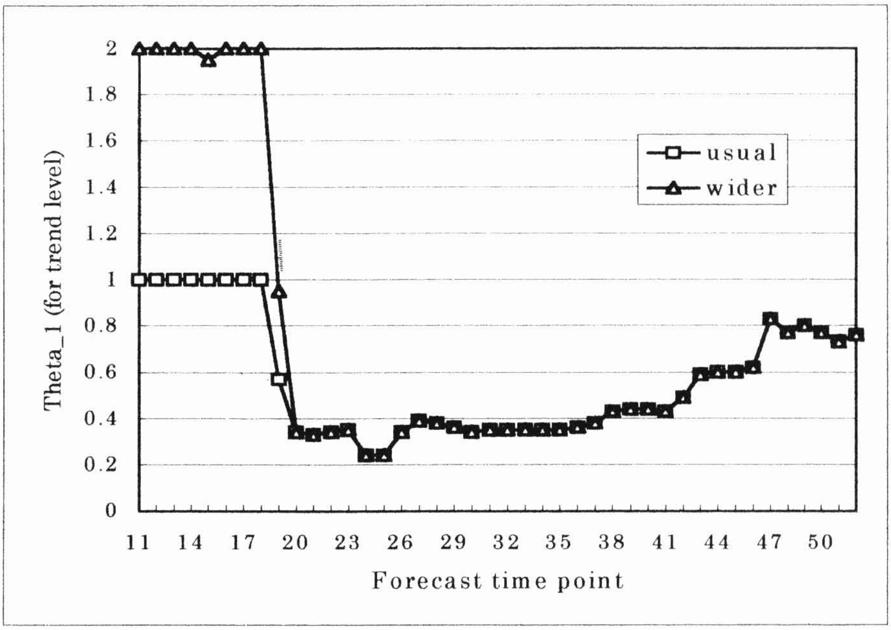

Figure 4-3. Estimated values of θ_1 for forecasts in Figure 4-2 using the usual range and the wider range.

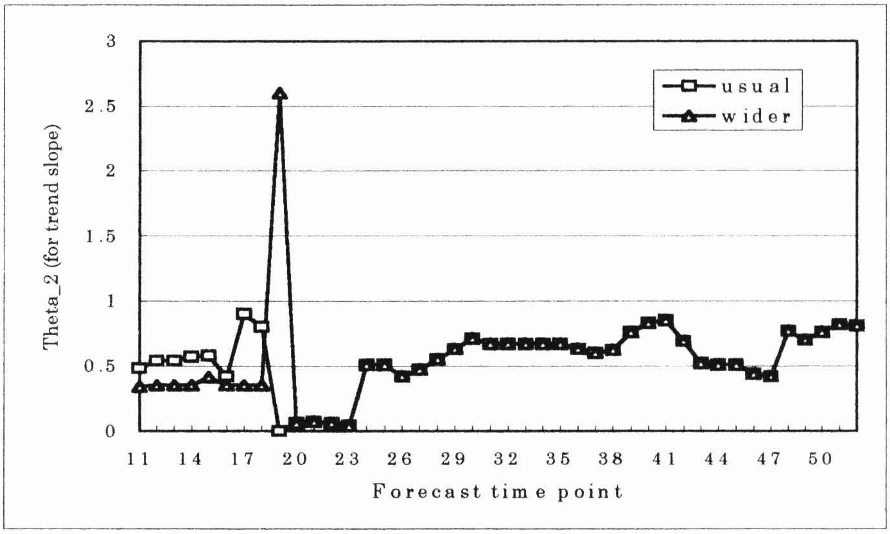

Figure 4-4. Estimated values of θ_2 for forecasts in Figure 4-3 using the usual range and the wider range.

from 1929 to 1980. This series shows a smooth trend except in 1946, where the obervation appears like an outlier.

The one-step forecasts of $Y_{11}, \cdots, Y_{52}$ were calculated using the Holt's method based on the usual range and the wider range. The forecasts and the actual values are shown in Figure 4-2. $RMSE$, MAE and $MAPE$ are listed in Table 4. The corresponding estimated values of θ_1 and θ_2 in those forecasts are shown in Figures 4-3 and 4-4 respectively. For this data, the Holt's method provides satisfactory one-step forecasts. However, the one-step forecast for 1947 using the wider range largely exeeds the actual value. The reason is as follows: when the wider range is used, the abnormal value in 1946 causes the value of θ_2 used to forecast 1947 going largely beyond the usual range (see Figure 4-4), and then the large value of θ_2 overshoots the sudden change in the trend due to the abnormal value and causes the forecast bad.

4. Concluding Remarks

- We have given theoretical results on the asymptotic forecast errors of the Holt's method for some stochastic processes. These results are helpful for evaluating the forecasting accuracy of the Holt's method from a theoretical point of view.

- Our limited empirical studies show that the Holt's method is very satisfactory for short-range forecasting of a time series that has a trend.

- Concerning the problem of the range of the smoothing parameters, our empirical studies show that there is no effects on improving the forecasting accuracy by using the wider range suggested by some previous works. When the series involes a smooth trend (see Case 1), the estimated values of the smoothing parameters do not go beyond the usual range. On the other hand, in the case when the trend in the series involves sudden changes (see Cases 2, 3 and 4), the estimated values of the smoothing parameters go slightly or largely beyond the usual range, depending on the characteristics of changes in the trend. When the estimated values of the smoothing parameters, especially of θ_2, go largely beyond the usual range, there is a tendency to overshoot the sudden changes in the trend and the forecasting accuracy using the wider range gets worse than that using the usual range.

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