# Impusity Effect in Superconductor

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Summary

We have investigated the effect of the correlation of impurities in a superconductor. Using the exact solution for the Hamiltonian without the spin dynamics, we have obtained the excited impurity levels dependent on R, which is the distance between impurities. Regarding this R as the inverse of the concentration of impurities, we have discussed the growth of the impurity band in the energy gap of the superconductor.

## § 1. Introduction

It is well known that the scattering of conduction electron by magnetic impurities in a metal becomes anomalous at low tmeperatures. This phenomenon has been investigated extensively by many authors.

As the classic paper, there is Abrikosov and Gorkov's one.[1] They have first pointed out the possibility of existence of the gapless superconductor, using temperature Green function method. Meanwhile, since the discovery of the very strong scattering of electrons from magnetic impurities,[2] it has become clear that the Abrikosov-Gorkov treatment of the scattering has to be corrected by going beyond the second order perturbation theory. First attempts in this direction were made by Liu[3] and Griffin[4] using higher order perturbation theory in the calculation of the shifts of the superconducting transition temperature $T_C$. Another effect on superconductors is the appearance of the bound state in the energy gap. This bound state has been first investigated by Soda, Matsuura, and Nagaoka,[5] using the methods of Yoshida[6] in the case of normal state. Fowler and Maki[7] have extended the dispersion theory of Suhl[8] to the cace of superconductors and obtained the impurity excited state. The first attempt in solving the Nagaoka's Green function equations[9] generalized to superconductors was undertaken by Takano and Matayoshi.[10] J. Zitartz and E. Müller-Hartmann,[11] using this Green function method, have given exact solution for this problem.

However, many authors have not taken the impurity correlation into consideration in detail. We believe that this correlation plays an essential role at low temperatures. We will discuss the superconductor containing one Ising impurity and two, respectively. Then, because of non-spin dynamics, we cannot expect the anomalous scattering of conduction electrons, the so called "Kondo effect".[2] In spite of this restriction of impurity spins, it is interesting to investigate the effect of the correlation between impurities. The scattering of electrons and the correlation of impurities are treated

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exactly, and the bound state and physical quantities will be discussed in this paper and also in the forthcoming series.

In § 2, we treat the case of one impurity and examine the dependence of impurity levels on the coupling constant, J, of the electron-impurity interaction. In § 3 and § 4, we discuss the case of two imputities.

## § 2. The case of one impurity

The Hamiltonian of the system can be written in the case of one impurity as.

$$H = H_0 + H', \tag{2-1}$$

where

$$H_0 = \sum_{l\sigma} \xi_l C^+_{l\sigma} C_{l\sigma} - \Delta \sum_l (C^+_{l\uparrow} C^+_{-l\downarrow} + C_{-l\downarrow} C_{l\uparrow}), \tag{2-1 a}$$

$$H' = -\frac{J}{2N} \sum_{ll'} (C^+_{l\uparrow} C_{l'\uparrow} - C^+_{l\downarrow} C_{l'\downarrow})\, S_z \tag{2-1 b}$$

and $\xi_k$ is the kinetic energy of conduction electrons measured from the Fermi energy, and $\Delta$ is the order parameter defined by

$$\Delta = g \sum_l \langle C^+_{l\uparrow} C^+_{-l\downarrow} \rangle = g \sum_l \langle C_{-l\downarrow} C_{l\uparrow} \rangle, \tag{2-2}$$

g being the coluping constant of the interaction. We completly neglect, as many authors do, the effect of the local spatial variation of the order parameter around an Ising impurity, which will be discussed in § 5.

