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On differentiable involutions on homotopy spheres †

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1. In this note we shall find a necessary condition for a homotopy sphere to admit a smooth involution with exactly two fixed points and obtain some corollaries.

Theorem 1. *If a homotopy sphere M^n ($n > 4$) admits a smooth involution with exactly two fixed points, then $RP^n \# M^n$ is diffeomorphic to RP^n (if n is odd then in orientation preserving fashion), where RP^n is the real projective space.*

According to Milnor [1], the inertia group of RP^7 is trivial. Hence we obtain

Corollary 1. *If a homotopy 7-sphere admits a smooth involution with exactly two fixed points, then it is diffeomorphic to the standard sphere S^7 .*

It is known [2] that every smooth involution on S^n ($n \geq 5$) which fixes exactly two points, is topologically equivalent to the linear involution. However, combining corollary 1 with the result of R. Lee [3, corollary 3], we have

Corollary 2. *There exists a smooth involution on S^7 with exactly two fixed points which is differentiably inequivalent to the linear involution.*

Let M^{2k+1} be a homotopy sphere of order greater than two. Then comparing 2-fold covering spaces it is easily seen that $RP^{2k+1} \# M^{2k+1}$ is not diffeomorphic to RP^{2k+1}

Corollary 3. *If a homotopy sphere M^{2k+1} ($k > 1$) admits a smooth involution with exactly two fixed points, then $2M^{2k+1} = S^{2k+1}$.*

On the other hand

Theorem 2. *Let M^{2k} be a homotopy sphere of even dimension. Then there is an involution on $2M^{2k}$ with exactly two fixed points.*

Corollary. *Let M^{10} be a generator of the 3-component of $\Theta_{10} \cong Z_2 + Z_3$. Then $RP^{10} \# M^{10}$ is diffeomorphic to RP^{10} .*

Concerning smooth actions of circle group, the following theorem holds.

Theorem 3. *If a homotopy sphere M^{2k} ($k > 2$) admits a smooth S^1 action which fixes exactly two points and acts freely otherwise,*

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then $CP^k \# M^{2k}$ is diffeomorphic to CP^k where CP^k is the complex projective space.

According to K. Kawakubo [4], the inertia group of CP^k is trivial for $k \leq 8$.

Corollary. Let M^{2k} be a homotopy sphere of dimension $2k$, $3 \leq k \leq 8$. If M^{2k} admits a smooth S^1 action which fixes exactly two points and acts freely otherwise, then M^{2k} is diffeomorphic to S^{2k} .

Let $\phi(2k)$ be the set of orientation-preserving equivariant diffeomorphism classes of homotopy $2k$ -spheres with semi-free S^1 -action which fixes exactly two points. Then it can be proved that $\phi(2k)$ ($k \geq 3$) forms a group (under the equivariant connected sum operation) isomorphic to the group of pseudo-diffeotopy classes of those diffeomorphisms of the $(k-1)$ -complex projective space onto itself that are homotopic to the identity map. (See [3].) Denote the latter group by $D_0(CP^{k-1})$.

Corollary. For $3 \leq k \leq 8$, the equivalence classes of semi-free S^1 -actions on S^{2k} with exactly two fixed points form a group isomorphic to $D_0(CP^{k-1})$.

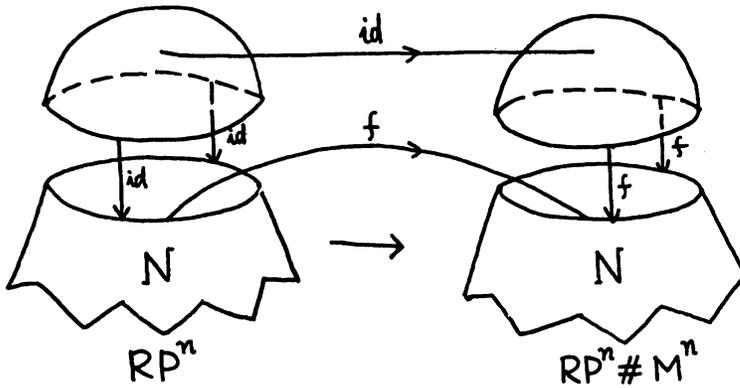
2. We prove theorem 1, 2.

