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与られた数の「節」や「枝」をもった「木」の数および「包括図」に関する諸定理

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Theorems on number of "trees" with a given number of "knots" or "branches" and on "spanning graphs"

By

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1. The number of trees with n branches or with m knots.

A tree with n branches has either 1, 2, 3, ..., or n main branches. If the tree has one main branch, it can only be formed by adding on to this main branch a tree with $n-1$ branches. If the tree has two main branches, then $p+q$ being a partition of $n-2$, the tree can be formed by adding onto one main branch a tree of p branches, and to the other main branch a tree of q branches, the number of trees so obtained is

$$\begin{aligned} &A_p A_q && \text{if } p \neq q \\ &\frac{1}{2} A_p (A_p + 1) && \text{if } p = q = \frac{1}{2}(n-2) \end{aligned}$$

where A_p and A_q are the numbers of trees that can be formed by p and q branches, respectively.

If the tree has three main branches, then if $p+q+r$ is any partition of $n-3$, A_n contains the part

$$\begin{aligned} &A_p A_q A_r && \text{if } p \neq q \neq r \\ &\frac{1}{2} A_p (A_p + 1) A_r && \text{if } p = q \neq r \\ &\frac{1}{6} A_p (A_p + 1) (A_p + 2) && \text{if } p = q = r = \frac{1}{3}(n-3) \end{aligned}$$

The preceding rule for the formation of the number A_n is completely expressed by the "generating function":

$$a(x) = A_0 + A_1 x + A_2 x^2 + \dots = (1-x)^{-1} (1-x^2)^{-A_1} (1-x^3)^{-A_2} \dots$$

Comparing the expanded right hand with the middle section of the equation, the number A_n of topologically distinct trees with n branches is obtained. The number of topologically distinct rooted trees, C_n , with n knots is equal to the number of topologically distinct trees with $n-1$ branches.

$$\begin{aligned} C(x) &= C_1 x + C_2 x^2 + C_3 x^3 + \dots = x(1-x)^{-c_1} (1-x^2)^{-c_2} (1-x^3)^{-c_3} \dots \\ &= x \left(1 + C_1 x + \frac{C_1(C_1+1)}{2!} x^2 + \frac{C_1(C_1+1)(C_1+2)}{3!} x^3 + \dots \right) \\ &\quad \times \left(1 + C_2 x^2 + \frac{C_2(C_2+1)}{2!} x^4 + \dots \right) \times \dots \end{aligned}$$

giving

$$\begin{aligned} C_1 &= 1 \\ C_2 &= C_1 = 1 \\ C_3 &= \frac{C_1(C_1+1)}{2!} + C_2 = 2 \\ C_4 &= \frac{C_1(C_1+1)(C_1+2)}{3!} + C_1 C_2 + C_3 = 4 \\ C_5 &= \frac{C_1(C_1+1)(C_1+2)(C_1+3)}{4!} + \frac{C_1(C_1+1)}{2!} + C_1 C_3 + C_4 + \frac{C_2(C_2+1)}{2!} = 9 \end{aligned}$$

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Thus

n	1	2	3	4	5	6	7	8	9	10	11	12
A_n	1	2	4	9	20	48	115	286	719	1842	4766	12486
C_n	1	1	2	4	9	20	48	115	286	719	1842	4766

The generating function $c(x)$ may be written

$$c(x) = xe^{\frac{c(x)}{1} + \frac{c(x)^2}{2} + \frac{c(x)^3}{3} + \dots}$$

$$= x \left[1 + \frac{c(x)}{1!} + \frac{c(x)^2 + c(x)^2}{2!} + \frac{c(x)^3 + 3c(x)c(x)^2 + 2c(x)^3}{3!} + \dots \right]$$

