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A Splitting Condition for Metabelian Groups

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1. Let G be a group and N a normal subgroup of G . If G contains a subgroup C such that $G=NC$ and $N \cap C=1$, then we shall say G splits over N and C is a complement of N in G .

The purpose of this paper is to prove the following theorem concerning a splitting condition for metabelian groups (a group G is called metabelian, if the commutator subgroup G' is Abelian) over its commutator subgroup.

Theorem 1. *Let G be a metabelian group whose commutator subgroup G' contains no non-trivial central elements. Suppose further that all ascending and descending chains of normal (relative to G) subgroups of G' are finite. Then (i) G splits over G' and (ii) the complements of G' in G form a conjugate class.*

Theorem 1 generalizes Theorem 3.3 and a part of Theorem 3.5 of Newman [3]. Another condition is given in Corollary of the Principal Ideal Theorem in [4, p.179].

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2. Let G be a group and $x, y, z \in G$. The commutators satisfy the following well-known identities:

$$x[x, y] = x^y = [y, x^{-1}]x, \quad (1)$$

$$[x, y]^{-1} = [y, x], \quad (2)$$

$$[xz, y] = [x, y]^z [z, y], \quad (3)$$

$$[x, yz] = [x, z][x, y]^z. \quad (4)$$

If z belongs to the centralizer $C(G')$ of the commutator subgroup G' , then (3) and (4) become, respectively,

$$[xz, y] = [x, y][z, y] \quad (5)$$

$$[x, yz] = [x, z][x, y]. \quad (6)$$

Lemma 1. *Let G be a metabelian group. Then, for $a \in G', x, y \in G$,*

$$(i) \quad [a, x] = [a, x^y],$$

$$(ii) \quad [a, x, y] = [a, y, x].$$

Proof. The normality and the Abelianess of G' are used repeatedly in the proof.

$$(i) \quad \begin{aligned} [a, x^y] &= a^{-1}y^{-1}x^{-1}yay^{-1}xy \\ &= a^{-1}x^{-1}(xy^{-1}x^{-1}y)ay^{-1}xy \\ &= a^{-1}x^{-1}a(xy^{-1}x^{-1}y)y^{-1}xy \\ &= a^{-1}x^{-1}ax = [a, x]. \end{aligned}$$

$$(ii) \quad \begin{aligned} [a, x, y] &= [x, a]y^{-1}[a, x]y \\ &= (x^{-1}a^{-1}xa)y^{-1}(a^{-1}x^{-1}ax)y \end{aligned}$$

$$\begin{aligned}
&=x^{-1}a^{-1}x(ay^{-1}a^{-1}y)y^{-1}x^{-1}axy \\
&=[a^{-1},y]x^{-1}a^{-1}(xy^{-1}x^{-1})a(xyx^{-1})x \\
&=[y,a]x^{-1}[a,y^{x^{-1}}]x \\
&=[y,a]x^{-1}[a,y]x \text{ (by (i))} \\
&=[a,y,x].
\end{aligned}$$

Lemma 2. *Let G be a metabelian group with $G' \cap Z(G) = 1$, where $Z(G)$ is the center of G . Let N be a minimal normal subgroup of G , contained in G' , and let x be an element of G but not contained in the centralizer $C(N)$ of N . Then the mapping $\phi: a\phi = [x, a]$, $a \in N$, (and, similarly, the mapping $\psi: a\psi = [a, x]$, $a \in N$) is an automorphism of N .*

Proof. By a result of Newman [3, Lemma 3.2, p.356], $N\phi$ and $C(x) \cap N$ are normal subgroups of G . Hence, by the minimality of N , $N\phi = N$ and $C(x) \cap N = 1$. Since G' is Abelian, it follows at once from (6) that ϕ is an automorphism of N .

The following lemma is a generalization of Theorem 3.3 and a part of Theorem 3.5 of Newman [3]. However the proof is those of Newman with slight modification.

