与られた数の「節」や「枝」をもった「木」の数および「包括図」に関する諸定理

Theorems on number of "trees" with a given number of "knots" or "branches" and on "spanning graphs"

By

Tadao INAMI*

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1. The number of trees with n branches or with m knots.

A tree with n branches has either $1, 2, 3, \cdots$, or n main branches. If the tree has one main branch, it can only be formed by adding on to this main branch a tree with n-1 branches. If the tree has two main branches, then $p+q$ being a partition of n-2, the tree can be formed by adding onto one main branch a tree of p branches, and to the other main branch a tree of q branches, the number of trees so obtained is

$$\begin{array}{ll} A_pA_q & \text{if} \quad p \neq q \\ \frac{1}{2}A_p(A_p+1) & \text{if} \quad p=q=\frac{1}{2}(n-2) \end{array}$$

where A_p and A_q are the numbers of trees that can be formed by p and q branches, respectively.

If the tree has three main branches, then if $p+q+r$ is any partition of $n-3$, A_n contains the part

$$\begin{array}{ll} A_pA_qA_r & \text{if} \quad p \neq q \neq r \\ \frac{1}{2}A_p(A_p+1)A_r & \text{if} \quad p=q \neq r \\ \frac{1}{6}A_p(A_p+1)(A_p+2) & \text{if} \quad p=q=r=\frac{1}{3}(n-3) \end{array}$$

The prereding rule for the formation of the number A_n is completely expressed by the "generating function":

$$a(x)=A_0+A_1x+A_2x^2+\cdots\cdots=(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}\cdots\cdots$$

Comparing the expanded right hand with the middle section of the equation, the number A_n of tropologically distinct trees with n branches is obtained. The number of topologically distinct rooted trees, C_n, with n knots is equal to the number of topologically distinct trees with $n-1$ branches.

$$\begin{aligned} C(x)&=C_1x+C_2x^2+C_3x^3+\cdots\cdots=x(1-x)^{-c_1}(1-x^2)^{-c_2}(1-x^3)^{-c_3}\cdots\cdots \\ &=x\left(1+C_1x+\frac{C_1(C_1+1)}{2!}x^2+\frac{C_1(C_1+1)(C_1+2)}{3!}x^2+\cdots\cdots\right) \\ &\quad\times\left(1+C_2x^2+\frac{C_2(C_2+1)}{2!}x^4+\cdots\cdots\right)\times\cdots\cdots \end{aligned}$$

giving

$$\begin{aligned} &C_1=1 \\ &C_2=C_1=1 \\ &C_3=\frac{C_1(C_1+1)}{2!}+C_2=2 \\ &C_4=\frac{C_1(C_1+1)(C_1+2)}{3!}+C_1C_2+C_3=4 \\ &C_5=\frac{C_1(C_1+1)(C_1+2)(C_1+3)}{4!}+\frac{C_1(C_1+1)}{2!}+C_1C_3+C_4+\frac{C_2(C_2+1)}{2!}=9 \end{aligned}$$

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* Agriculture, Home Economics and Engineering Division, University of the Ryukyus.

Thus

n	1	2	3	4	5	6	7	8	9	10	11	12
A_n	1	2	4	9	20	48	115	286	719	1842	4766	12486
C_n	1	1	2	4	9	20	48	115	286	719	1842	4766

The generating function $c(x)$ may be written

$$c(x)=xe^{\frac{c(x)}{1}+\frac{c(x^2)}{2}+\frac{c(x^3)}{3}+\cdots}$$
$$=x\left[1+\frac{c(x)}{1!}+\frac{c(x)^2+c(x^2)}{2!}+\frac{c(x)^3+3c(x)c(x^2)+2c(x^3)}{3!}+\cdots\cdot\right]$$

