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#### On orbit spaces of semi-free SO(2) actions\*

#### Hiroshi MAEHARA

A differentiable action of SO(2) on a smooth manifold is said to be *semi-free* if there exists a non-vacuous set of fixed points, outside of which the action is free, that is, there are two types of isotropy subgroups, the identity group  $\{e\}$  and the whole group SO(2). Let M be a compact connected smooth manifold with a semi-free SO(2) action. Let  $M^*=M/SO(2)$  be the orbit space and  $\pi:M\longrightarrow M^*$  the natural projection. Denote by  $\pi_1(M)$ ,  $\pi_1(M^*)$  the fundamental groups of M,  $M^*$ , respectively. In this note we prove the following result.

**Theorem 1.**  $\pi_{\#}$ :  $\pi_1(M) \longrightarrow \pi_1(M^*)$  is an isomorphism.

If M is the *n*-sphere  $S^n$ , the set of fixed points is a homology sphere by Smith theory. Let k be its dimmension, then n-k is even, say 2r. We also prove

**Theorem 2.**  $S^n/SO(2) \simeq \Sigma^{k+1}CP(r-1)$  (homotopy equivalence), where  $\Sigma^{k+1}CP$  (r-1) denotes the (k+1)-iterated suspension space of the (r-1)-complex projectiv espace.

## 1. The fixed point set and its neighborhood.

Let M be a compact smooth manifold with a  $semi-free\ SO(2)$  action. By averaging a given Riemannian metric on M, we have a new metric for which SO(2) acts as a group of isometries. Let  $T_x$  be the tangent space of M at a fixed point x. The SO(2) action on  $T_x$  via the differential  $dg:T_x \longrightarrow T_x.g \in SO(2)$ , is an orthogonal action and the exponential map  $T_x \longrightarrow M$  is an "equivariant" diffeomorphism near x with respect to the actions. Therefore the set of fixed points is locally Euclidean, and a connected component F of the set of fixed points is a smooth submanifold of M. Moreover, as usual, we can choose an invariant tubular neighborhood of F. Let  $N_x$  denote the normal space of F at x, then we see the dimmension of  $N_x$  is even, say 2r. Since the action is semi-free it follows easily that the induced action of SO(2) on  $N_x$  is equivalent to the action defined by the representation  $\phi: SO(2) \longrightarrow O(2r)$ , which takes  $A \in SO(2)$  to the  $(2 \times 2)$ -block matrix

$$\phi(\mathbf{A}) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ & \cdot \\ \mathbf{0} & \mathbf{A} \end{bmatrix}$$

Hence, as easily seen, the orbit space of the SO(2) action on the normal sphere bundle of F is a complex projective space bundle over F.

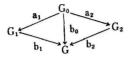
#### 2. Van Kampen's theorem.

We recall here Van Kampen's theorem. The proof is given in [1]. Let X be a

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topological space and  $X_0$ ,  $X_1$ ,  $X_2$  path connected open subsets of X such that  $X_1 \cup X_2 = X$ ,  $X_{1\cap}X_2 = X_0$ ; \* a point of  $X_0$ ;  $G = \pi_1(X, *)$ ,  $G_i = \pi_1(X_i, *)$  (i = 0, 1, 2) the fundamental groups of X,  $X_i$  (i = 0, 1, 2) based at \*, respectively. The inclusion maps induce the commutative diagram



Van Kampen's theorem

- i)  $b_i$   $G_i$  (i=1, 2) generate  $G_i$
- ii) Given a group H and a homomorphism  $c_i: G_i \longrightarrow H$  (i=0, 1, 2) satisfying  $c_0 = c_1a_1 = c_2a_2$ , then there is a homomorphism  $d: G \longrightarrow H$  satisfying  $c_i = db_i$  (i=0, 1, 2).

Denote by  $K_0$  the kernel of  $a_2$  and by  $K_1$  the minimal normal subgroup of  $G_1$  containing  $a_1K_{0}$ .

Lemma 1. (corollary of Van Kampen's theorem)

If  $a_2$  is onto, then so is  $b_1$  and its kernel is  $K_{1}$ .

