抵抗函数の性質について

On Some Properties of Resistance Functions*

By

Tadao INAMI**

Introduction

Although convexity and concavity are mathematical notions, the theorems of this thesis are concerned with examining certain resistance functions which arise from the study of electrical networks. Certain concepts from physics will be used without definition, among these are the concept of an electrical network, the concept of potential at a point (and also potential difference between points) of the network, and the concept of a loop and of a nodal point of the network.

The two basic laws due to Kirchhoff concerning currents and potentials in an electrical network will be assumed.

Kirchhoff's Laws

1. The sum of the currents flowing into a point of a network is zero.
2. The sum of the potential differences around a closed loop of a network is zero.

Before theorems are presented and proved, several definitions must be given.

DEFINITION 1. Power. If N is a network and x is an element of N, the statement that P is the power consumed in x means that P is the rate with respect to time at which the external energy is consumed, i.e., absorbed or transformed, by x; and the statement that P' is the power consumed in N means that P' is the rate at which the external energy is consumed by N.

DEFINITION 2. Resistance. If N is a network, x is an element of N, I is the current through x, and P is the power consumed by x, the statement that R is the resistance of x means that

$$R=\frac{P}{I^2}.$$

DEFINITION 3. Impepance. If N is a network, x is an element of N, I is the current through x, and E is the potential difference across x, the statement that Z is the impedance of x means that

$$Z=\frac{E}{I}.$$

DEFINITION 4. Conductance. If N is a network, x is an element of N, E is the potential difference across x, and P is the power consumed in x, the statement that G is the conductance of x means that

$$G=\frac{P}{E^2}.$$

DEFINITION 5. Admittance. If N is a network, x is an element of N, I is the current through x, and E is the potential difference across x, the statement that Y is the admittance

* This paper is based on the master's thesis presented by the author to the faculty of the graduate school of the University of Texas.

** Agriculture, Home Economics and Engineering Division, University of the Ryukyus.

of x means that

$$Y=\frac{I}{E}.$$

DEFINITION 6. Linear Network. The statement that a network N is linear means that the quantities, such as resistance, impedance, conductance and admittance, of each element of N are not influenced by external constraints, i.e., conditions imposed upon N from outside.

DEFINITION 7. Duality. Suppose that M is a set of variables with range R, that M' is a set of variables with range R', and that a relation f holds among the elements of M for all values of the variables in R, then the stalement that M and M' are duals with respect to f means that

1. the relation f holds among the elements of M' for all values of the valuables in R'.
2. there exists a reversible transformation T that throws each element of M into some element of M', and
3. if K is a subset of M and a relation g holds among the elements of K for all values of those variables in R, there is a subset K' of M' such that the relation g holds among the elements of M' for all values of them in R';

and the statement that an element m of M and an element m' of M' are duals means that m' is the image of m under the transformation T. If A is a set of equations, inequalities, or theorems expressing the relation of f among the elements of M in the range R, and B is a corresponding set of equations, inequalities, or theorems expressing the relation f among elements of M' in the range R', then A and B are said to be duals with respect to f.

NOTE: It may be noted that impedance of an element of a network is the reciprocal of admittance of the element, and vice versa. Also note that resistance and impedance have the same numerical value for an element of a network when $P/I=E$; and that conductance and admittance have the same numerical value for an element of a network $P/E=I$. In discussions to follow, resistance, impedance, couductance, and admittance will be assumed to be constants with respect to time for a given element of a network, i.e., only linear networks will be treated.

The definition of duality given here is a fairly general one. Several other definitions of duality, however, are possible. For instance, under certain restricting conditions, a relation f', distinct from f, that obtains among the elements of M' may be viewed as a dual of f. The duality of addition and multiplication in Boolean algebra is one such example.

On the basis of the foregoing laws and definitions, it is now possible to present and prove some theorems.

THEOREM 1. If N is either a linear network of impedances and of electromotive forces, containing no negative resistance, or a part of such a network, the currents in branches of N distribute themselves in such a way that the function

$$f=\sum_{j=1}^{m} R_jI_j^2-2\sum_{j=1}^{m} E_jI_j$$

is a minimum; where R_j is the resistance of jth branch, I_j is the current through jth branch, E_j is the potential difference across jth branch of N, and m is the total number of the branches (numbered 1, 2, $\cdots$, m) of N.

PROOF. Suppose that N is a linear network of impedances and of electromotive forces, containing no negative resistance, or a part of such a network. Let the loop currents of N be i_1, i_2, $\cdots$, i_n with signs positive or negative according to whether the sense of the currents is toward an arbitrarily chosen direction, say, counter-clockwise in the plane of respective loops, or against it, respectively.

Then the current in branch j $(0<j\leq m)$ of N is expressed as

$$I_j=a_{j1}i_1+a_{j2}i_2+a_{j3}i_3+\cdots+a_{jn}i_n\equiv\sum_{k=1}^{n} a_{jk}i_k$$

where, for each positive integer k not greater than n, a_{jk} is a constant having a value of either 1, -1, or 0.