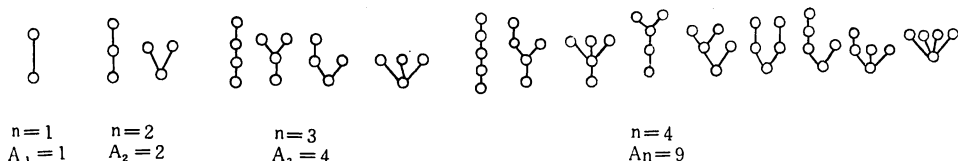


Fig. 1. A_n

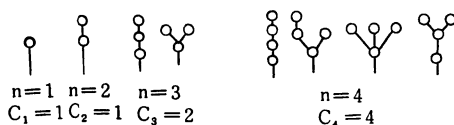


Fig. 2. C_n

Both forms of expansion have their advantages: The first form serves as the starting point for asymptotic computation of C_n and the number, C'_n , of topologically distinct unrooted trees with n knots. The second serves as the starting point for generalization [In the general term of the series, the cycle-index of the symmetrical groups of n elements is recognized.]

The number of trees D_n which can be formed with n given knots labelled $\alpha, \beta, \gamma, \dots$ is given by n^{n-2} (See Fig. 3)

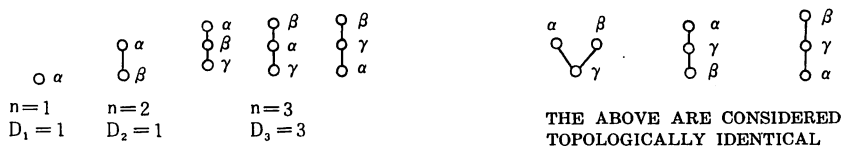


Fig. 3. D_n

2. Number of trees with a given number of free branches

The number of trees B_n with a given number, n , of free branches, bifurcations at least, is given, according to Cayley¹⁾ by

$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3} \dots = 1+x+2B_2x^2+2B_3x^3+\dots$$

to give, for $n=2, 3, \dots, 7$,

n	2	3	4	5	6	7
B_n	1	2	5	12	33	90



Fig. 4. B_n

3. Multiple-operators and Labelled trees.

If U is an operand and P, Q, R, \dots are operators, then there exists a certain relationship between decomposition of multiple-operators and trees labelled by operators and the operand¹⁾, as shown in Fig. 5; where $Q \times P$ denotes the mere algebraic product of Q and P (so does the bifurcations of branches Q and P), while QP (and the succession of Q and P nodes in cascade) denotes the result of operation performed upon P as operand.

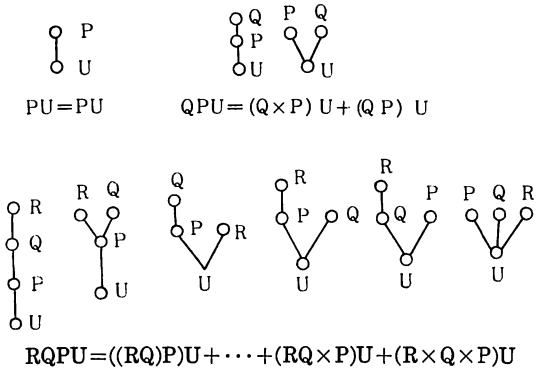


Fig. 5. The relationship between multiple-operators and labelled trees where no transposition of order $PQR \dots$ occurs from root to free branches.

4. Number of bifurcation-trees with n end-points.

By a bifurcation-tree, we mean a tree with non-terminal knots of three branches. The number D_n of bifurcation tree with a given number, n , of end points is, according to Cayley², expressed in terms of D_1, D_2, \dots, D_{n-1} , as

$$D_n = D_1 D_{n-1} + D_2 D_{n-2} + \dots + D_{n-1} D_1$$

if we let, arbitrarily

$$D_1 = 1.$$

If we consider a function

$$f(x) = D_1 + xD_2 + x^2D_3 + \dots$$

we have

$$\begin{aligned} f^2 &= D_1 D_1 + x(D_1 D_2 + D_2 D_1) + x^2(D_1 D_3 + D_2 D_2 + D_3 D_1) + \dots \\ &= D_2 + xD_3 \qquad \qquad \qquad + x^2 D_4 \qquad \qquad \qquad + \dots \end{aligned}$$