Following the usual procedure, we construct the double time Green function in the matrix form,

$$\hat{G}_{kk'}(\omega) = \begin{vmatrix} \langle\langle C_{k\uparrow} \mid C^+_{k'\uparrow} \rangle\rangle & \langle\langle C_{k\uparrow} \mid C_{-k'\downarrow} \rangle\rangle \\ \langle\langle C^+_{-k\downarrow} \mid C^+_{k'\uparrow} \rangle\rangle & \langle\langle C^+_{-k\downarrow} \mid C_{-k'\downarrow} \rangle\rangle \end{vmatrix}, \tag{2-3}$$

and

$$\hat{\Gamma}_{kk'}(\omega) = \begin{vmatrix} \langle\langle C_{k\uparrow} S_z \mid C^+_{k'\uparrow} \rangle\rangle & \langle\langle C_{k\uparrow} S_z \mid C_{-k'\downarrow} \rangle\rangle \\ \langle\langle C^+_{-k\downarrow} S_z \mid C^+_{k'\uparrow} \rangle\rangle & \langle\langle C^+_{-k\downarrow} S_z \mid C_{-k'\downarrow} \rangle\rangle \end{vmatrix}, \tag{2-4}$$

The equations of motion for these Green functions can be written as

$$\begin{aligned} &\hat{\Omega}_k(\omega)\hat{G}_{kk'}(\omega) + \frac{J}{2N}\sum_l \hat{\Gamma}_{lk'}(\omega) = \delta_{kk'}, \\ &\Omega_k(\omega)\hat{\Gamma}_{kk'}(\omega) + \frac{J}{8N}\sum_l' \hat{G}_{lk'}(\omega) = 0, \end{aligned} \tag{2-5}$$

which are exact equations without decoupling.

The coefficient in Equation (2—5) is given by

$$\hat{\Omega}_k(\omega) = \begin{pmatrix} \omega - \xi_k & \Delta \\ \Delta & \omega + \xi_k \end{pmatrix} \tag{2-6}$$

Equation (2—5) can be solved easily to give the solution,

$$\hat{G}_{kk'}(\omega) = \delta_{kk'}\hat{\Omega}_k^{-1}(\omega) + \hat{\Omega}_k^{-1}(\omega)\,\hat{t}_1(\omega)\,\hat{\Omega}_{k'}^{-1}(\omega),$$

$$\hat{\Gamma}_{kk'}(\omega)=\hat{\Omega}_k^{-1}(\omega)\,\hat{t}_2\,(\omega)\hat{\Omega}_{k'}^{-1}(\omega), \tag{2-7}$$

$$\hat{t}_1(\omega)=\frac{J^2}{16N^2}\left[1-\frac{J^2}{16}\hat{F}_0^{\,2}(\omega)\right]^{-1}\hat{F}_0(\omega),$$

$$\hat{t}_2(\omega)=-\frac{J}{8N}\left[1-\frac{J^2}{16}\hat{F}_0^{\,2}(\omega)\right]^{-1}, \tag{2-8}$$

where the matrix $\hat{F}_0$ is defined as

$$\hat{F}_0(\omega)=\frac{1}{N}\Sigma_l\,\hat{\Omega}_l^{-1}(\omega), \tag{2-9}$$

and, after summing over wave number $l$'s, is explicitly expressed as follows;

$$\hat{F}_0^{\pm}(\omega)=\pm\frac{i\pi\rho}{N}\frac{1}{\sqrt{\omega^2-\Delta^2}}\begin{pmatrix}\omega & -\Delta\\ -\Delta & \omega\end{pmatrix}\quad \text{for}\,|\omega\geq\Delta,$$

$$=-\frac{\pi\rho}{N}\frac{1}{\sqrt{\Delta^2-\omega^2}}\begin{pmatrix}\omega & -\Delta\\ -\Delta & \omega\end{pmatrix}\quad \text{for}\,|\omega|<\Delta, \tag{2-10}$$

Upper and lower sign corresponds to retarded and advanced case, respectively.

The position of the bound state is given by poles of the $\hat{t}_1(\omega)$ or zero points of the determinant $|\,1-\frac{J^2}{16}\hat{F}_0^{\,2}(\omega)\,|$, that is, by

$$\omega=\pm\frac{1-\left(\frac{J\pi\rho}{4N}\right)^2}{1+\left(\frac{J\pi\rho}{4N}\right)^2}\Delta\equiv\omega_1 \tag{2-11}$$

which coincide with Shiba's[12] result. $\omega-J$ curves are shown in Fig. 1. This figure is naturally different especially for positive J from the figure in Zittartz and Müller-Hartmann's[11] paper, who used the Kondo Hamiltonian.