Let the electromotive forces in the branches of N be $E_1, E_2, \cdots, E_m$, with signs positive or negative according that the sense of the electromotive forces are toward or against the predetermined direction, and let the resistances in branches of N be $R_1, R_2, \cdots, R_m$, each of which is nonnegative. Then

$$f=\sum_{j=1}^{m} R_j I_j^2-2\sum_{j=1}^{m} E_j I_j=\sum_{j=1}^{m} R_j(\sum_{k=1}^{n} a_{jk} i_k)^2-2\sum_{j=1}^{n} E_j(\sum_{k=1} a_{jk} i_k).$$

Assume, now, that it is possible to obtain a different arrangement of current. The most general of such alterations that does not violate the Kirchhoff's first law consists of superposition of loop currents $\Delta i_1, \Delta i_2, \cdots, \Delta i_n$ on $i_1, i_2, \cdots, i_n$. Then the function becomes

$$f+\Delta f=\sum_{j=1}^{m} R_j \left[\sum_{k=1}^{n} a_{jk}(i_k+\Delta i_k)\right]^2-2\sum_{j=1}^{m} E_j \left[\sum_{k=1}^{n} a_{jk}(i_k+\Delta i_k)\right].$$

Then

$$\Delta f=\sum_{j=1}^{m}\sum_{k=1}^{n} R_j(a_{jk}\Delta i_k)^2+2\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\Delta i_k(a_{jk}R_j i_k-E_j)$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{n} R_j(a_{jk}\Delta i_k)^2+2\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\Delta i_k(R_j I_j-E_j).$$

By Kirchhoff's second law,

$$\sum_{j=1}^{m}(R_j I_j-E_j)=0.$$

Therefore

$$\Delta f=\sum_{j=1}^{m}\sum_{k=1}^{n} R_j(a_{jk}\Delta i_k)^2\geq 0,$$

since, for each positive integer j not greater than n, R_j is nonnegative by hypothesis, and $(a_{jk}\Delta i_k)^2$ is nonnegotive, because a_{jk} and Δi_k are real quantities. This proves that the original distribution of currents was such that f is a minimum.

COROLLARY 1-1. If N is either a linear network of impedances containing no source of electromotive force or no negative resistance, or a part of such a network, the currents in the branches of N distribute themselves in such a way that the total power dissipated by N is a minimum.

PROOF: Shppose that N is either a linear network of impedances containing no source of electromotive force or no negative resistance, or a part of such a network. Then, by theorem 1,

$$f=\sum_{j=1}^{m} R_j I_j^2-2\sum_{j=1}^{m} E_j I_j$$

is a minimum. Since N contains no source of electromotive force,

$$\sum_{j=1}^{m} E_j I_j=0.$$

Hence

$$f=\sum_{j=1}^{m} R_j I_j^2.$$

But

$$\sum_{j=1}^{m} R_j I_j^2$$

is the total power dissipated by N.

NOTE: If, in theorem 1, we substitute admittance for impedance, conductance for resistance, potential difference for current, and current for electromotive force, we obtain our next theorem, theorem 2. It is seen that theorem 2 is a dual of theorem 1, for, let

$$M=N_1[Z_j,\ E_j,\ R_j,\ I_j,]$$

$$M'=N_2[Y_j,\ i_j,\ G_j,\sum_{k=1}^{m}(v_j-v_k)]$$

$$f=\sum_{j=1}^{m}(A_jB_j^2-2C_jB_j)$$

where $N_1[Z_j,$ (etc.)] is a set of all impedance (etc.) variables in the network of theorem 1. Substitution of appropriate variables, first from M and then from M', for A, B, and C yields the relations of theorem 1 and of the following theorem.

THEOLEM 2. If N is either a linear network of admittances and of current sources containing no negative conductance, or a part of such a network, the potentials at the nodes of N are distributed in such a way that

$$f=\sum_{j=1}^{m}\sum_{l=1}^{m}[g_{jk}(v_j-v_k)^2-2i_{jk}(v_j-v_k)]$$

is a minimum; where g_{jk} is the conductance between the jth and kth nodes, i_{jk} is the current from the jth node to the kth node due to the current source between the jth and kth nodes, v_j is the potential of jth node, of N, and m is the total number of the nodes of N, considering each end of each branch admittance and current-source as a node.

PROOF. Suppose that N is a linear network of admittances and of current sources containing no negative conductance or a part of such a network.

Let each end of each branch admittance and of each branch current-source be considered a node, numbered 1, 2, $\cdots$, m. Let the node potentials at nodes 1, 2, $\cdots$, m, be v_1, v_2, $\ldots$, v_m, respectively, with respect to an arbitrarily chosen potential. Then the potential at a node $j(0<j\leq n)$ is expressed as

$$v_j=a_{j1}v_1+a_{j2}v_2+a_{j3}v_3+\cdots+a_{jn}v_m\equiv\sum_{k=1}^{m}a_{jk}v_k$$

where, for each positive integer k not greater than m, a_{jk} is a constant having a value of either 1, -1, or 0.