Lemma. If a homotopy sphere M^n ($n > 5$) admits a smooth involution with exactly two fixed points, then M^n is diffeomorphic to a twisted sphere obtained by gluing two copies of disks D_i^n ($i=1,2$) together along their boundaries by such a diffeomorphism f of S^{n-1} onto itself as commutes with the antipodal map of S^{n-1} .

Proof. Let T be a smooth involution on M^n which fixes two points P_1, P_2 . We can choose small invariant disk-neighborhoods D_i^n of P_i ($i=1,2$) so that the restricted involutions $T|D_i^n$ may be regarded as the linear involutions. The quotient spaces $\partial D_i / T$ ($i=1,2$) are the real projective spaces RP^{n-1} and $(M^n - \text{int } D_1^n - \text{int } D_2^n) / T$ forms an h -cobordism between them. Since $Wh(\mathbb{Z}_2) = 0$, $(M^n - \text{int } D_1^n - \text{int } D_2^n) / T$ is diffeomorphic to $RP^{n-1} \times [0, 1]$. Let $g: RP^{n-1} \rightarrow RP^{n-1}$ be a diffeomorphism. Then the diffeomorphism $g: S^{n-1} \rightarrow S^{n-1}$ which covers g , commutes with the antipodal map of S^{n-1} . Now lemma follows easily.

Proof of theorem 1. Let $W = \{ (x_1, \dots, x_{n+1}) \in S^n; -\frac{1}{2} \leq x_{n+1} \leq \frac{1}{2} \}$. Then W is invariant under the antipodal map of S^n , which is denoted by $A(n+1)$. Let N be the quotient space of W , then $RP^n - \text{int } N$ is a disk. Let M^n be a homotopy sphere which admits a smooth involution with two fixed points and let $f: S^{n-1} \rightarrow S^{n-1}$ be a diffeomorphism with $f \circ A(n) = A(n)$

of f such that $M^n = D^n \cup_f D^n$ (lemma). Identify ∂N with S^{n-1} . Then to prove theorem I it is sufficient to show the diffeomorphism $f : \partial N \rightarrow \partial N$ can be extended to a diffeomorphism $N \rightarrow N$. (See the figure) We regard



W as $S^{n-1} \times [-\frac{1}{2}, \frac{1}{2}]$. Then $A(n+1)(X, t) = (A(n)(X), -t)$, $X \in S^{n-1}$, $t \in [-\frac{1}{2}, \frac{1}{2}]$. Let $(f \times id) : S^{n-1} \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow S^{n-1} \times [-\frac{1}{2}, \frac{1}{2}]$ be the diffeomorphism which sends (X, t) to $(f(X), t)$. Since $f \circ A(n) = A(n) \circ f$, it follows that $A(n+1) \circ (f \times id) = (f \times id) \circ A(n+1)$. Hence $(f \times id)$ induces a diffeomorphism $F : N \rightarrow N$ such that $F|_{\partial N} = f$. Thus $f : \partial N \rightarrow \partial N$ can be extended to a diffeomorphism $N \rightarrow N$. This proves theorem 1.

Proof of theorem 2.

We construct an involution on $2M^{2k}$ as follows.

Let T be the involution on S^{2k} defined by $T(x_1, \dots, x_{2k}, x_{2k+1}) = (-x_1, \dots, -x_{2k}, x_{2k+1})$. It is clear that T preserves orientation. Let V^{2k} be a sufficiently small disk such that $V^{2k} \cap T(V^{2k}) = \emptyset$. Now make M^{2k} connected sum to S^{2k} in V^{2k} and in $T(V^{2k})$ so that the resulting manifold preserves the involution. Then we have a desired involution on $S^{2k} \# M^{2k} \# M^{2k} = 2M^{2k}$.

Reference

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