*Proof.* The ontoness of  $b_1$  is obvious by i) of Van Kampen's theorem. Let  $e_i:G_i \longrightarrow G_i/K_i$  (i=0, 1) be the natural projections and  $f_1:G_0/K_0 \longrightarrow G_1/K_1$ ,  $f_2:G_0/K_0 \longrightarrow G_2$  be the natural homomorphisms induced by  $a_1, a_2, f_2$  is an isomorphism. We put  $H=G_1/K_1$ ,  $c_1=e_1$ ,  $c_0=f_1e_0$  and  $c_2=f_1f_2^{-1}$ . Since the diagram

is commutative, there is a homomorphism  $d: G \longrightarrow H = G_1/K_1$  satisfying  $c_i = db_i$  (i = 0, 1, 2) by ii) of Van Kampen's theorem. Since  $b_1a_1K_0 = b_2a_2K_0 = 1$  (1 the unit of G), the normal subgroup, Kernel  $b_1$ , contains  $a_1K_0$  and hence contains  $K_1$ .

Therefore  $b_1$  induces a homomorphism  $g: G_1/K_1 \longrightarrow G$  satisfying  $ge_1 = b_1$ . From

$$dg(e_1x) = d(ge_1)x = db_1x = c_1x = e_1x, x \in G,$$

it follows that dg = id:  $G_1/K_1 \longrightarrow G_1/K_1$ . Hence g is a monomorphism and  $Kernel\ b_1 = Kernel\ (ge_1) = Kernel\ e_1 = K_1$ .

This proves lemma 1.

## 3. Proof of theorem 1.

Let F be a connected component of the set of fixed points and T a small open invariant tubular neighborhood of F such that T-F has no fixed point;  $\dot{T}=T\cap (M-F)$ . Denote by  $T^*$ ,  $\dot{T}^*$  the orbit spaces of T.  $\dot{T}$ , respectively.  $\dot{T}$ , T,  $\dot{T}$ ,  $T^*$  are fibre bundles over F.

i) The inclusion map induces an isomorphism

$$\pi_1(M^*-F) \longrightarrow \pi_1(M^*)$$
.

**Proof.** We denote by  $p:T^*\longrightarrow F$ ,  $\dot{p}:\dot{T}^*\longrightarrow F$  the bundle projections and by  $j:\dot{T}^*\longrightarrow T^*$  the inclusion. p is a homotopy equivalence and the fibre of  $\dot{p}$  has the homotopy type of a complex projective space. Consider the commutative diagram

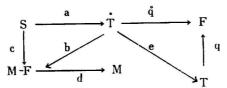
$$\pi_{1}(\dot{\mathbf{T}}^{*}) \xrightarrow{\dot{\mathbf{J}}_{\sharp}} \pi_{1}(\mathbf{T}^{*})$$

$$\dot{\mathbf{p}}_{\sharp}$$

$$\pi_{1}(\mathbf{F})$$

 $p_{\#}$  is an isomorphism. Since the fibre of  $p:\mathring{T}^*\longrightarrow F$  is simply connected,  $\mathring{p}_{\#}$  is also an isomorphism. Hence  $j_{\#}$  is an isomorphism. Applying lemma 1 to  $M^*=(M^*-F)\bigcap T^*$ ,  $(M^*-F)\bigcap T^*=\mathring{T}^*$  we obtain  $\mathring{i}$ ).

Let S be an orbit of the SO(2) action in T and a, b, c, d, e the inclusion maps, q,  $\dot{q}$  the bundle projections in the following diagram



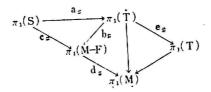
ii)  $e_{\#}$ :  $\pi_1$   $(\mathring{T}) \longrightarrow \pi_1(T)$  is onto and its kernel equals  $a_{\#}$   $\pi_1(S)$ .

*Proof.* If the codimmension of F is greater than two,  $e_{\#}$  is an isomorphism and  $a_{\#}$   $\pi_1(S) = 1$  by the general position argument.

If the codimmension of F is two, then S is a deformation retract of a fibre of  $q^{\bullet}: \mathring{T} \longrightarrow F$ ,  $i_{\bullet} e_{\bullet}$ ,  $S \xrightarrow{a} \mathring{T} \xrightarrow{q} F$  is a fibration. Hence  $\mathring{q}_{\#}: \pi_{1}(\mathring{T}) \longrightarrow \pi_{1}(F)$  is onto and Kernel  $\mathring{q}_{\#} = a_{\#}\pi_{1}(S)$ . Since q is a homotopy equivalence and  $qe = \mathring{q}$ , ii is now clear.

iii) If there remains any fixed point in M-F, then  $d_{\#}: \pi_1(M-F) \longrightarrow \pi_1(M)$  is an isomorphism, otherwise  $d_{\#}$  is onto and Kernel  $d_{\#}=c_{\#}\pi_1(S)$ .