For each pair of positive integers j and k both not greater than m, let g_{jk} be the conductance between nodes j and k and let i_{jk} be the current from node j to node k, due to a current source between the nodes. Then

$$f=\sum_{j=1}^{m}\sum_{k=1}^{m}[(v_j-v_k)^2g_{jk}-2i_{jk}(v_j-v_k)].$$

Assume, now, that it is possible to obtain a different arrangement of potentials. The most general rearrangement which does not violate Kirchhoff's second law consists of superposition of potentials Δv_1, Δv_2, $\cdots$, Δv_m at each node of the loops of N.

Then the new value of the function becomes

$$f+\Delta f=\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}[v_j-v_k+\Delta v_l(a_{jl}-a_{kl})]^2g_{jk}-2\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}[v_j-v_k+\Delta v_l(a_{jl}-a_{kl})]i_{jk}.$$

Thus

$$\Delta f=\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}[(a_{jl}-a_{kl})\Delta v_l]^2g_{jk}+2\Delta v_l(a_{jk}-a_{kl})[(v_j-v_k)g_{jk}-i_{jk}].$$

By Kirchhoff's first law

$$\sum_{j=1}^{m}\sum_{k=1}^{m}[(v_j-v_k)g_{jk}-i_{kj}]=0.$$

Therefore

$$\Delta f=\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}[(a_{jl}-a_{kl})\Delta v_l]^2g_{kj}\geq 0,$$

since, for each set of positive integers j, k, and l all not greater than m, g_{kj} is nonnegative and $[(a_{jl}-a_{kl})\Delta v_l]^2$ is nonnegative, because a_{jl}, a_{kl}, and Δv_l are real quantities.

COROLLARY 2-1. If N is either a linear network of electrical admittances containing no source of electric current nor a negative conductance, or a part of such a network, the potentials at nodes on N are such that the power dissipated in N is a minimum.

THEOREM 3. If two points A, B are connected by a linear network of conductors, a decrease in the resistance of any one of these conductors will not increase the resistance between A and B.

PROOF: Suppose that two points A, B are connected by a network of conductors, that I is the current from A to B, and that R is the resistance between A and B.

The power consumed by the network is

$$P=I^2R.$$

Suppose that the resistance of any single conductor j in the network is reduced from R_j to R'_j and let I_j be the current flowing through R_j before the reduction. Assume that the currents flowing in the branches of the network remain unaltered in spite of this reduction in the resistance R_j. There will be a decrease in power consumed by the network equal to

$$\Delta P=(R_j-R'_j)I_j^2.$$

The currents which are assumed to exist in the network after the reduction in the resistance are not the ones that are possible but the ones that were imagined. Suppose that the currents are allowed to distribute themselves in a manner that is possible in nature. By theorem 1, there is a further reduction in power consumed by the network.

Thus a decrease in the resistance of any single conductor results in the reduction of power consumed, i.e., if we denote the new resistance between A and B by R',

$$R'I^2\leq RI^2$$

or

$$R'\leq R.$$

COROLLARY 3-1. In a linear two terminal resistance network, closing a switch in any branch of the network does not increase the resistance between terminals.

PROOF: In a linear two terminal resistance network, closing a switch in any branch of the network represents a reduction of resistance in the switch-circuit from infinite value to a finite value. By theorem 3, this does not increase the resistance between terminals.

NOTE: This is an extension of the second theorem presented by C. E. Shannon and D. W. Hagelbarger. (See the primary reference at the end of this paper.) They proved that this theorem obtains when only the direct-current constraints are to be applied at the terminals of a passive network. Here, however, it is proved that the theorem is true for alternating-current resistance and that the restrictions imposed by Shannon and Hagelbarger are quite unnecessary.

COROLLARY 3-2. In a linear four-terminal resistance network, a reduction in the resistance of any one section of the network does not increase the input resistance of the network.

PROOF: An input resistance of a four-terminal network is the resistance between two input terminals. Therefore, by theorem 3, a reduction in the resistance of any one section of the network does not increase the input resistance of the network.

COROLLARY 3-3. In a linear four-terminal resistance network, a decrease in the resistance of any one section of the network does not increase the output resistance of the network.

COROLLARY 3-4. In a linear four-terminal resistance network, closing a switch in any branch of the network does not increase the input nor out-put resistance.