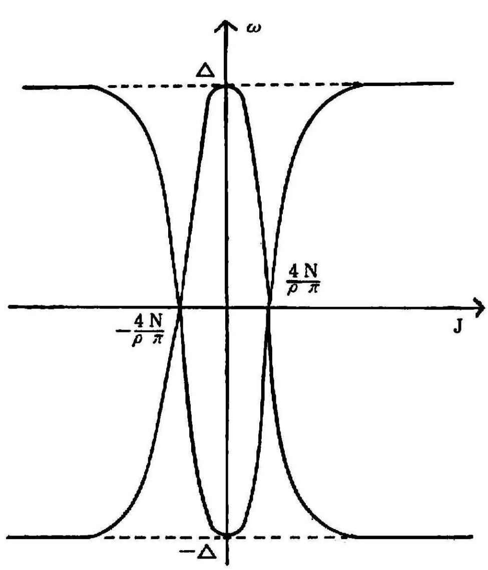



Fig. 1. $\omega-J$ curve for the case of one impurity.
$\omega$: bound state energy;
J : coupling constant between electron and impurity.
$\Delta$: order parameter.

§ 3. Formulation for the case of two impurities

In this section we treat the superconductor containing two Ising impurities which are located at the positions $R_1$ and $R_2$. The derivation of the equation of motion for the Green functions proceeds along the same lines as the case of one impurity. The Hamiltonian of this system is different in the perturbation term H′ from the one impurity system. Instead of Equation (2—1 b), H′ is written as

$$H' = -\frac{J}{2N} \Sigma_{ll'} e^{i(l-l')R_1} (C_{l\uparrow}^{+} C_{l'\uparrow} - C_{l\downarrow}^{+} C_{l'\downarrow}) S_z^{(1)}$$
$$-\frac{J}{2N} \Sigma_{ll'} e^{i(l-l')R_2} (C_{l\uparrow}^{+} C_{l'\uparrow} - C_{l\downarrow}^{+} C_{l'\downarrow}) S_z^{(2)}; \qquad (3-1)$$

where $S_z^{(1)}$ and $S_z^{(2)}$ are both spin $\frac{1}{2}$ operators.

The Green function $\hat{G}_{kk'}(\omega)$ is defined in the same form as Equation (2—3), and the other Green functions are defined as

$$\hat{\Gamma}_{kk'}^{(n)}(\omega) = \begin{pmatrix} \langle\langle C_{k\uparrow} S_z^{(n)} \mid C_{k'\uparrow}^{+} \rangle\rangle & \langle\langle C_{k\uparrow} S_z^{(n)} \mid C_{-k'\downarrow} \rangle\rangle \\ \langle\langle C_{-k\downarrow}^{+} S_z^{(n)} \mid C_{k'\uparrow}^{+} \rangle\rangle & \langle\langle C_{-k\downarrow}^{+} S_z^{(n)} \mid C_{-k'\downarrow} \rangle\rangle \end{pmatrix} \qquad (3-2\ a)$$

for n= 1 and 2, and

$$\hat{\Lambda}_{kk'}(\omega) = \begin{pmatrix} \langle\langle C_{k\uparrow} S_z^{(1)} S_z^{(2)} \mid C_{k'\uparrow}^{+} \rangle\rangle & \langle\langle C_{k\uparrow} S_z^{(1)} S_z^{(2)} \mid C_{-k'\downarrow} \rangle\rangle \\ \langle\langle C_{-k\downarrow}^{+} S_z^{(1)} S_z^{(2)} \mid C_{k'\uparrow}^{+} \rangle\rangle & \langle\langle C_{-k\downarrow}^{+} S_z^{(1)} S_z^{(2)} \mid C_{-k'\downarrow} \rangle\rangle \end{pmatrix} \qquad (3-2\ b)$$

