On orbit spaces of semi-free SO(2) actions

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A differentiable action of $SO(2)$ on a smooth manifold is said to be *semi-free* if there exists a non-vacuous set of fixed points, outside of which the action is free, that is, there are two types of isotropy subgroups, the identity group $\{e\}$ and the whole group $SO(2)$. Let M be a compact connected smooth manifold with a semi-free $SO(2)$ action. Let $M^*=M/SO(2)$ be the orbit space and $\pi: M \longrightarrow M^*$ the natural projection. Denote by $\pi_1(M)$, $\pi_1(M^*)$ the fundamental groups of M, M^*, respectively. In this note we prove the following result.

Theorem 1. $\pi_\#: \pi_1(M) \longrightarrow \pi_1(M^*)$ *is an isomorphism.*

If M is the n-sphere S^n, the set of fixed points is a homology sphere by Smith theory. Let k be its dimmension, then $n-k$ is even, say $2r$. We also prove

Theorem 2. $S^n/SO(2) \simeq \Sigma^{k+1} CP(r-1)$ *(homotopy equivalence), where* $\Sigma^{k+1} CP(r-1)$ *denotes the* $(k+1)$*-iterated suspension space of the* $(r-1)$*-complex projectiv espace.*

1. *The fixed point set and its neighborhood.*

Let M be a compact smooth manifold with a *semi-free* $SO(2)$ action. By averaging a given Riemannian metric on M, we have a new metric for which $SO(2)$ acts as a group of isometries. Let T_x be the tangent space of M at a fixed point x. The $SO(2)$ action on T_x via the differential $dg: T_x \longrightarrow T_x$, $g \in SO(2)$, is an orthogonal action and the exponential map $T_x \longrightarrow M$ is an "*equivariant*" diffeomorphism near x with respect to the actions. Therefore the set of fixed points is locally Euclidean, and a connected component F of the set of fixed points is a smooth submanifold of M. Moreover, as usual, we can choose an invariant tubular neighborhood of F. Let N_x denote the normal space of F at x. then we see the dimmension of N_x is even, say $2r$. Since the action is semi-free it follows easily that the induced action of $SO(2)$ on N_x is equivalent to the action defined by the representation $\phi : SO(2) \longrightarrow O(2r)$, which takes $A \in SO(2)$ to the (2×2)-block matrix

$$\phi(A) = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}$$

Hence, as easily seen, the orbit space of the $SO(2)$ action on the normal sphere bundle of F is a complex projective space bundle over F.

2. *Van Kampen's theorem.*

We recall here Van Kampen's theorem. The proof is given in [1]. Let X be a

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topological space and X_0, X_1, X_2 path connected open subsets of X such that $X_1 \cup X_2 = X$, $X_1 \cap X_2 = X_0$; $*$ a point of X_0 ; $G = \pi_1(X, *)$, $G_i = \pi_1(X_i, *)$ $(i = 0, 1, 2)$ the fundamental groups of X, X_i $(i=0, 1, 2)$ based at $*$, respectively. The inclusion maps induce the commutative diagram

$$\begin{tikzcd} & G_0 \arrow[dl, "a_1"'] \arrow[dr, "a_2"] \arrow[d, "b_0"] & \\ G_1 \arrow[r, "b_1"'] & G & G_2 \arrow[l, "b_2"] \end{tikzcd}$$

Van Kampen's theorem

i) $b_i G_i$ $(i=1, 2)$ *generate* G.

ii) Given a group H *and a homomorphism* $c_i : G_i \longrightarrow H$ $(i=0, 1, 2)$ *satisfying* $c_0 = c_1 a_1 = c_2 a_2$, *then there is a homomorphism* $d : G \longrightarrow H$ *satisfying* $c_i = db_i$ $(i=0, 1, 2)$.

Denote by K_0 the kernel of a_2 and by K_1 the minimal normal subgroup of G_1 containing $a_1 K_0$.

Lemma 1. *(corollary of Van Kampen's theorem)*

If a_2 *is onto, then so is* b_1 *and its kernel is* K_1.

Proof. The ontoness of b_1 is obvious by *i)* of Van Kampen's theorem. Let $e_i : G_i \longrightarrow G_i/K_i$ $(i=0, 1)$ be the natural projections and $f_1 : G_0/K_0 \longrightarrow G_1/K_1$, $f_2 : G_0/K_0 \longrightarrow G_2$ be the natural homomorphisms induced by a_1, a_2. f_2 is an isomorphism. We put $H = G_1/K_1$, $c_1 = e_1$, $c_0 = f_1 e_0$ and $c_2 = f_1 f_2^{-1}$. Since the diagram

$$\begin{tikzcd} G_1 \arrow[d, "e_1"'] & G_0 \arrow[l, "a_1"'] \arrow[d, "e_0"] \arrow[dr, "a_2"] & \\ G_1/K_1 & G_0/K_0 \arrow[l, "f_1"] \arrow[r, "f_2"'] & G_2 \end{tikzcd}$$

is commutative, there is a homomorphism $d : G \longrightarrow H = G_1/K_1$ satisfying $c_i = db_i$ $(i = 0, 1, 2)$ by *ii)* of Van Kampen's theorem. Since $b_1 a_1 K_0 = b_2 a_2 K_0 = 1$ (1 the unit of G), the normal subgroup, Kernel b_1, contains $a_1 K_0$ and hence contains K_1.

