On differentiable involutions on homotopy spheres

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Hiroshi MAEHARA *

1. In this note we shall find a necessary condition for a homotopy sphere to admit a smooth involution with exactly two fixed points and obtain some corollaries.

Theorem 1. *If a homotopy sphere* M^n $(n>4)$ *admits a smooth involution with exactly two fixed points, then* $RP^n \# M^n$ *is diffeomorphic to* RP^n *(if n is odd then in orientation preserving fashion), where* RP^n *is the real projective space.*

According to Milnor [1], the inertia group of RP^7 is trivial. Hence we obtain

Corollary 1. *If a homotopy 7-sphere admits a smooth involution with exactly two fixed points, then it is diffeomorphic to the standard sphere* S^7.

It is known [2] that every smooth involution on S^n $(n \geqq 5)$ which fixes exactly two points, is topologically equivalent to the linear involution. However, combining corollary 1 with the result of R. Lee [3, corollary 3], we have

Corollary 2. *There exists a smooth involution on* S^7 *with exactly two fixed points which is differentiably inequivalent to the linear involution.*

Let M^{2k+1} be a homotopy sphere of order greater than two. Then compairing 2-fold covering spaces it is easily seen that $RP^{2k+1} \# M^{2k+1}$ is not diffeomorphic to RP^{2k+1}

Corollary 3. *If a homotopy sphere* M^{2k+1} $(k>1)$ *admits a smooth involution with exactly two fixed points, then* $2M^{2k+1} = S^{2k+1}$.

On the other hand

Theorem 2. Let M^{2k} *be a homotopy sphere of even dimmension. Then there is an involution on* $2M^{2k}$ *with exactly two fixed points.*

Corollary. Let M^{10} *be a generator of the 3-component of* $\Theta_{10} \doteqdot Z_2 + Z_3$. *Then* $RP^{10} \# M^{10}$ *is diffeomorphic to* RP^{10}.

Concerning smooth actions of circle group, the following theorem holds.

Theorem 3. If a homotopy sphere M^{2k} $(k>2)$ *admits a smooth* S^1 *action which fixes exactly two points and acts freely otherwise,*

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* Dept. of Math., Univ. of the Ryukyus.

then $CP^k \# M^{2k}$ *is diffeomorphic to* CP^k *where* CP^k *is the complex projective space.*

According to K. Kawakubo [4], the inertia group of CP^k is trivial for $k \leqq 8$.

Corollary. Let M^{2k} *be a homotopy sphere of dimmension 2k,* $3 \leqq k \leqq 8$. *If* M^{2k} *admits a smooth* S^1 *action which fixes exactly two points and acts freely otherwise, then* M^{2k} *is diffeomorphic to* S^{2k}.

Let $\phi(2k)$ be the set of orientation-presrving equivariant diffeomorphism classes of homotopy 2k-spheres with semi-free S^1-action which fixes exactly two points. Then it can be proved that $\phi(2k)$ $(k \geqq 3)$ forms a group (under the equivariant connected sum operation) isomorphic to the group of pseudo-diffeotopy classes of those diffeomorphisms of the (k-1) -complex projecctive space onto itself that are homotopic to the identity map. (See [3].) Donote the latter group by $D_o(CP^{k-1})$.

Corollary. For $3 \leqq k \leqq 8$, *the equivalence classes of semi-free* S^1*-actions on* S^{2k} *with exactly two fixed points form a group isomorphic to* $D_o(CP^{k-1})$.

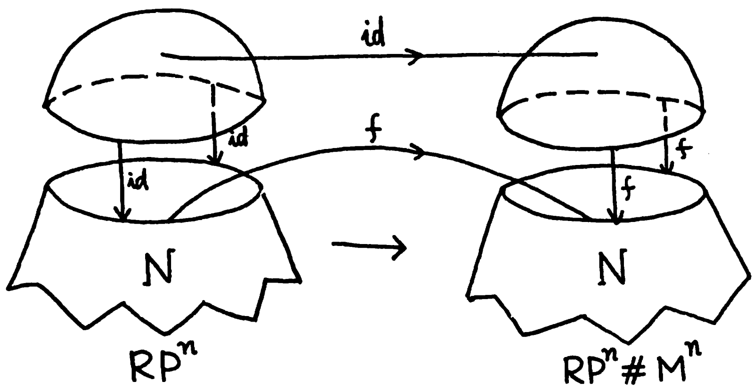
2. We prove theorem 1, 2.

