A Splitting Condition for Metabelian Groups

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Masanobu YONAHA

1. Let G be a group and N a normal subgroup of G. If G contains a subgroup C such that $G=NC$ and $N\cap C=1$, then we shall say G splits over N and C is a complement of N in G.

The purpose of this paper is to prove the following theorem concerning a splitting condition for metabelian groups (a group G is called metabelian, if the commutator subgroup G' is Abelian) over its commutator subgroup.

Theorem 1. *Let G be a metabelian group whose commutator subgroup G' contains no non-trivial contral elements. Suppose further that all ascending and descending chains of normal (relative to G) subgroups of G' are finite. Then (i) G splits over G' and (ii) the complements of G' in G form a conjugate class.*

Theorem 1 generalizes Theorem 3.3 and a part of Theorem 3.5 of Newman [3]. Another condition is given in Corollary of the Principal Ideal Theorem in [4, p. 179].

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2. Let G be a group and x, y, $z\in G$. The commutators satisfy the following well-known indentities:

$$x[x,y]=x^y=[y,x^{-1}]x, \tag{1}$$
$$[x,y]^{-1}=[y,x], \tag{2}$$
$$[xz,y]=[x,y]^z[z,y], \tag{3}$$
$$[x,yz]=[x,z][x,y]^z. \tag{4}$$

If z belongs to the centralizer $C(G')$ of the commutator subgroup G', then (3) and (4) become, respectively,

$$[xz,y]=[x,y][z,y] \tag{5}$$
$$[x,yz]=[x,z][x,y]. \tag{6}$$

Lemma 1. *Let G be a metabelian group. Then, for $a\in G'$, $x,y\in G$,*

(i) $[a,x]=[a,x^y]$,

(ii) $[a,x,y]=[a,y,x]$.

Proof. The normality and the Abelianess of G' are used repeatedly in the proof.

(i)
$$\begin{aligned}[a,x^y]&=a^{-1}y^{-1}x^{-1}yay^{-1}xy\\&=a^{-1}x^{-1}(xy^{-1}x^{-1}y)ay^{-1}xy\\&=a^{-1}x^{-1}a(xy^{-1}x^{1-}y)y^{-1}xy\\&=a^{-1}x^{-1}ax=[a,x].\end{aligned}$$

(ii)
$$\begin{aligned}[a,x,y]&=[x,a]y^{-1}[a,x]y\\&=(x^{-1}a^{-1}xa)y^{-1}(a^{-1}x^{-1}ax)y\end{aligned}$$

$$\begin{aligned}&=x^{-1}a^{-1}x(ay^{-1}a^{-1}y)y^{-1}x^{-1}axy\\&=[a^{-1},y]x^{-1}a^{-1}(xy^{-1}x^{-1})a(xyx^{-1})x\\&=[y,a]x^{-1}[a,y^{x^{-1}}]x\\&=[y,a]x^{-1}[a,y]x \text{ (by (i))}\\&=[a,y,x].\end{aligned}$$

Lemma 2. *Let G be a metabelian group with $G'\cap Z(G)=1$, where $Z(G)$ is the center of G. Let N be a minimal normal subgroup of G, contained in G', and let x be an element of G but not contained in the centralizer $C(N)$ of N. Then the mapping $\phi: a\phi=[x,a]$, $a\in N$, (and, similarly, the mapping $\psi: a\psi=[a,x]$, $a\in N$) is an automorphism of N.*

Proof. By a result of Newman [3, Lemma 3.2, p.356], $N\phi$ and $C(x)\cap N$ are normal subgroups of G. Hence, by the minimality of N, $N\phi=N$ and $C(x)\cap N=1$. Since G' is Abelian, it follows at once from (6) that ϕ is an automorphism of N.

The following lemma is a generalization of Theorem 3.3 and a part of Theorem 3.5 of Newman [3]. However the proof is those of Newman with slight modification.