These Green functions satisfy the following equations,

$$\hat{\Omega}_k(\omega)\, \hat{G}_{kk'}(\omega) + \frac{J}{2N} \Sigma_l\, e^{i(k-l)R_1}\, \hat{\Gamma}_{lk'}^{(1)}(\omega) + \frac{J}{2N} \Sigma_l\, e^{i(k-l)R_2}\, \Gamma_{lk'}^{(2)}(\omega) = \delta_{kk'},$$

$$\Omega_k(\omega)\, \hat{\Gamma}_{kk'}^{(1)}(\omega) + \frac{J}{8N} \Sigma_l\, e^{i(k-l)R_1}\, \hat{G}_{lk'}(\omega) + \frac{J}{2N} \Sigma_l\, e^{i(k-l)R_2}\, \hat{\Lambda}_{lk'}(\omega) = 0,$$

$$\hat{\Omega}_k(\omega)\, \hat{\Gamma}_{kk'}^{(2)}(\omega) + \frac{J}{2N} \Sigma_l\, e^{i(k-l)R_1}\, \hat{\Lambda}_{lk'}(\omega) + \frac{J}{8N} \Sigma_l\, e^{i(k-l)R_2}\, \hat{G}_{lk'}(\omega) = 0,$$

$$\hat{\Omega}_k(\omega) \hat{\Lambda}_{kk'}(\omega) + \frac{J}{8N} \Sigma_l\, e^{i(k-l)R_1}\, \hat{\Gamma}_{lk'}^{(2)}(\omega) + \frac{J}{8N} \Sigma_l\, e^{i(k-l)R_2}\, \Gamma_{lk'}^{(1)}(\omega) = \delta_{kk'}$$
$$\times \langle S_z^{(1)} S_z^{(2)} \rangle, \qquad (3-3)$$

where $\hat{\Omega}_k(\omega)$ is given by Equation (2—6).

In order to obtain the solution of Equation (3—3), it is convenient to define the following matrices,

$$\hat{X}_1^1 = \frac{1}{N} \Sigma_l\, e^{-ilR_1}\, \hat{\Gamma}_{lk'}^{(1)}(\omega), \qquad \hat{X}_1^2 = \frac{1}{N} \Sigma_l\, e^{-ilR_2}\, \hat{\Gamma}_{lk'}^{(1)}(\omega),$$
$$\hat{X}_2^1 = \frac{1}{N} \Sigma_l\, e^{-ilR_1}\, \hat{\Gamma}_{lk'}^{(1)}(\omega), \qquad \hat{X}_2^2 = \frac{1}{N} \Sigma_l\, e^{-ilR_2}\, \hat{\Gamma}_{lk'}^{(2)}(\omega), \qquad (3-4)$$

The equation for these matrices can be obtained through Equation (3—3), that is,

$$\hat{a}\, \hat{X}_1^1 + \hat{b} \hat{X}_1^2 + \hat{d}\, \hat{X}_2^1 + \hat{c}\, \hat{X}_2^2 = \hat{A},$$
$$\hat{d}\, \hat{X}_1^1 + \hat{a}\, \hat{X}_1^2 + \hat{c}\, \hat{X}_2^1 + \hat{b} \hat{X}_2^2 = \hat{B},$$

$$\hat{b}\hat{X}_1^1+\hat{c}\hat{X}_1^2+\hat{a}\hat{X}_2^1+\hat{d}\hat{X}_2^2=\hat{C},$$
$$\hat{c}\hat{X}_1^1+\hat{b}\hat{X}_1^2+\hat{d}\hat{X}_2^1+\hat{a}\hat{X}_2^2=\hat{D}, \qquad (3-5)$$

where

$$\hat{a}=1-\frac{J^2}{16}\hat{F}_0\hat{F}_0, \qquad \hat{c}=-\frac{J^2}{16}\hat{F}_0\hat{F},$$
$$\hat{b}=-\frac{J^2}{16}\hat{F}\hat{F}, \qquad \hat{d}=-\frac{J^2}{16}\hat{F}\hat{F}_0, \qquad (3-6)$$