Lemma. If a homotopy sphere M^n $(n>5)$ *admits a smooth involution with exactly two fixed points, then* M^n *is diffeomorphic to a twisted sphere obtained by gluing two copies of disks* D^n_i $(i=1,2)$ *together along their boundaries by such a diffeomorphism f of* S^{n-1} *onto itself as commutes with the antipodal map of* S^{n-1}.

Proof. Let T be a smooth involution on M^n which fixes two points P_1, P_2. We can choose small invariant disk-neighborhoods D^n_i of P_i $(i=1,2)$ so that the restricted involutions $T \mid D^n_i$ may be regarded as the linear in-volutions. The quotient spaces $\partial D_i / T$ $(i=1,2)$ are the real projective spaces RP^{n-1} and $(M^n - \text{int } D^n_1 - \text{int } D^n_2)/T$ forms an h-cobordism between them. Since $Wh(Z_2)=o$, $(M^n - \text{int } D^n_1 - \text{int } D^n_2)/T$ is diffeomorphic to $RP^{n-1} \times [0, 1.]$. Let $g: RP^{n-1} \to RP^{n-1}$ be a diffeomorphism. Then the diffeomorphism $g: S^{n-1} \to S^{n-1}$ which covers g, commutes with the antipodal map of S^{n-1}. Now lemma follows easily.

Proof of theorem 1. Let $W = \{(x_1, \dots, x_{n+1}) \in S^n ; -1/2 \leqq x_{n+1} \leqq 1/2\}$. Then W is invariant under the antipodal map of S^n, which is denoted by $A(n+1)$. Let N be the quotient space of W, then $RP^n - \text{int } N$ is a disk. Let M^n be a homotopy sphere which admits a smooth involution with two fixed points and let $f: S^{n-1} \to S^{n-1}$ be a diffeomorphism with $f \circ A(n) = A(n)$

$\circ$ f such that $M^n = D^n \cup_f D^n$ (lemma) . Identify ∂N with S^{n-1}. Then to prove theorem I it is sufficient to show the diffeomorphism f : $\partial N \to \partial N$ can be extended to a diffeomorphism N→N. (See the figure) We regard



W as $S^{n-1} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then $A(n+1)(X, t) = (A(n)(X), -t)$, $X \in S^{n-1}$, $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $(f \times id) : S^{n-1} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \to S^{n-1} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ be the diffeomorphism which sends (X, t) to (f(X), t). Since $f \circ A(n) = A(n) \circ f$, it follows that $A(n+1) \circ (f \times id) = (f \times id) \circ A(n+1)$. Hence (f x id) induces a diffeomorphism F : N→N such that $F \mid \partial N = f$. Thus f : $\partial N \to \partial N$ can be extended to a diffeomorphism N→ N. This proves theorem 1.

Proof of theorem 2.

We construct an involution on $2M^{2k}$ as follows.
Let T be the involution on S^{2k} defined by $T(x_1, \ldots, x_{2k}, x_{2k+1}) = (-x_1, \ldots, -x_{2k}, x_{2k+1})$. It is clear that T preserves orientation. Let. V^{2k} be a sufficiently small disk such that $V^{2k} \cap T(V^{2k}) = \phi$. Now make M^{2k} connected sum tó S^{2k} in V^{2k} and in $T(V^{2k})$ so that the resulting manifold preserves the involution. Then we have a disired involution on $S^{2k} \# M^{2k} \# M^{2k} = 2M^{2k}$.

Reference

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