Lemma 3. *Let G be a metabelian group with $G' \cap Z(G) = 1$. If G' is a minimal normal subgroup of G , then (i) G splits over G' and (ii) the complements of G' are maximal subgroups and they form a conjugate class.*

Proof. (i) Let x be an element of G not belonging to the centralizer $C(G')$ of G' , then, by Lemma 2, the mapping: $a \rightarrow [x, a]$, $a \in G'$, is an automorphism of G' . Hence, for every element g of G , there is one, and only one, element a in G' , such that $[x, a] = [g, x]$ and, by (6) and (2),

$$[x, ga] = [x, a][x, g] = 1.$$

That is, there is exactly one element of the centralizer $C(x)$ of x in each coset of G' in G . Thus

$$G = G'C(x) \text{ and } G' \cap C(x) = 1.$$

(ii) Let C be a complement of G' . Let M be a subgroup of G , which contains C properly. Then, by the modular law, $M = (G' \cap M)C$. Since $G' \cap M \leq G'$ and G' is Abelian and normal in G , $G' \cap M$ is normal in G . Now, by the minimality of G' , $G' \cap M = G'$, and hence $M = G$. Therefore C is maximal.

Let C_1 and C_2 be two complements of G' in G . Let x, y be elements of C_1 and C_2 , respectively, which are not in $C(G')$ (such elements exist, since $G' \cap Z(G) = 1$). Then, since C_1 and C_2 are Abelian and maximal, $C_1 = C(x)$ and $C_2 = C(y)$ (centralizers of x and y , respectively). The mappings: $a \rightarrow [x, a]$ and $b \rightarrow [b, y]$, $a, b \in G'$, are, by Lemma 2, automorphisms of G' , and hence the mapping: $a \rightarrow [x, a, y]$, $a \in G'$, is an automorphism of G' . Thus there is a unique element $a \in G'$, such that $[x, a, y] = [y, x]$. Hence, by (1) and (5),

$$\begin{aligned}
[x^a, y] &= [x[x, a], y] \\
&= [x, y][x, a, y] \\
&= 1,
\end{aligned}$$

so that $x^a \in C_2$. Moreover, x^a does not belong to $C(G')$. For, otherwise, x would

be contained in $C(G')$, a contradiction. Now $x^a \notin C(G')$ implies, by a previous remark, that $C_2 = C(x^a)$, the centralizer of x^a . Therefore C_2 is conjugate to C_1 .

3. We now prove our theorem. If $G' = 1$, then the theorem is trivially true. On the other hand, if G' is a minimal normal subgroup of G , then the conclusions follow from Lemma 3.

Therefore suppose that G' is not a minimal normal subgroup of G . Let N be a minimal normal subgroup of G , which is contained in G' (there is such N by our hypothesis on the descending chains of normal subgroups in G'). We consider G/N . Clearly $(G/N)' = G'/N$. We first show that

$$G'/N \cap Z(G/N) = 1.$$

Deny. Then there is an element a in G' , but not in N , such that Na is a central element of G/N , that is, $[a, g] \in N$ for all g in G . Let $A = N\langle a \rangle$, then $A \leq G'$. Let x be an element of G , not in the centralizer $C(N)$ of N (there is such x since $G' \cap Z(G) = 1$). Then the mapping $\phi: b \rightarrow [b, x]$, $b \in A$, is a homomorphism of A onto N . For, by (5),

$$[b, x] = [na^i, x] = [n, x] [a^i, x] = [n, x] [a, x]^i \in N,$$

and $[b_1 b_2, x] = [b_1, x] [b_2, x]$,

where $b, b_1, b_2 \in A$ and $n \in N$. Thus ϕ is a homomorphism from A into N . But, by Lemma 2, $N\phi = N$ and hence ϕ is onto. Let K be the kernel of ϕ , then $A = NK$. Moreover, since each element of K commutes with x , whereas no elements, except the identity, of N commute with x , we have $N \cap K = 1$. Now, $a = nk$ for some $n \in N$ and $k \in K, k \neq 1$ (since $a \notin N$). Also, by (5),