Lemma 3. *Let G be a metablian group with $G'\cap Z(G)=1$. If G' is a minimal normal subgroup of G, then (i) G splits over G' and (ii) the complements of G' are maximal subgroups and they form a conjugate class.*

Proof. (i) Let x be an element of G not belonging to the centralizer $C(G')$ of G', then, by Lemma 2, the mapping: $a\to[x,a]$, $a\in G'$, is an automorphism of G'. Hence, for every element g of G, there is one, and only one, element a in G', such that $[x,a]=[g,x]$ and, by (6) and (2),

$$[x,ga]=[x,a][x,g]=1.$$

That is, there is exactly one element of the centralizer $C(x)$ of x in each coset of G' in G. Thus

$$G=G'C(x) \text{ and } G'\cap C(x)=1.$$

(ii) Let C be a complement of G'. Let M be a subgroup of G, which contains C properly. Then, by the modular law, $M=(G'\cap M)C$.
Since $G'\cap M\leqq G'$ and G' is Abelian and normal in G, $G'\cap M$ is normal in G. Now, by the minimality of G',$G'\cap M=G'$, and hence $M=G$. Therefore C is maximal.

Let C_1 and C_2 be two complements of G' in G. Let x, y be elements of C_1 and C_2, respectively, which are not in $C(G')$ (such elements exist, since $G'\cap Z(G)=1$). Then, since C_1 and C_2 are Abelian and maximal, $C_1=C(x)$ and $C_2=C(y)$ (centralizers of x and y, respectively). The mappings: $a\to[x,a]$ and $b\to[b,y]$, $a,b\in G'$, are, by Lemma 2, automorphisms of G', and hence the mapping: $a\to[x,a,y]$, $a\in G'$, is an automorphism of G'. Thus there is a unique element $a\in G'$, such that $[x,a,y]=[y,x]$. Hence, by (1) and (5),

$$\begin{aligned}[x^a,y]&=[x[x,a],y]\\&=[x,y][x,a,y]\\&=1,\end{aligned}$$

so that $x^a\in C_2$. Moreover, x^a does not belong to $C(G')$. For, otherwise, x would be contained in $C(G')$, a contradiction. Now $x^a \notin C(G')$ implies, by a previous remark, that $C_2 = C(x^a)$, the centralizer of x^a. Therefore C_2 is conjugate to C_1.

3. We now prove our theorem. If $G'=1$, then the theorem is trivially true. On the other hand, if G' is a minimal normal subgroup of G, then the conclusions follow from Lemma 3.

Therefore suppose that G' is not a minimal normal subgroup of G. Let N be a minimal normal subgroup of G, which is contained in G' (there is such N by our hypothesis on the descending chains of normal subgroups in G'). We consider G/N. Clearly $(G/N)'=G'/N$. We first show that

$$G'/N \cap Z(G/N)=1.$$

Deny. Then there is an element a in G', but not in N, such that Na is a central element of G/N, that is, $[a,g] \in N$ for all g in G. Let $A=N\{a\}$, then $A \leqq G'$. Let x be an element of G, not in the centralizer $C(N)$ of N (there is such x since $G' \cap Z(G)=1$). Then the mapping ϕ: $b \rightarrow [b,x]$, $b \in A$, is a homomorphism of A onto N. For, by (5),

$$[b,x]=[na^i,x]=[n,x]\ [a^i,x]=[n,x][\ a,x]^i \in N,$$

and $$[b_1\ b_2,x]=[b_1,x][b_2,x],$$

where b, b_1, $b_2 \in A$ and $n \in N$. Thus ϕ is a homomorphism from A into N. But, by Lemma 2, $N\phi=N$ and hence ϕ is onto. Let K be the kernel of ϕ, then $A=NK$. Moreover, since each element of K commutes with x, whereas no elements, except the indentity, of N commute with x, we have $N \cap K=1$. Now, $a=nk$ for some $n \in N$ and$k \in K, k \neq 1$ (since $a \notin N$). Also, by (5),