Thus

$$x f^2 = f - 1$$

$$\therefore f = \frac{1 - \sqrt{1 - 4x}}{2x}$$

But

$$\begin{aligned} \sqrt{1-4x} &= 1 - \frac{1}{2}4x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \cdot 2}(4x)^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2 \cdot 3}(4x^3) + \dots \\ &= 1 - 2x - 2x^2 - 4x^3 - 10x^4 - \dots \\ \therefore f &= 1 + x + 2x^2 + 5x^3 + \dots \end{aligned}$$

The coefficients of $x^0, x^1, x^2, x^3, \dots$ are equal to $D_1, D_2, D_3, D_4, \dots$. The expression for general term is seen to be

$$D_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} 2^{n-1}$$

giving

n	1	2	3	4	5	6	7
D_n	1	1	2	5	14	42	132

Illustrations for $n=2, 3, 4$ are shown in Figs. 6, 7, 8, respectively.

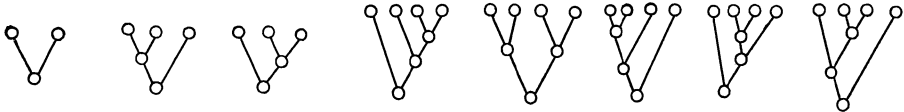


Fig. 6. D_2

Fig. 7. D_3

Fig. 8. D_4

If A, B, C, D are symbols capable of successive binary combinations, but do not satisfy the associative law, number of the different significations of the ambiguous expression $ABC \dots N$ is equal to D_n . For instance, AB has only one meaning; ABC may mean either $A \cdot BC$ or $AB \cdot C$; $ABCD$ may mean $A(B \cdot CD)$, $AB \cdot CD$, $(AB \cdot C)D$, $(A \cdot BC)D$, or $A(BC \cdot D)$.

5. Spanning graphs.

Def. Spanning sub-graph. A sub-graph spans a graph if it contains all the vertices of the graph.

Def. Maximal graph. A graph is maximal if it is not contained in any larger graph of the same sort.

Def. Minimal graph. A graph is minimal if it does not contain any smaller graph of the same sort.

Def. Forest. A forest is a graph with no loops.

Problem. Give a practical method for constructing a spanning subtree of minimum length.

Solution. There is no loss of generality in assuming that the given connected graph G is complete, i.e., that every pair of vertices is connected by an edge. For if any edge of G is "missing", it is possible to consider the "missing" edge as an edge of infinite length.

Construction A. Perform the following step as many times as possible: Among the edges of G not yet chosen, choose the shortest edge which does not form any loops with those edges already chosen. Clearly the set of edges eventually chosen must form a spanning tree of G , and in fact it forms a shortest spanning tree.

Construction A'. (Dual of A) Perform the following step as many times as possible: Among the edges not yet chosen, choose the longest edge whose removal will not disconnect them. Clearly the set of edges not eventually chosen forms a spanning tree of G , and in fact it forms a shortest spanning tree.

Construction B. Let V be an arbitrary but fixed (nonempty) subset of the vertices of G . Then perform the following step as many times as possible: Among the edges of G which are not yet chosen but which are connected either to a vertex of V or to an edge

already chosen, pick the shortest edge which does not form any loops with the edges already chosen. Clearly the set of edges eventually chosen forms a spanning tree of G , and in fact it forms a shortest spanning tree. In case V is the set of all vertices of G , then Construction B reduces to Construction A .

Theorem 1. If G is a connected graph with n vertices, and T is a subgraph of G , then the following conditions are all equivalent:

- (a) T is a spanning tree of G ;
- (b) T is a maximal forest in G ;
- (c) T is a minimal connected spanning graph of G ;
- (d) T is a forest with $n-1$ edges.
- (e) T is a connected spanning graph with $n-1$ edges.

Theorem 2. If the edges of G all have distinct lengths, then T is unique, where T is any shortest spanning tree of G .

Proof. In Construction A in the above problem, let the edges chosen be called a_1, \dots, a_{n-1} in the order chosen. Let A_i be the forest consisting of edges a_1 through a_i . From the hypothesis that the edges of G have distinct lengths, it is easily seen that Construction A proceeds in a unique manner. Thus the A_i are unique, and hence also T .