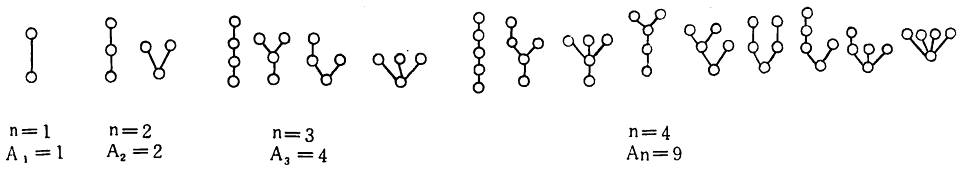


Fig. 1. A_n

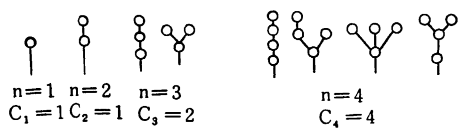


Fig. 2. C_n

Both forms of expansion have their advantages: The first form serves as the starting point for asymptotic computation of C_n and the number, C_n', of topologically distinct unrooted trees with n knots. The second serves as the starting point for generalization [In the general term of the series, the cycle-index of the symmetrical groups of n elements is recognized.]

The number of trees D_n which can be formed with n given knots labelled $\alpha, \beta, \gamma, \cdots$ is given by n^{n-2} (See Fig. 3)

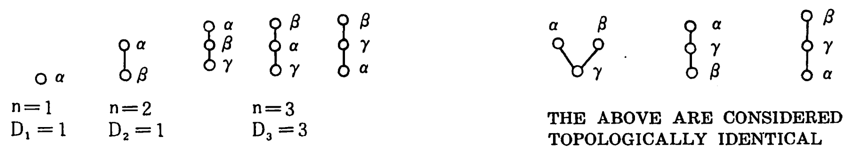


Fig. 3. D_n

2. Number of trees with a given number of free branches

The number of trees B_n with a given number, n, of free branches, bifurcations at least, is given, according to Cayley[1] by

$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^2)^{-B_3}\cdots=1+x+2B_2x^2+2B_3x^3+\cdots$$

to give, for $n=2,3,\cdots,7$,

n	2	3	4	5	6	7
B_n	1	2	5	12	33	90

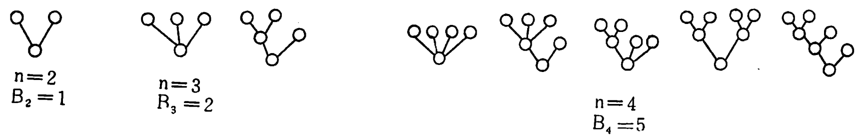
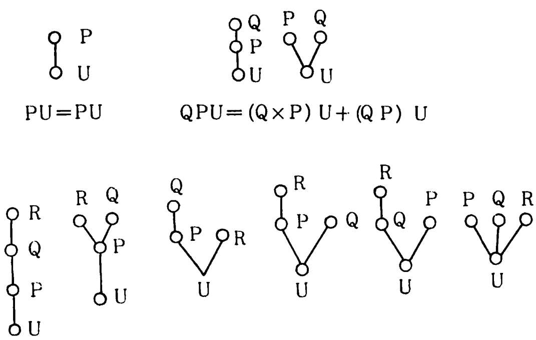


Fig. 4. B_n

3. Multiple-operators and Labelled trees.

If U is an operand and P, Q, R,$\cdots$ are operators, then there exists a certain relationship between decomposition of multiple-operators and trees labelled by operators and the operand[1)], as shown in Fig. 5; where Q×P denotes the mere algebraic product of Q and P (so does the bifurcations of branches Q and P), while QP (and the succession of Q and P nodes in cascade) denotes the result of operation performed upon P as operand.



RQPU=((RQ)P)U+$\cdots$+(RQ×P)U+(R×Q×P)U

Fig. 5. The relationship between multiple-operators and labelled trees where no transposition of order PQR$\cdots$ occurs from root to free branches.

4. Number of bifuraction-trees with *n* end-points.

By a bifurcation-tree, we mean a tree with non-terminal knots of three branches. The number D_n of bifurcation tree with a given number, n, of end points is, according to Cayley[2], expressed in terms of D_1, D_2, $\cdots$, D_{n-1}, as

$$D_n=D_1D_{n-1}+D_2D_{n-2}+\cdots+D_{n-1}D_1$$

if we let, arbitrarily

$$D_1=1.$$

If we consider a function

$$f(x)=D_1+xD_2+x^2D_3+\cdots$$

we have

$$\begin{aligned} f^2&=D_1D_1+x(D_1D_2+D_2D_1)+x^2(D_1D_3+D_2D_2+D_3D_1)+\cdots \\ &=D_2+xD_3+x^2D_4+\cdots \end{aligned}$$