DEFINITION 8. Concave (downward) function. If f is a function of n nonnegative real variables, the statement that f is a concave (downward) function means that, for any two sets of values $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ in the range of f,

$$f\left(\frac{x_1+x_1'}{2}, \frac{x_2+x_2'}{2}, \cdots, \frac{x_n+x_n'}{2}\right)\geq\frac{1}{2}[f(x_1, x_2, \cdots, x_n)+f(x_1', x_2', \cdots, x_n')].$$

DEFINITION 9. Convex (downward) function. If f is a function of n nonnegative real variables, the statement that f is a convex (downward) function means that, for any two sets of values $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ in the range of f,

$$f\left(\frac{x_1+x_1'}{2}, \frac{x_2+x_2'}{2}, \cdots, \frac{x_n+x_n'}{2}\right) \leq \frac{1}{2}[f(x_1, x_2, \cdots, x_n)+f(x_1', x_2', \cdots, x_n')].$$

DEFINITION 10. Linear function. If f is a function of n nonnegative real variables, the statement that f is a linear function means that, for any two sets of values $x_1, x_2, \ldots, x_n$ and $x_1', x_2', \cdots, x_n'$ in the range of f,

$$f\left(\frac{x_1+x_1'}{2}, \frac{x_2+x_2'}{2}, \cdots, \frac{x_n+x_n'}{2}\right) = \frac{1}{2}[f(x_1, x_2, \cdots, x_n)+f(x_1', x_2', \cdots, x'_n)].$$

NOTE: It should be obvious that, if f is a convex function, then $-f$ is a concave function, and vice-versa. For other elementary properties of convex and concave functions, a book[1] by Hardy, *et al.*, gffers a good summary.

With the preliminary theorems and definitions, we can now proceed to prove several other theorems, one of which is an extension of a theorem by C. E. Shannon and D. H. Hagelbarger.

LEMMA 1. If N is a linear network of nonnegative resistances $R_1, R_2, R_3, \cdots, R_n$ in series, and R is the resistance between terminals of N, then R is a linear function of $R_1, R_2, \cdots, R_n$.

PROOF: Let N_1 be a network of nonnegative resistances $R_1, R_2, \cdots, R_n$ in series, and R_{01} be the resistance between terminals of N_1. Let N_2 be a network of nonnegative resistances $R_1', R_2', \cdots, R_n'$ in series and R_{02} be the resistance between the terminals of N_2. Let N_0 be the network of nonnegative resistances $(R_1+R_1'), (R_2+R_2'), \cdots, (R_n+R_n')$ in series and R_0 be the resistance between terminals of N_0. Then

$$R_{01}=R_1+R_2+\cdots+R_n=\sum_{i=1}^{n} R_i$$

$$R_{02}=R_1'+R_2'+\cdots+R_n'=\sum_{i=1}^{n} R_i'$$

$$R_0=(R_1+R_1')+(R_2+R_2')+\cdots+(R_n+R_n')=\sum_{i=1}^{n}(R_i+R_i')=\sum_{i=1}^{n} R_i+\sum_{i=1}^{n} R_i'.$$

$$\therefore \quad R_0=R_{01}+R_{02}.$$

Therefore, if N is a network of nonnegative resistances $R_1, R_2, \cdots, R_n$ in series and R is the resistance between terminals of N, then R is a linear function of $R_1, R_2, \cdots, R_n$.

LEMMA 2. If N is a linear network of nonnegative conductances $G_1, G_2, \cdots, G_n$ in parallel, and G is the conductance between terminals of N, then G is a linear function of $G_1, G_2, \cdots, G_n$.

PROOF: This is a dual of lemma 1. The proof may be given in exactly similar manner as in lemma 1.

THEOREM 4. If N is a two-terminal linear network of nonnegative resistances, $R_1, R_2, \cdots, R_n$ and R is the resistance between terminals of N, then R is a concave (downward) function of $R_1, R_2, \cdots, R_n$.

PROOF: Suppose that N is a linear two-terminal network of nonnegative resistances $R_1, R_2, \cdots, R_n$ and that R is the resistance between terminals of N. Form a network N' of similar configuration by replacing R_1 by R_1', R_2 by R_2', $\cdots$, R_n by R_n', where R_1' R_2', $\cdots$, R_n' are arbitrary linear nonnegative resistances and let R' be the resistance between terminals of N'. Form a similar network N_0 with $\frac{R_1+R_1'}{2}, \frac{R_2+R_2'}{2}, \cdots, \frac{R_n+R_n'}{2}$, and let R_0 be the resistance between terminals of N_0.