Therefore b_1 induces a homomorphism $g : G_1/K_1 \longrightarrow G$ satisfying $ge_1 = b_1$.

From

$$dg(e_1 x) = d(ge_1)x = db_1 x = c_1 x = e_1 x, \ x \in G,$$

it follows that $dg = id : G_1/K_1 \longrightarrow G_1/K_1$. Hence g is a monomorphism and

$$Kernel\ b_1 = Kernel\ (ge_1) = Kernel\ e_1 = K_1.$$

This proves lemma 1.

3. ***Proof of theorem*** 1.

Let F be a connected component of the set of fixed points and T a small open invariant tubular neighborhood of F such that $T - F$ has no fixed point ; $\dot{T} = T \cap (M - F)$. Denote by T^*, $\dot{T}^*$ the orbit spaces of T, $\dot{T}$, respectively. $\dot{T}$, T, $\dot{T}^*$, T^* are fibre bundles over F.

i) The inclusion map induces an isomorphism

$$\pi_1(M^* - F) \longrightarrow \pi_1(M^*).$$

Proof. We denote by $p : T^* \longrightarrow F$, $\dot{p} : \dot{T}^* \longrightarrow F$ the bundle projections and by $j : \dot{T}^* \longrightarrow T^*$ the inclusion. p is a homotopy equivalence and the fibre of $\dot{p}$ has the homotopy type of a complex projective space. Consider the commutative diagram

$$\begin{tikzcd} \pi_1(\dot{T}^*) \arrow[rr, "j_\#"] \arrow[dr, "\dot{p}_\#"'] & & \pi_1(T^*) \arrow[dl, "p_\#"] \\ & \pi_1(F) & \end{tikzcd}$$

$p_\#$ is an isomorphism. Since the fibre of $\dot{p} : \dot{T}^* \longrightarrow F$ is simply connected, $\dot{p}_\#$ is also an isomorphism. Hence $j_\#$ is an isomorphism. Applying lemma 1 to $M^* = (M^*-F) \cap T^*$, $(M^*-F) \cap T^* = \dot{T}^*$ we obtain $i)$.

Let S be an orbit of the $SO(2)$ action in $\dot{T}$ and a, b, c, d, e the inclusion maps, q, $\dot{q}$ the bundle projections in the following diagram

$$\begin{tikzcd} S \arrow[r, "a"] \arrow[d, "c"] & \dot{T} \arrow[rr, "\mathring{q}"] \arrow[dl, "b"] \arrow[drr, "e"] & & F \\ M-F \arrow[r, "d"] & M & & T \arrow[u, "q"] \end{tikzcd}$$

ii) $e_\# : \pi_1(\dot{T}) \longrightarrow \pi_1(T)$ is onto and its kernel equals $a_\# \pi_1(S)$.

Proof. If the codimmension of F is greater than two, $e_\#$ is an isomorphism and $a_\# \pi_1(S) = 1$ by the general position argument.
If the codimmension of F is two, then S is a deformation retract of a fibre of $\dot{q} : \dot{T} \longrightarrow F$, i. e., $S \xrightarrow{a} \dot{T} \xrightarrow{\dot{q}} F$ is a fibration. Hence $\dot{q}_\# : \pi_1(\dot{T}) \longrightarrow \pi_1(F)$ is onto and Kernel $\dot{q}_\# = a_\# \pi_1(S)$. Since q is a homotopy equivalence and $qe = \dot{q}$, $ii)$ is now clear.

iii) If there remains any fixed point in $M-F$, then $d_\# : \pi_1(M-F) \longrightarrow \pi_1(M)$ is an isomorphism, otherwise $d_\#$ is onto and Kernel $d_\# = c_\# \pi_1(S)$.

Proof. Consider the commutative diagram induced by the inclusion maps

$$\begin{tikzcd} \pi_1(S) \arrow[rr, "a_\#"] \arrow[dr, "c_\#"'] & & \pi_1(\dot{T}) \arrow[dl, "b_\#"] \arrow[dd] \arrow[dr, "e_\#"] & \\ & \pi_1(M-F) \arrow[dr, "d_\#"'] & & \pi_1(T) \arrow[dl] \\ & & \pi_1(M) & \end{tikzcd}$$

If there is any fixed point in $M-F$, then S is contractible to a point in $M-F$. Hence, $b_\# a_\# \pi_1(S) = c_\# \pi_1(S) = 1$, and lemma 1, $ii)$ above, show that $d_\#$ is an isomorphism. If $M-F$ has no fixed point, then

$$S \xrightarrow{c} (M-F) \xrightarrow{\pi} (M^*-F)$$

is a fibration. Therefore $c_\# \pi_1(S) = \text{Kernel } [\pi_\# : \pi_1(M-F) \longrightarrow \pi_1(M)]$,
which is a normal subgroup of $\pi_1(M-F)$. The rest oi $iii)$ follows from lemma 1.

