Finite Solvable Groups with Nilpotent Maximal A-invariant Subgroups

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Masanobu YONAHA*

Let G be a finite group with a nilpotent maximal subgroup M. A theorem of Deskins, Janko and Thompson [3] states that if the Sylow 2-subgroup of M has class at most 2, then G is solvable. Recently, as a partial generalization of this theorem, Brown [1] has proved the following result.

(*) *If a solvable group A acts on G and G contains a nilpotent maximal A-invariant subgroup M whose Sylow 2-subgroup is Abelian, then G is solvable.*

A group A acts on G via φ if φ is a homomorphism from A into the automophism group of G and $g^a = g^{\varphi(a)}$ for all $g \in G$, $a \in A$. We say that A acts irreducibly on G if the only A-invariant subgroups of G are G and 1. In this note we prove the following theorem which gives a necessary and sufficient condition for such a group to be solvable.

Theorem. *Suppose a solvable group A acts on G and G contains a nilpotent maximal A-invariant subgroup M. Let N be the maximal normal 2-subgroup in G. Then G is solvable if and only if G/N is 2-normal.*

A finite group is p-normal if the center of a Sylow p-subgroup is the center of any other Sylow p-subgroup which contains it. If we take $A=1$ in the above theorem, then as an immediate consequence we have

Corollary. *Suppose G contains a nilpotent maximal subgroup M. Then G is solvable if and only if G/N is 2-normal, where N is the maximal normal 2-subgroup of G.*

Another characterization of such a solvable group is given by Simmon [5].

Proof of the theorem. We first assume that G is solvable and show that G/N is 2-normal. Since N is characteristic in G, we may assume that $N=1$ and hence G contains no normal 2-subgroups. If M is normal in G, then since M is maximal A-invariant, A acts irreducibly on G/M. By Lemma 1 of [1] G/M is an elementary Abelian p-group The order $|M|$ of M must be odd. Otherwise the Sylow 2-subgroup M_2 of M, being characteristic in M, is normal in G, a contradiction. If $p=$odd, then $|G|=$odd and we have nothing to prove. If $p=2$, then for any Sylow 2-subgroup G_2 of G we have $G=MG_2$, $M \cap G_2 = 1$ and G_2 is elementary Abelian. Hence G is 2-normal.

Suppose M is not normal in G. Then, since M is maximal A-invariant, $N(M)=M$, where $N(M)$ is the normalizer of M in G. Let H be the maximal normal (relative to G) A-invariant subgroup of M, then $|H|=$ odd. Consider G/H. Since M/H is nilpotent and maximal A-invariant and has no non-trivial normal A-invariant subgroups, the normalizers of the Sylow subgroups of M/H are M/H. Hence M/H is a nilpotent Hall subgroup. Let K/H be a minimal non-trivial characteristic subgroup of G/H,

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* Mathematics Dept., Sciences & Engineering Div

then K/H is an elementary Abelian p-group. Moreover, since M/H is a Hall subgroup and has no normal A-invariant subgroups, we have $K/H \cap M/H = 1$. By the maximality of M/H,

$$G/H = K/H \cdot M/H.$$

If $p = 2$, then K/H is the Sylow 2-subgroup of G/H and therefore a Sylow 2-subgroup K_2 of K is also a Sylow 2-subgroup of G. Since $(2, |H|) = 1$, we have $K = HK_2$, $H \cap K = 1$ and K_2 is Abelian. Hence G is 2-normal. Assume p=odd. If $|M/H|$ is odd, then $|G|$ is odd and again we have nothing to prove. If $2 \mid |M/H|$, then since

$$HM_2/H \cong M_2/H \cap M_2 = M_2$$

M_2 is a Sylow 2-subgroup of G. Further, HM_2/H has a normal complement KM_2'/H, where M_2', is the 2-complement in M. Hence M_2 has a normal complement KM_2', in G, i. e., G has a normal 2-complement. By Theorem 13. 5. 13 of [4] G is 2-normal.

We now assume that G/N is 2-normal and show that G is solvable. (This part of the proof is essentially contained in [1].) Again we assume that $N = 1$ and G contains no normal 2-subgroups. If $|M|$ is odd, then by (*) G is solvable. Note that in this case condition "2-normality" is not needed.

If $2 \mid |M|$, then the Sylow 2-subgroup M_2 of M is a Sylow 2-subgroup of G and

$$N(Z(M_2)) = M = M_2 \times M_2',$$

where $Z(M_2)$ is the center of M_2. Hence by a theorem of Grun [4], there exists a normal subgroup K such that $G = KM_2$ and $K \cap M_2 = 1$. K is characteristic in G. By Lemma 2 of [1] the semi-direct product AM_2, which is solvable, acts on K. Let $M' = K \cap M$, then M' must be a maximal AM_2-invariant subgroup of K. AM_2-invariantness is clear. If M' is not maximal, then there is an A-invariant and M_2-invariant subgroup L (by the same lemma) such that

$$M' < L < K.$$

LM_2 is an A-invariant subgroup of G. Since $G = KM_2 = KM$ and $M_2 \cong G/K \cong M/M \cap K$, $|M| = |M \cap K|\,|M_2|$ and hence $M = (M \cap K)M_2$. Furthermore,

$$M = M'M_2 < LM_2 < KM_2 = G,$$

which contradict the maximality of M. Since M' is nilpotent of odd order, by (*) again K is solvable. Hence G is solvable.

Remark. G itself may not be 2-normal. for example, the symmetric group S_4 on four letters.

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