$$[a,g]=[nk,g]=[n,g]\ [k,g],$$

which is in N. Hence $[k, g]$ is in N. It suffices to show that $[k, g]=1$ for all g in G, i.e., $k \in Z(G)$. For, then, since $k \in G'$, $G \cap Z(G) \neq 1$, contradicting our assumption. (We could have taken $a=k$, to begin with.) Since k commutes with x, $[k, x]=1$, and hence $[k, x, g]=1$, for all g in G. By Lemma 1 (ii),

$$1=[k,\ x,\ g]=[k,\ g,\ x].$$

But, by Lemma 2, the mapping: $n \rightarrow [n,\ x]$, $n \in N$, is an automorphism of N. Therefore $[k,\ g]=1$.

(i) Now, G/N is, of course, metabelian and $G'/N \cap Z(G/N)=1$. Moreover, our hypothesis on the ascending and descending chains of normal subgroups in G', which also carries over to G'/N, implies that the length of a chief series of G between 1 and G' is finite. Thus, by induction on its length, there is a subgroup H of G, such that $G/N=G'/N \cdot H/N$, $G'/N \cap H/N=1$, that is, $G=G'H$, $G' \cap H \leqq N$. Since $H' \leqq G' \cap H \leqq G'$, which is Abelian, and H' is normal in H, H' is normal in G. Hence, either $H'=N$ or $H'=1$, by the minimality of N. If $H'=1$ (and hence H is Abelian), then each element of N commutes elementwise with both H and G' and hence is in the center $Z(G)$ of G, a contradiction (since $N \leqq G'$). Thus, $H'=N$. If there were a normal subgroup N_1 of H, such that $1 < N_1 < N$, then N_1 would be normal in G, contracticting the minimality of N. Hence N $(=H')$ is minimal normal in H. Also, $N \cap Z(H)=1$ is clear. Therefore, by Lemma 3 (i),

H possesses a subgroup C such that $H=NC$ and $N\cap C=1$. We now have

$$G=G'H=G'NC=G'C$$

and

$$G'\cap C\leqq G'\cap H\cap C\leqq N\cap C=1.$$

(ii) Let C_1 and C_2 be two complements of G'. Then $N\ C_1/N$ and $N\ C_2/N$ are complements of G'/N in G/N. Hence, as in (i), by induction, they are conjugate, that is, there is an element x in G, such that

$$H=NC_1=N(x^{-1}C_2x).$$

Since C_1 and $x^{-1}C_2x$ are Abelian and $N\cap Z(H)=1$, $N\cap C_1=N\cap(x^{-1}C_2x)=1$. Hence C_1 and $x^{-1}C_2x$ are complements of $N(=H')$ in H. Since N is minimal normal in H, by Lemma 3 (ii), they are conjugate. In orther words, there is an element y in H, such that

$$C_1=(xy)^{-1}C_2(xy).$$

This completes the proof.

4. It follows, as an immediate consequence, from Theorem 1 that if G is finite metabelian group whose commutator subgroup G' contains no non-trivial central elements, i.e., $G'\cap Z(G)=1$, then G splits over G'. On the other hand, the converse also holds. The following lemma is a slightly more general case.

Lemma 4. *Let G be a finite metabelian group and G' its commutator subgroup. If G possesses an Abelian subgroup A such that $G=G'A$, then (i) G' contains no non-trivial central elements and (ii) A is a complement of G' in G.*

Proof. We have

$$[G',G]=[G',G'A]=[G',A]$$

Hence, by a theorem of Ito [2, Satz 1, p. 400], $[G',G]=G'$. Now, $1<G'<G$ is the lower nilpotent series of G, and G' is Abelian by assumption. Therefore, by a theorem of Carter [1, Theorem 1, p. 90],

$$G'\cap Z(G)=1.$$

The second part follows from (i) immediately, for, otherwise, $1\neq G'\cap A\leqq Z(G)$, a contradiction.

We have thus shown

Theorem 2. *A finite metabelian group G splits over its commutator subgroup G' if, and only if,*

$$G'\cap Z(G)=1.$$

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University of Kansas
University of the Ryukyus