$$\hat{A}=-\frac{J}{8}\,e^{-ik'R_1}\,\hat{F}_0\hat{\Omega}_{k'}^{-1}-\frac{J}{2}\,e^{-ik'R_1}<S_z^{(1)}S_z^{(2)}>\hat{F}\hat{\Omega}_{k'}^{-1},$$
$$\hat{B}=-\frac{J}{8}\,e^{-ik'R_1}\,\hat{F}\hat{\Omega}_{k'}^{-1}-\frac{J}{2}\,e^{-ik'R_1}<S_z^{(1)}S_z^{(2)}>\hat{F}_0\hat{\Omega}_{k'}^{-1},$$
$$\hat{C}=-\frac{J}{2}\,e^{-ik'R_1}<S_z^{(1)}S_z^{(2)}>\hat{F}_0\hat{\Omega}_{k'}^{-1}-\frac{J}{8}\,e^{-ik'R_2}\,\hat{F}\hat{\Omega}_{k'}^{-1},$$
$$\hat{D}=-\frac{J}{2}\,e^{-ik'R_1}<S_z^{(1)}S_z^{(2)}>\hat{F}\hat{\Omega}_{k'}^{-1}-\frac{J}{8}\,e^{-ik'R_2}\,\hat{F}_0\hat{\Omega}_{k'}^{-1}, \qquad (3-7)$$

and

$$\hat{F}_0(\omega)=\frac{1}{N}\sum_l\hat{\Omega}_l^{-1}(\omega),$$
$$F(\omega)=\frac{1}{N}\sum_l e^{il(R_1-R_2)}\hat{\Omega}_l^{-1}(\omega)=\frac{1}{N}\sum_l e^{-il(R_1-R_2)}\hat{\Omega}_l^{-1}(\omega), \qquad (3-8)$$

The solutions of Equation (3—5) are written as follows:

$$\hat{X}=\frac{1}{4}(\hat{a}+\hat{b}+\hat{c}+\hat{d})^{-1}(\hat{A}+\hat{B}+\hat{C}+\hat{D})$$
$$+\frac{\varepsilon_1}{4}(\hat{a}+\hat{b}-\hat{c}-\hat{d})^{-1}(\hat{A}-\hat{B}+\hat{C}-\hat{D})$$
$$+\frac{\varepsilon_2}{4}(\hat{a}-\hat{b}-\hat{c}+\hat{d})^{-1}(\hat{A}+\hat{B}-\hat{C}-\hat{D})$$
$$+\frac{\varepsilon_3}{4}(\hat{a}-\hat{b}+\hat{c}-\hat{d})^{-1}(\hat{A}-\hat{B}-\hat{C}+\hat{D}), \qquad (3-9)$$

where

$$\hat{X}=\hat{X}_1^1 \quad \text{for} \quad \varepsilon_1=\varepsilon_2=\varepsilon_3=1,$$
$$=\hat{X}_1^2 \quad \text{for} \quad \varepsilon_2=1,\ \varepsilon_1=\varepsilon_3=-1,$$
$$=\hat{X}_2^1 \quad \text{for} \quad \varepsilon_1=1,\ \varepsilon_2=\varepsilon_3=-1,$$
$$=\hat{X}_2^2 \quad \text{for} \quad \varepsilon_3=1,\ \varepsilon_1=\varepsilon_2=-1, \qquad (3-9\text{ a})$$

Making use of these results, the solutions for the Green functions are, for example, given as

$$\hat{G}^{kk'}(\omega)=-\frac{J}{2}\,e^{ikR_1}\hat{\Omega}_k^{-1}\hat{X}_1^1-\frac{J}{2}\,e^{ikR_2}\hat{\Omega}_k^{-1}\hat{X}_2^2+\delta_{kk'}\hat{\Omega}_k^{-1}, \qquad (3-10)$$

and so on.

## § 4. Calculation of the bound states

The impurity levels are obtained by seeking singular points of four inverse matrices $(\hat{a}+\hat{b}+\hat{c}+\hat{d})^{-1}$, $(\hat{a}+\hat{b}-\hat{c}-\hat{d})^{-1}$, etc., which appear in Equation (3—9): namely we can obtain the desired values of $\omega$ by solving four determinant equations,

$$\det|\hat{a}+\hat{b}+\hat{c}+\hat{d}|=0,$$
$$\det|\hat{a}+\hat{b}-\hat{c}-\hat{d}|=0, \qquad \text{etc.}$$