Let I denote the current through terminals of N_0 and I_j denote the current in the jth

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities", Cambridge University Press, London, 1934.

branch $R_j+R_j'(0<j\leq n)$ of N_0. Then

$$R_0=\sum_{j=1}^{n}\frac{(R_j+R_j')I_j^2}{2I^2}=\sum_{j=1}^{n}\frac{R_jI_j^2}{2I^2}+\sum_{j=1}^{n}\frac{R_j'I_j^2}{2I^2}.$$

Let i_j denote the current in the jth branch R_j $(0<j\leq n)$ of N under the same constraint as before, and let i_j' denote the current in the jth branch $R_j'(0<j\leq n)$ of N' under the same constraint as before. Then

$$\frac{R}{2}=\sum_{j=1}^{n}\frac{R_ji_j^2}{2I^2}$$

$$\frac{R'}{2}=\sum_{j=1}^{n}\frac{R_j'i_j'^2}{2I^2}.$$

By Corollary 1-1,

$$\sum_{j=1}^{n}R_ji_j^2\leq\sum_{j=1}^{n}R_jI_j^2$$

$$\sum_{j=1}^{n}R_j'i_j'^2\leq\sum_{j=1}^{n}R_j'I_j^2.$$

Hence

$$R_0\left(\frac{R_1+R_1'}{2},\ \frac{R_2+R_2'}{2},\ \cdots,\ \frac{R_n+R_n'}{2}\right)\geq\frac{1}{2}[R(R_1,\ R_2,\ \cdots,\ R_n)+R'(R_1',\ R_2',\ \cdots,\ R_n')]$$

Quod erat demonstrandum.

NOTE: This is an extension of the first theorem presented by C. E. Shannon and D. W. Hagelbarger in the *Journal of Applied Physics*, Vol. 27, No. 1, January, 1956.

COROLLARY 4-1. If N is a four-terminal network of linear nonnegative resistances R_1, R_2, $\cdots$, R_n with input resistance R, then R is a concave (downward) function of R_1, R_2, $\cdots$, R_n.

THEOREM 5[2)]. If N is a linear two-terminal network of nonnegative conductances G_1, G_2, $\cdots$, G_n and if G is the conductance between the terminals of N, then G is a concave (downward) function of G_1, G_2, $\cdots$, G_n.

PROOF: Suppose that N is a linear two-terminal network of nonnegative conductances G_1, G_2, $\cdots$, G_n and that G is the conductance between the terminals of N. Form a lineor network N' by similar configuration by replacing G_1 by G_1', G_2 by G_2', $\cdots$, G_n by G_n', where G_1', G_2', $\cdots$, G_n' are arbitrary linear nonnegative conductances, and let G' be the conductance between the terminals of N'. Form a linear network N_0 of $\frac{G_1+G_1'}{2}$, $\frac{G_2+G_2'}{2}$, $\cdots$, $\frac{G_n+G_n'}{2}$ in a similar manner, and let G_0 be the conductance between terminals of N_0.

Let V denote the potential difference across the terminals of N_0 and let V_j denote the voltage across the jth branch $G_j+G_j'(0<j\leq n)$ of N_0. Then

$$G_0=\sum_{j=1}^{n}\frac{(G_j+G_j')V_j^2}{2V^2}=\sum_{j=1}^{n}\frac{G_jV_j^2}{2V^2}+\sum_{j=1}^{n}\frac{G_j'V_j^2}{2V^2}$$

Let v_j denote the potential differentce across the jth branch $G_j(0<j\leq n)$ of N under the same constraint as before, and let v_j' denote the potential difference across the jth branch $G_j'(0<j\leq n)$ of N' under the same constraint as before. Then

$$\frac{G}{2}=\sum_{j=1}^{n}\frac{G_jv_j'^2}{2V^2}$$

$$\frac{G'}{2}=\sum_{j=1}^{n}\frac{G_j'v_j'^2}{2V^2}.$$

By corollary 2-1,

[2)] This is a dual of the previous theorem.

$$\sum_{j=1}^{n} G_j v_j^2 \leq \sum_{j=1}^{n} G_j V_j^2$$

$$\sum_{j=1}^{n} G_j' v_j'^2 \leq \sum_{j=1}^{n} G_j' V_j^2.$$

Hence

$$G_0\left(\frac{G_1+G_1'}{2}, \frac{G_2+G_2'}{2}, \cdots, \frac{G_n+G_n'}{2}\right) \geq \frac{1}{2}[G(G_1, G_2, \cdots, G_n)+G'(G_1', G_2', \cdots, G_n')].$$

COROLLARY 5-1. If N is a four-terminal network of linear nonnegative conductance $G_1, G_2, \cdots, G_n$ with input conductance G, then G is a concave (downward) function of $G_1, G_2, \cdots, G_n$.

THEOREM 6. If N is a multi-terminal balanced linear network of nonnegative resistances, the input resistance of N is a concave (downward) function of resistances in N.

COROLLARY 6-1. If N is a multi-terminal balanced network of linear nonnegative conductances, the input conductance of N is a concave (downward) function of conductances of N.

THEOREM 7. The input conductance for an electromagnetic wave propagating through a medium is:

(a) a concave (downward) function of the conductance encountered by the wave if
  (i) the intrinsic impedance of the medium is real, and
  (ii) the admittance encountered by the wave has a real value such that $G\,G' \leq 4$ for every G and G' in the range.

(b) a convex (downward) function of the conductance encountered by the wave if
  (i) intrinsic impedance of the medium is real, and
  (ii) the admittance encountered by the wave has a real value such that $G\,G' \geq 4$ for every G and G' in the range.