Thus

$$xf^2=f-1$$

$$\therefore\quad f=\frac{1-\sqrt{1-4x}}{2x}$$

But

$$\sqrt{1-4x}=1-\frac{1}{2}4x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}(4x)^2-\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3}(4x^3)+\cdots$$

$$=1-2x-2x^2-4x^3-10x^4-\cdots\cdot$$

$$\therefore\quad f=1+x+2x^2+5x^3+\cdots$$

The coefficients of $x^0, x^1, x^2, x^3, \cdots$ are equal to $D_1, D_2, D_3, D_4, \cdots$. The expression for general term is seen to be

$$D_n=\frac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-3)}{1\cdot 2\cdot 3\cdot\ \cdots\ \cdot n}2^{n-1}$$

giving

n	1	2	3	4	5	6	7
D_n	1	1	2	5	14	42	132

Illustrations for $n=2, 3, 4$ are shown in Figs. 6, 7, 8, respectively.

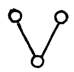
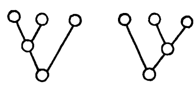
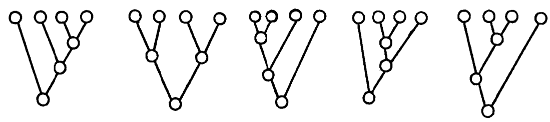




Fig. 6. D_2 Fig. 7. D_3 Fig. 8. D_4

If A, B, C, D are symbols capable of successive binary combinations, but do not satisfy the associative law, number of the different significations of the ambiguous expression $ABC\cdots\cdot N$ is equal to D_n. For instance, AB has only one meaning; ABC may mean either $A\cdot BC$ or $AB\cdot C$; $ABCD$ may mean $A(B\cdot CD)$, $AB\cdot CD$, $(AB\cdot C)D$, $(A\cdot BC)D$, or $A(BC\cdot D)$.

5. Spanning graphs.

Def. Spanning sub-graph. A sub-graph spans a graph if it contains all the vertices of the graph.

Def. Maximal graph. A graph is maximal if it is not contained in any larger graph of the same sort.

Def. Minimal graph. A graph is minimal if it does not contain any smaller graph of the same sort.

Def. Forest. A forest is a graph with no loops.

Problem. Give a practical method for constructing a spanning subtree of minimum length.

Solution. There is no loss of generality in assuming that the given connected graph G is complete, i.e., that every pair of vertices is connected by an edge. For if any edge of G is "missing", it is possible to consider the "missing" edge as an edge of infinite length.

Construction A. Perform the following step as many times as possible: Among the edges of G not yet chosen, choose the shortest edge which does not form any loops with those edges already chosen. Clearly the set of edges eventually chosen must form a spanning tree of G, and in fact it forms a shortest spanning tree.

Construction A'. (Dual of A) Perform the following step as many times as possible: Among the edges not yet chosen, choose the longest edge whose removal will not disconnect them. Clearly the set of edges not eventually chosen forms a spanning tree of G, and in fact it forms a shortest spanning tree.

Construction B. Let V be an arbitrary but fixed (nonempty) subset of the vertices of G. Then perform the following step as many times as possible: Among the edges of G which are not yet chosen but which are connected either to a vertex of V or to an edge already chosen, pick the shortest edge which does not form any loops with the edges already chosen. Clearly the set of edges eventually chosen forms a spanning tree of G, and in fact it forms a shortest spanning tree. In case V is the set of all vertices of G, then Conssruction B reduces to Construction A.

Theorem 1. If G is a connected graph with n vertices, and T is a subgraph of G, then the following conditions are all equivalent:

(a) T is a spanning tree of G;

(b) T is a maximal forest in G;

(c) T is a minimal connected spanning graph of G;

(d) T is a forest with n-1 edges.

(e) T is a connected spanning graph with n-1 edges.

Theorem 2. If the edges of G all have distinct lengths, then T is unique, where T is any shortest spanning tree of G.