$$[a, g] = [nk, g] = [n, g] [k, g],$$

which is in N . Hence $[k, g]$ is in N . It suffices to show that $[k, g] = 1$ for all g in G , i.e., $k \in Z(G)$. For, then, since $k \in G'$, $G' \cap Z(G) \neq 1$, contradicting our assumption. (We could have taken $a = k$, to begin with.) Since k commutes with x , $[k, x] = 1$, and hence $[k, x, g] = 1$, for all g in G . By Lemma 1 (ii),

$$1 = [k, x, g] = [k, g, x].$$

But, by Lemma 2, the mapping: $n \rightarrow [n, x]$, $n \in N$, is an automorphism of N . Therefore $[k, g] = 1$.

(i) Now, G/N is, of course, metabelian and $G'/N \cap Z(G/N) = 1$. Moreover, our hypothesis on the ascending and descending chains of normal subgroups in G' , which also carries over to G'/N , implies that the length of a chief series of G between 1 and G' is finite. Thus, by induction on its length, there is a subgroup H of G , such that $G/N = G'/N \cdot H/N$, $G'/N \cap H/N = 1$, that is, $G = G'H$, $G' \cap H \leq N$. Since $H' \leq G' \cap H \leq G'$, which is Abelian, and H' is normal in H , H' is normal in G . Hence, either $H' = N$ or $H' = 1$, by the minimality of N . If $H' = 1$ (and hence H is Abelian), then each element of N commutes elementwise with both H and G' and hence is in the center $Z(G)$ of G , a contradiction (since $N \leq G'$). Thus, $H' = N$. If there were a normal subgroup N_1 of H , such that $1 < N_1 < N$, then N_1 would be normal in G , contradicting the minimality of N . Hence $N (=H')$ is minimal normal in H . Also, $N \cap Z(H) = 1$ is clear. Therefore, by Lemma 3 (i),

H possesses a subgroup C such that $H=NC$ and $N\cap C=1$. We now have

$$G=G'H=G'NC=G'C$$

and

$$G'\cap C\leq G'\cap H\cap C\leq N\cap C=1.$$

(ii) Let C_1 and C_2 be two complements of G' . Then $N C_1/N$ and $N C_2/N$ are complements of G'/N in G/N . Hence, as in (i), by induction, they are conjugate, that is, there is an element x in G , such that

$$H=NC_1=N(x^{-1}C_2x).$$

Since C_1 and $x^{-1}C_2x$ are Abelian and $N\cap Z(H)=1$, $N\cap C_1=N\cap(x^{-1}C_2x)=1$.

Hence C_1 and $x^{-1}C_2x$ are complements of $N(=H')$ in H . Since N is minimal normal in H , by Lemma 3 (ii), they are conjugate. In other words, there is an element y in H , such that

$$C_1=(xy)^{-1}C_2(xy).$$

This completes the proof.

4. It follows, as an immediate consequence, from Theorem 1 that if G is finite metabelian group whose commutator subgroup G' contains no non-trivial central elements, i. e., $G'\cap Z(G)=1$, then G splits over G' . On the other hand, the converse also holds. The following lemma is a slightly more general case.

Lemma 4. *Let G be a finite metabelian group and G' its commutator subgroup. If G possesses an Abelian subgroup A such that $G=G'A$, then (i) G' contains no non-trivial central elements and (ii) A is a complement of G' in G .*

Proof. We have

$$[G',G]=[G',G'A]=[G',A]$$

Hence, by a theorem of Ito [2, Satz 1, p. 400], $[G',G]=G'$. Now, $1<G'<G$ is the lower nilpotent series of G , and G' is Abelian by assumption. Therefore, by a theorem of Carter [1, Theorem 1, p. 90],

$$G'\cap Z(G)=1.$$

The second part follows from (i) immediately, for, otherwise, $1\neq G'\cap A\leq Z(G)$, a contradiction.

We have thus shown

Theorem 2. *A finite metabelian group G splits over its commutator subgroup G' if, and only if,*

$$G'\cap Z(G)=1.$$

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