(c) a linear function of the conductance encountered by the wave if
  (i) intrinsic impedance of the medium is real, and
  (ii) the admittance encountered by the wave has a real value such that $G\,G' = 4$ for every G and G' in the range.

PROOF: If η is the intrinsic impedance of the medium, the input impedance for an electromagnetic wave propagating through the medium is given by the following equation as a function of impedance Z encountered by the wave at a point at the distance l from the source of the wave:[3)]

$$Z_i = \frac{Z \cos kl + j\eta \sin kl}{\eta \cos kl + jZ \sin kl}.$$

The input resistance is, then,

$$R_i = \eta \frac{\eta R \cos^2 kl + \eta R \sin^2 kl}{\eta^2 \cos^2 kl + R^2 \sin^2 kl} = \frac{\eta^2 R}{\eta^2 \cos^2 kl + R^2 \sin^2 kl}$$

$$R_i = \frac{1}{\frac{1}{R} \cos^2 kl + \frac{R}{\eta^2} \sin^2 kl} = \frac{1}{G \cos^2 kl + \frac{1}{\eta^2 G} \sin^2 kl}.$$

The input conductance is

$$G_i(G) = G \cos^2 kl + \frac{1}{\eta^2 G} \sin^2 kl.$$

Also

$$G_i(G') = G' \cos^2 kl + \frac{1}{\eta^2 G'} \sin^2 kl$$

and

3) See Simon Ramo and John R. Whinnery: "Fields and Waves in Modern Radio," John Wiley and Sons, 1953.

$$G_i\left(\frac{G+G'}{2}\right)=\frac{G+G'}{2}\cos^2 kl+\frac{2}{\eta^2(G+G')}\sin^2 kl$$

$$\frac{G_i(G)+G_i(G')}{2}=\frac{G+G'}{2}\cos^2 kl+\frac{GG'}{2\eta^2(G+G')}\sin^2 kl.$$

Hence,

$$\text{if}\quad GG'\leq 4,\ G_i\left(\frac{G+G'}{2}\right)\geq\frac{G_i(G)+G_i(G')}{2}$$

$$\text{if}\quad GG'\geq 4,\ G_i\left(\frac{G+G'}{2}\right)\leq\frac{G_i(G)+G_i(G')}{2}$$

$$\text{if}\quad GG'=4,\ G_i\left(\frac{G+G'}{2}\right)=\frac{G_i(G)+G_i(G)}{2}$$

THEOREM 8. The input conductance of a transmission line is a

(a) concave (downward) function of the load conductance if
  (i) the characteristic impedance of the line is real, and
  (ii) the load admittance has a real value such that $GG'\leq 4$ for every G and G' in the range.

(b) convex (downward) function of the load conductance if
  (i) the characteristic impedance of the line is real, and
  (ii) the load admittance has a real value such that $GG'\geq 4$ for every G and G' in the range.

(c) linear function of the load conductance if
  (i) the characteristic impedance of the line is real, and
  (ii) the load admittance has a real value such that $GG'=4$ for every G and G' in the range.

PROOF. If Z_0 is the characteristic impedance of the transmission line, the input impedance is given by the following equation as a function of load impedance Z_L placed at a distance l from the sending end of the line:[4]

$$Zi=\frac{Z_L\cos\beta l+jZ_0\sin\beta l}{Z_0\cos\beta l+jo_L\sin\beta l}.$$

Thus the theorem can be proved in a similar manner as in the previous theorem.

Conclusion

Theorems 1 and 2 are examples of that Principle of Extremum which is manifested so often in nature. (For examples of manifestations of the Principle, see, for instance D'Abro's "The Rise of New Physics," Vol. I, Dover.) The theorems here might be called the Minimum Power Theorems in Electrical Circuit. If we introduce the concept of duality, theorem 2 may be stated simply as "the dual of theorem 1 is also true." If a theorem is proved, the dual of the theorem can easily be proved in exactly similar manner as in the original theorem. Thus, the introduction of duality-concept enables us not only to simplify the statement and the proof of theorems, but also to find out new theorems.

Corollary 3-1 and Theorem 4 are extensions of the theorems due to Shannon and Hagelbarger. Their theorems apply only to passive linear resistance networks where a direct-current constrainst is to be applied externally. Their approach of proof makes it difficult to extend their theorems any further.

But, by proving theorems 1 and 2 and their corollaries, and by adopting an entirely new method of proof, I have extended their theorems to apply to any linear network containing no negative resistance. My theorems apply for active as well as passive networks, when

[4] See, for instance, Ramo and Whinnery, "Fields and Waves in Modern Radio."

direct or alternating current constraints are imposed. My theorems also apply when transformers are involved in the networks.

In addition, several other theorems on properties of resistance functions have been presented and proved (eight theorems, nine corollaries, and two lemmas in all). Theorem 7 concerns with electromagnetic waves in space, and theorem 8 with transmission lines with distributed-constants. All other theorems concern with electrical networks with lumped-constants.