Suppose $T \neq A$. Let a_i be the first edge of A_{n-1} which is not in T . Then a_1, \dots, a_{i-1} are in T . $T \cup a_i$ must have exactly one loop, which must contain a_i . This loop must also contain some edge e which is not in A_{n-1} . Then $T \cup a_i - e$ is a forest with $n-1$ edges.

As $A_{i-1} \cup e$ is contained in the last named forest, it is a forest, so from Construction A ,
 $\text{length}(e) > \text{length}(a_i)$

But then $T \cup a_i - e$ is shorter than T . This contradicts the definition of T , and hence proves indirectly that $T = A_{n-1}$. *Q. E. D.*

References

1. Arthur Cayley: "On the theory of the analytical forms called trees," *Philosophical Magazine*, vol. XIII, pp. 172-176; 1857.
2. Arthur Cayley: "On the analytical forms called trees," *Philosophical Magazine*, vol. XIII, pp. 374-378; 1859.

与られた数の「節」や「枝」をもった「木」の数および 「包括図」に関する諸定理 (摘要)

伊 波 直 朗

この論文では、 n 個の「枝」または m 個の「節」をもったトポロギー的に異なる「木」の数、 $\alpha, \beta, \gamma, \dots$ とレットテルをはってある n 個の「節」をもったトポロギー的に異なる「木」の数、 n 個の「枝端」をもったトポロギー的に異なる「木」の数、多演算子とレットテルばり木の数との関係、 n 個の「枝端」をもったトポロギー的に異なる「分岐木」の数、「包括図」、「最大図」、「最小図」、「森」に関する諸定理を提起、証明した。

n 個の「枝」をもったトポロギー的に異なる「木」の数 A_n および n 個の「節」をもったトポロギー的に異なる「木」の数 C_n は、それぞれ次の「生成函数」 $a(x), c(x)$ によってあらわすことができる。

$$\begin{aligned} a(x) &= A_0 + A_1x + A_2x^2 + \dots = (1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2} \dots \\ c(x) &= C_1x + C_2x^2 + C_3x^3 + \dots = x(1-x)^{-C_1}(1-x^2)^{-C_2}(1-x^3)^{-C_3} \dots \end{aligned}$$

すなわち、 $n=1, 2, 3, \dots, 12$ に対する A_n の値は、1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486 で、 C_n の値は、1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766 である。

n 個の「枝端」をもったトポロギー的に異なる「木」の数 B_n は次の「生成函数」 $b(x)$ によってあらわすことができる。

$$b(x) = (1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3} = 1+x+2B_2x^2+2B_3x^3+\dots$$

すなわち、 $n=1, 2, 3, \dots, 9$ に対する B_n の値はそれぞれ 0, 1, 2, 5, 12, 33, 90 である。

n 個の「枝端」をもったトポロギー的に異なる「分岐木」の数 D_n は次の「生成函数」 $d(x)$ によってあらわすことができる。

$$d(x) = D_1 + D_2x + D_3x^2 + \dots = \frac{1-\sqrt{1-4x}}{2x}$$

すなわち、 $n=1, 2, 3, \dots, 7$ に対する D_n の値は 1, 1, 2, 5, 14, 42, 132 である。

「図」の全頂点を含む「部分図」はその図を「包括する」という。同一種類のそれより大きな「図」に含まれない「図」は「最大」であるという。同一種類のそれより小さな「図」を含まない「図」は「最小」であるという。ループを含まない「図」を「森」という。

上の定義にしたがえば、次の定理が成立する。

定理 1 もし G が n 個の頂点をもつ連結した「図」であり、 T が G の部分図であれば、次の条件は等価である。

- (a) T は G の包括木である。
- (b) T は G の最大森である。
- (c) T は G の最小連結包括図である。
- (d) T は $n-1$ 個の枝をもった森である。
- (e) T は $n-1$ 個の枝をもった連結包括図である。

定理 2 G の枝が全部ちがった長さであれば前定理の条件を満足する T は一意的にさだまる。このとき T は G の任意の最短包括木である。

G の最短包括木を作るに当たっての実際的な方法も示してある。