Now we complete the proof of theorem 1.

Let F_i $(i=1, \ldots, s)$ be connected components of the set of fixed points. Put $M_0=M$, $M_i=M_{i-1}-F_i$ $(i=1, \ldots, s-1)$ and $M_i^*=M_i/SO(2)$ $(i=0, \ldots, s-1)$. The inclusion maps and the natural projections induce the commutative diagram

$$\begin{array}{ccccccc} \pi_1(M_{s-1}) & \longrightarrow & \pi_1(M_{s-2}) & \longrightarrow & \cdots & \longrightarrow & \pi_1(M_0) \\ \downarrow{\scriptstyle \pi_\#} & & \downarrow{\scriptstyle \pi_\#} & & & & \downarrow{\scriptstyle \pi_\#} \\ \pi_1(M_{s-1}^*) & \longrightarrow & \pi_1(M_{s-2}^*) & \longrightarrow & \cdots & \longrightarrow & \pi_1(M_0^*) \end{array}$$

in which the horizontal maps are isomorphisms by *i*) and *iii*). Let S be an orbit in $M_{s-1}-F_s$ and $c: S \longrightarrow (M_{s-1}-F_s)$ the inclusion. Since $M_{s-1}-F_s$ has no fixed point,

$$S \xrightarrow{c} (M_{s-1}-F_s) \xrightarrow{\pi} (M_{s-1}^*-F_s)$$

is a fibration. The commutative diagram

$$\begin{array}{ccccccc} \longrightarrow \pi_1(S) & \xrightarrow{c_\#} & \pi_1(M_{s-1}-F_s) & \xrightarrow{\pi_\#} & \pi_1(M_{s-1}^*-F_s) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \cong\ i)} & & \\ & & \pi_1(M_{s-1}) & \xrightarrow{\pi_\#} & \pi_1(M_{s-1}^*) & & \end{array}$$

(in which the vertical maps are induced by the inclusions) and *iii*) yield the commutatve diagram

$$\begin{array}{ccc} \pi_1(M_{s-1}-F_s)/c_\#\pi_1(S) & \xrightarrow{\cong} & \pi_1(M_{s-1}^*-F_s) \\ \downarrow{\scriptstyle \cong\ iii)} & & \downarrow{\scriptstyle \cong} \\ \pi_1(M_{s-1}) & \xrightarrow{\pi_\#} & \pi_1(M_{s-1}^*) \end{array}$$

Therefore $\pi_\#: \pi_1(M_{s-1}) \longrightarrow \pi_1(M_{s-1}^*)$ is an isomorphism, and so is $\pi_\#: \pi_1(M_0) \longrightarrow \pi_1(M_0^*)$. This complete the proof.

4. Proof of theorem 2.

Lemma 2. Let X be a compact acyclic manifold with a semi-free $SO(2)$ action. Then $X/SO(2)$ is acyclic.

Proof. The set of fixed points is acyclic by P. A. Smith theory [2]. Let T be a closed invariant tubular neighborhood of the set of fixed points, W the closure of $X-F$, $\dot{T}=W \cap T$. Denote by W^*, $\dot{T}^*$ the orbit spaces of W, $\dot{T}$, respectively. In the Mayer-Vietoris sequence

$$\longrightarrow H_{i+1}(X) \longrightarrow H_i(\dot{T}) \longrightarrow H_i(W)+H_i(T) \longrightarrow H_i(X) \longrightarrow,$$

$H_i(\dot{T}) \longrightarrow H_i(W)$ are isomorphisms for all i. It follows then easily from Gysin homology sequence that $H_i(\dot{T}^*) \longrightarrow H_i(W^*)$ are isomorphisms for all i. Using the Mayer-Vietoris sequence again, we obtain lemma 2.

Proof of theorem 2. Choose a small invariant disk neighborhood D of a fixed point and denote by ∂D its boundary. Since the action on D may be regarded as a linear action, it is easy to see that $\partial D/SO(2)$ and $D/SO(2)$ are homeomorphic to $\Sigma^k CP(r-1)$ and its cone, respectively. Let V be the cloure of S^n-D and V^* its orbit space. Since V is acyclic, V^* is also acyclic by lemma 2. Let Y be the space obtained from $S^n/SO(2)$ by collapsing V^* to a single point. Clearly Y is homeomorphic to $\Sigma^{k+1}CP(r-1)$. Let $w : S^n/SO(2) \longrightarrow Y$ be the identification map. Since $S^n/SO(2)$ is simply connected by Theorem 1 and triangulable by C. T. Yang [3], it follows now easily from the theorem of J. H. C. Whitehead that w is a homotopy equivalence. This complete the proof.

Reference

[1] R. Crowell and R. Fox, *An introduction to knot theory*, Ginn and co., 1963.

[2] P. A. Smith, *Fixed point of periodic transformations*, Appendix B in Lefschetz, Algebraic Topology, 1942.

[3] C. T. Yang, *The triangulability of the orbit space of a differentiable transformation group*, Bull. Amer. Math. Soc. (3) 69 (1963), 405-408.