Primary Reference

C. E. Shannon and D. W. Hagelberger: Concavity of Resistance Functions, Journal of Applied Physics, vol. 27, no. 1, Jan. 1956,

Secondary References

1. E. F. Beckenbach: Convex Functions, Bulletin of American Methematical Society, vol. 54, 1948, pp. 439-460.
2. T. Bonnesen and W. Fenchel: Theorie der Konvexen Körper, Berlin, Springer, 1934.
3. E. F. Beckenbach and R. H. Bing: On generalized convex functions, Transaction of American Mathematical Society, vol. 58, 1945, pp. 220-230.
4. G. Pólya: Untersuchungen Über Lücken und Singularitaten von Potenzreihen, Mathematik Zeitung, vol. 29, 1929, pp. 549-560.
5. Mauricio Matos-Peixoto: On the existence of derivatives of generalized convex functions, Summa Brasiliensis Mathematicae, vol. 2, fasc. 3, 1948, pp. 35-42.
6. E. F. Bonsall: The Characterization of Generalized Convex Functions, Quarterly Journal of Mathematics, Oxford Series 2, vol. 1, 1950, pp. 100-111.
7. J. W. Green: Sets subtending a constanding a constant angle on a circle, Duke Mathematical Journal, vol. 17, 1950, pp. 263-267.
8. R. A. Rosenbaum: Sub-additive functions, Duke Mathematical Journal, vol. 17, 1950, pp. 227-247.
9. J. W. Green: Approximately convex functions, Duke Mathematical Journal, vol. 19, 1952, pp. 499-504.
10. D. H. Hyers and S. N. Ulam: Approximately convex functions, Proceedings of American Mathematical Society, vol. 3, 1952, pp. 821-828.
11. J. W. Green: Support, differentiability, and convergence properties of generalized convex functions, Proceedings of American Mathematical Society, vol. 4, 1953, pp. 391-396.
12. J. W. Green: Length and area of a convex curve under area preserving affine transformation, Pacific Journal of Mathematics, vol. 3, 1953, pp. 393-402.
13. J. W. Green: Recent Applieations of Convex Functions, The American Mathematical Monthly, vol. 61, no. 7, 1954, pp. 449-454.
14. M. Motos-Peixoto: Generalized convex functions and second order differential inequalities, Bulletin of the American Mathematical Society, vol. 55, no. 6, 1949, pp. 563-572.
15. E. F. Beckenbach: Generalized convex functions, Bulletin of the American Mathematical Society, vol. 43, 1937, pp. 363-371.
16. J. E. Wilkins: The converse of a theorem of Tchaplygin on differential inequalities, Bulletin of the American Mathematical Society, vol. 53, 1947, pp. 126-129.
17. G. Valiron: Fonction convexes et fonctions entieres, Bull. Soc. Math. France, vol. 60, 1932, pp. 278-287.
18. Harold Shniad: On the convexity of mean value functions, Bulletin of the American Mathematical Society, vol. 54, 1948, pp. 770-776.
19. F. Bernstein and G. Doetsch: Zur Theorie der konvexen Functionen, Math. Ann., vol. 76, 1915, pp. 514-526.

20. H. Blumberg: Convex functions, Trans. Amer. Math. Soc., vol. 20, 1919, pp. 40-44.
21. T. Bonnesen: En bemaerkning om konvekse functioner, Mathematisk Tidsskrift B., 1928, pp. 18-20.
22. J. L. W. V. Jensen: Om konvexe Funktioner og Uligheder mellem Middelvaerdier, Nyt. Tidsskrift for Mathematik, vol. 16B, 1905, pp. 49-69.
23. J. L. W. V. Jensen: Sur les fonctions convexes et les inegalites entre les valeur moyennes, Acta Math., vol. 30, 1906, pp. 175-193.
24. P. Montel: Sur les fonctions convexes et les fonctions sousharmoniques, J. Math. Pures Appl., (9) vol. 7, 1928, pp. 29-60.
25. N. Norris: Convexity properties of generalized mean value functions, Ann. Math. Statist., vol. 8, 1937, pp. 118-120.
26. A. Astrowski: Zur Theorie der konvexen Functionen, Comment. Math. Helvetici, vol. 1, 1929, pp. 157-159.
27. T. Popoviciu: Les fonctions convexes, Paris, Hermann, 1945.
28. T. Radó: On convex functions, Trans. Amer. Math. Soc., vol. 37, 1935, pp. 266-285.
29. W. Sierpinski: Sur les fonctions convexes mesurables, Fund. Math., vol. 1, 1920, pp. 125-129.
30. G. Szekeres: On a property of monotone and convex functions, Proceedings of the American Mathematical Society, vol. 7, 1956, pp. 351-353.
31. R. F. Gabriel: The Schwarzian derivative and convex functions, Proceedings of the American Mathematical Society, vol. 6, 1955, pp. 58-60.
32. Leonard Tornheim: On n parameter families and associated convex functions, Transactions of the American Mathematical Society, vol. 69, no. 3, (1950), pp. 457-467.
33. T. Popoviciu: Notes sur les fonctions convexes d'ordre superieur (1), Mathematica vol. 12, (1936), pp. 81-92.
34. G. G. Lorentz: Problem No. 4517, "Logarithmic Concavity," American Mathematical Monthly, vol. 61 (March, 1954), p. 205.