Proof. In Construction A in the above problem, let the edges chosen be called $a_1, \cdots,$ a_{n-1} in the order chosen. Let A_i be the forest consisting of edges a_1 through a_i. From the hypothesis that the edges of G have distinct lengths, it is easily seen that Construction A proceeds in a unique manner. Thus the A_i are unique, and hence also T.

Suppose $T \neq A$. Let a_i be the first edge of A_{n-1} which is not in T. Then $a_1, \cdots,$ a_{i-1} are in T. $TU\ a_i$ must have exactly one loop, which must contain a_i. This loop must also contain some edge e which is not in A_{n-1}. Then $TU\ a_i - e$ is a forest with $n-1$ edgef.

As $A_{i-1}\ Ue$ is contained in the last named forest, it is a forest, so from Construction A,

$$\text{length}(e) > \text{length}(a_i)$$

But then $TU\ a_i - e$ is shorter than T. This contradicts the definitign of T, and hence proves indirectry that $T = A_{n-1}$. *Q. E. D.*

References

1. Arthur Cayley: "On the theory of the analytical forms called trees," Philosophical Magazine, vol. XIII, pp. 172–176; 1857.
2. Arthur Cayley: "On the analytical forms called trees," Philosophical Magazine, vol. XIII, pp. 374–378; 1859.

与られた数の「節」や「枝」をもった「木」の数および「包括図」に関する諸定理（摘要）

伊 波 直 朗

この論文では，n 個の「枝」または m 個の「節」をもったトポロギー的に異なる「木」の数，$\alpha, \beta, \gamma, \cdots$ とレッテルをはってある n 個の「節」をもったトポロギー的に異なる「木」の数，n 個の「枝端」をもったトポロギー的に異なる「木」の数，多演算子とレッテルばり木の数との関係，n 個の「枝端」をもったトポロギー的に異なる「分岐木」の数，「包括図」，「最大図」，「最小図」，「森」に関する諸定理を提起，証明した。

n 個の「枝」をもったトポロギー的に異なる「木」の数 A_n および n 個の「節」をもったトポロギー的に異なる「木」の数 C_n は，それぞれ次の「生成函数」$a(x)$, $c(x)$ によってあらわすことができる。

$$a(x)=A_0+A_1x+A_2x^2+\cdots=(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}\cdots$$
$$c(x)=C_1x+C_2x^2+C_3x^3+\cdots=x(1-x)^{-C_1}(1-x^2)^{-C_2}(1-x^3)^{-C_3}\cdots$$

すなわち，$n=1, 2, 3, \cdots, 12$ に対する A_n の値は，1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486 で，C_n の値は，1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766 である。

n 個の「枝端」をもったトポロギー的に異なる「木」の数 B_n は次の「生成函数」$b(x)$ によってあらわすことができる。

$$b(x)=(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}=1+x+2B_2x^2+2B_3x^3+\cdots$$

すなわち，$n=1, 2, 3, \cdots, 9$ に対する B_n の値はそれぞれ 0, 1, 2, 5, 12, 33, 90 である。

n 個の「枝端」をもったトポロギー的に異なる「分岐木」の数 D_n は次の「生成函数」$d(x)$ によってあらわすことができる。

$$d(x)=D_1+D_2x+D_3x^2+\cdots=\frac{1-\sqrt{1-4x}}{2x}$$

すなわち，$n=1, 2, 3, \cdots, 7$ に対する D_n の値は 1, 1, 2, 5, 14, 42, 132 である。

「図」の全頂点を含む「部分図」はその図を「包括する」という。同一種類のそれより大きな「図」に含まれない「図」は「最大」であるという。同一種類のそれより小さな「図」を含まない「図」は「最小」であるという。ループを含まない「図」を「森」という。

上の定義にしたがえば，次の定理が成立する。

定理 1　もし G が n 個の頂点をもつ連結した「図」であり，T が G の部分図であれば，次の条件は等価である。

(a)　T は G の包括木である。
(b)　T は G の最大森である。
(c)　T は G の最小連結包括図である。
(d)　T は $n-1$ 個の枝をもった森である。
(e)　T は $n-1$ 個の枝をもった連結包括図である。

定理 2　G の枝が全部ちがった長さであれば前定理の条件を満足する T は一意的にさだまる。このとき T は G の任意の最短包括木である。

G の最短包括木を作るに当っての実際的な方法も示してある。