Proof. Consider the commutative diagram induced by the inclusion maps



If there is any fixed point in M-F, then S is contractible to a point in M-F. Hence,  $b_{\#}a_{\#}\pi_1(S)=c_{\#}\pi_1(S)=1$ , and lemma 1, ii) above, show that  $d_{\#}$  is an isomorphism. If M-F has no fixed point, then

$$S \xrightarrow{c} (M-F) \xrightarrow{\pi} (M^*-F)$$

is a fibration. Therefore  $c_{\#}$   $\pi_1(S) = \text{Kernel } [\pi_{\#} : \pi_1(M-F) \longrightarrow \pi_1(M)]$ ,

which is a normal subgroup of  $\pi_1(M-F)$ . The rest oi iii) follows from Iemma 1. Now we complete the proof of theorem 1.

Let  $F_i$  ( $i=1,\ldots,s$ ) be connected components of the set of fixed points. Put  $M_0=M$ ,  $M_i=M_{i-1}-F_i$  ( $i=1,\ldots,s-1$ ) and  $M_i^*=M_i/SO(2)$  ( $i=0,\ldots,s-1$ ).

The inclusion maps and the natural projections induce the commutative diagram

in which the horizontal maps are isomorphisms by i) and iii). Let S be an orbit in  $M_{s-1}-F_s$  and  $c: S \longrightarrow (M_{s-1}-F_s)$  the inclusion. Since  $M_{s-1}-F_s$  has no fixed point,

$$S \xrightarrow{c} (M_{s-1} - F_s) \xrightarrow{\pi} (M_{s-1}^* - F_s)$$

is a fibration. The commutative diagram

(in which the vertical maps are induced by the inclusions) and iii) yield the commutatve diagram

$$\pi_{1}(M_{S-1}-F_{S})/c \neq \pi_{1}(S) \xrightarrow{\cong} \pi_{1}(M_{S-1}^{*}-F_{S})$$

$$\downarrow \cong iii) \qquad \qquad \downarrow \cong$$

$$\pi_{1}(M_{S-1}) \xrightarrow{\pi_{\#}} \pi_{1}(M_{S-1}^{*})$$

Therefore  $\pi_{\#}: \pi_1(M_{*-1}) \longrightarrow \pi_1(M_{*-1})$  is an isomorphism, and so is  $\pi_{\#}: \pi_1(M_0) \longrightarrow \pi_1(M_0^*)$ . This complete the proof.

#### 4. Proof of theorem 2.

Lemma 2. Let X be a compact acyclic manifold with a semi-free SO(2) action. Then X/SO(2) is acyclic.

**Proof.** The set of fixed points is acyclic by P. A. Smith theory [2]. Let T be a closed invariant tubular neighborhood of the set of fixed points. W the closure of X-F,  $\dot{T}=W\cap T$ . Denote by  $W^*$ ,  $\dot{T}^*$  the orbit spaces of W,  $\dot{T}$ , respectively. In the Mayer-Vietoris sequence

$$\longrightarrow H_{i+1}(X) \longrightarrow H_i(\dot{T}) \longrightarrow H_i(W) + H_i(T) \longrightarrow H_i(X) \longrightarrow.$$

 $H_i(\dot{T}) \longrightarrow H_i(W)$  are isomorphisms for all i. It follows then easily from Gysin homology sequence that  $H_i(\dot{T}^*) \longrightarrow H_i(W^*)$  are isomorphisms for all i. Using the Mayer-Vietoris sequence again, we obtain lemma 2.

**Proof of theorem 2.** Choose a small invariant disk neighborhood D of a fixed point and denote by  $\partial D$  its boundary. Since the action on D may be regarded as a linear action, it is easy to see that  $\partial D/SO(2)$  and D/SO(2) are homeomorphic to  $\sum^k CP(r-1)$  and its cone, respectively. Let V be the cloure of  $S^n-D$  and  $V^*$  its

orbit space. Since V is acyclic,  $V^*$  is also acyclic by lemma 2. Let Y be the space obtained from  $S^n/SO(2)$  by collapsing  $V^*$  to a single point. Clearly Y is homeomorphic to  $\sum_{k=1}^{k+1} CP(r-1)$ . Let  $w:S^n/SO(2) \longrightarrow Y$  be the identification map. Since  $S^n/SO(2)$  is simply connected by Theorem 1 and triangulable by C. T. Yang [3], it follows now easily from the theorem of J. H. C. Whitehead that w is a homotopy equivalence. This complete the proof.

#### Reference

- [1] R. Crowell and R. Fox. An introduction to knot theory, Ginn and co., 1963.
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