A sequence a_n, $n=0, 1, \cdots$ is called logarithmically concave if $a_n^2-a_{n-1}a_{n+1}\geq 0$ $n+1, 2, \cdots$ The following theorem is proved:

If a_n is logarithmically concave, then their convolution

$$c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$$

has the same properites.

35. R. J. Evey and P. C. Hammer; "Compositions of Convex and Concave Functions," Bulletin of American Mathematical Society, V. 63 (January 1957), p. 31.

G. Szekeres recently gave necessary and sufficient conditions that an increasing function with positive derivations and continuous second derivative be representable as a composition of two increasing functions one of which is concave, the other convex. In this paper, there are established the necessary and sufficient conditions for any increasing function to be represented as the composition of convex fuctions over the entire range, both functions being piece-wise linear and properly increasing.

抵抗函数の性質について（摘要）

伊　波　直　朗

たくさんの学者が，いままでに凸函数について相当な研究をしてきている。凹函数については，しかし，いままであまり研究されていない。これは，一つには，おなじ理論が凸函数についても，凹函数についても適用できるからである。なぜなら，$f(x)$ を凸函数とすれば，$-f(x)$ は凹函数となるからである。凹函数の研究があまりされていないもう一つの理由は，凹函数の応用性があまり発見されていないという点にある。

1956年に，C. E. Shannon と D. W. Hagelbarger が，ある種の抵抗函数は，直流回路網において，凹函数であるということを証明した。この論文は，この定理が，任意の線型集中定数回路網に適用できるということを証明しようとするものである。ほかに，任意の線型集中定数回路網，線型分布定数回路網又は電磁理論における抵抗函数の性質についての定理をいくつか発表し，証明してある。

この論文中であたらしく証明された定理の中には，次のようなものがある。

定理 1　Nがインピーダンスと起電力の線型回路網で負抵抗を含んでいないとき，またはそのような回路網の一部であるときは，Nの各枝の電流は

$$f=\sum_{j=1}^{m} R_jI_j^2-2\sum_{j=1}^{m} E_jI_j$$

が最小となるように分布する。

系 1—1　Nが起電力も負抵抗もふくまない線型のインピーダンス回路網であるときは，Nの各枝の電流は，Nによって消費される全電力が最小となるように分布する。

定理 2　Nが負コンダクタンスを含まないアドミッタンスと電流源の線型回路網であるとき，またはそのような回路網の一部であるときは，N各節点のポテンシャルは，

$$f=\sum_{j=1}^{m}\sum_{k=1}^{m}[g_{jk}(v_j-v_k)^2-2i_{jk}(v_j-v_k)]$$

が最小となるように分布する。

系 2—1　Nが電流源や負コンダクタンスを含まない線型のアドミッタンス回路網であるとき，または，このような回路網の一部であるときは，Nの各節点のポテンシャルは，Nに消費される全電力が最小となるように分布する。

定理 3　二点 A, B が線型回路網によって連結されているときは，この回路網中の一抵抗がその抵抗値を減少することによって A, B 間の抵抗値が増大することはない。

系 3—1　任意の二端子抵抗回路網において，任意のスウィッチを入れるとき，端子間の抵抗が増大することはない。

系 3—2　任意の四端子抵抗回路網において，任意の抵抗の抵抗値の減少によって回路の入力または出力抵抗が増加することはない。

定理 4　Nが非負抵抗 $R_1, R_2, R_3, \cdots, R_n$ の二端子線型回路網で，R が N の両端子間の抵抗であるときは，R は $R_1, R_2, R_3, \cdots, R_n$ の凹函数である。

定理 6　Nが非負抵抗の多端子平衡線型回路網であるとき，Nの入力抵抗はN中に含まれている抵抗の凹函数である。

定理 7　任意の媒質中を伝播している電磁波の入力コンダクタンスは

（a）　その媒質の固有インピーダンスが実で波が受けるアドミッタンスが実で且領域内のあらゆる G, G′ について $GG'\leq 4$ であれば，波が遭遇するコンダクタンスの凹函数であり

（b）　その媒質の固有インピーダンスが実で波が遭遇するアドミッタンスが実で且領域内のあらゆる G, G′ について $GG'\geq 4$ であれば，波が遭遇するアドミッタンスの凸函数である。

全部で2つの補助定理，8つの定理，9つの系が証明されている。