These four determinant equations are expressed in the following form,

$$\begin{aligned}&[1-\gamma^2\{(\omega^2+\Delta^2)f_0^2+\varepsilon_1(\omega^2+\Delta^2+\lambda^2+2\lambda\omega)f^2+(\varepsilon_2+\varepsilon_3)(\omega^2+\Delta^2+\lambda\omega)f_0 f\}]\\ &\times[1-\gamma^2\{(\omega^2+\Delta^2)f_0^2+\varepsilon_1(\omega^2+\Delta^2+\lambda^2-2\lambda\omega)f^2+(\varepsilon_2+\varepsilon_3)(\omega^2+\Delta^2-\lambda\omega)f_0 f\}]\\ &-4\gamma^4[\omega\Delta f_0^2+\varepsilon_1\omega\Delta f^2+\varepsilon_2(\omega\Delta-\frac{\lambda\Delta}{2})f_0 f+\varepsilon_3(\omega\Delta+\frac{\lambda\Delta}{2})f_0 f]\\ &\times[\omega\Delta f_0^2+\varepsilon_1\omega\Delta f^2+\varepsilon_2(\omega\Delta+\frac{\lambda\Delta}{2})f_0 f+\varepsilon_3(\omega\Delta-\frac{\lambda\Delta}{2})f_0 f]\\ &=0, \qquad (4-1)\end{aligned}$$

where we must consider next four cases,

$$\begin{aligned}&\text{Case (1)} && \varepsilon_1=\varepsilon_2=\varepsilon_3=1,\\ &\text{Case (2)} && \varepsilon_1=\varepsilon_2=-1,\ \varepsilon_3=1,\\ &\text{Case (3)} && \varepsilon_1=1,\ \varepsilon_2=\varepsilon_3=-1,\\ &\text{Case (4)} && \varepsilon_1=\varepsilon_3=-1,\ \varepsilon_2=1, \qquad (4-2)\end{aligned}$$

and

$$\begin{aligned}f_0&=\frac{1}{N}\sum_l\frac{1}{\omega^2-\xi_l^2-\Delta^2},\\ f&=\frac{1}{N}\sum_l\frac{e^{ilR}}{\omega^2-\xi_l^2-\Delta^2},\\ \lambda f&=\frac{1}{N}\sum_l\frac{\xi_l\, e^{ilR}}{\omega^2-\xi_l^2-\Delta^2}, \qquad (4-3)\end{aligned}$$

where $R=R_1-R_2$ and $\gamma=\frac{J}{4}$.

In the limiting case, $R\to 0$, we can take $f=f_0$ and $\lambda=0$, and solve Equation (4—1). The result corresponding to Case (1) is as follows;

$$\omega=\pm\frac{1-4\left(\frac{\pi J\rho}{4N}\right)^2}{1+4\left(\frac{\pi J\rho}{4N}\right)^2}\Delta\equiv\omega_2. \qquad (4-4)$$

It is easily understood in this limiting case that the Case (2), (3), and (4) have no solution in the range $|\omega|<\Delta$. We consider that this result is reasonable, coinciding with that of single impurity with resultant spin 1.

In the case, $k_F R\gg 1$, we can obtain the explicit form of matrix $\hat{F}$ using the usual technique.[13] The results are as follows;

$$\hat{F}^{\pm}(\omega) = -\frac{\pi\rho}{N k_F R} e^{\pm i\frac{R}{v_F}\sqrt{\omega^2-\Delta^2}} \begin{pmatrix} \cos k_F R \pm i\frac{\omega \sin k_F R}{\sqrt{\omega^2-\Delta^2}} & \mp i\frac{\Delta \sin k_F R}{\sqrt{\omega^2-\Delta^2}} \\ \mp i\frac{\Delta \sin k_F R}{\sqrt{\omega^2-\Delta^2}} & -\cos k_F R \pm i\frac{\omega \sin k_F R}{\sqrt{\omega^2-\Delta^2}} \end{pmatrix}$$

for $|\omega| \geq \Delta$;

$$= -\frac{\pi\rho}{N k_F R} e^{\pm i\frac{R}{v_F}\sqrt{\Delta^2-\omega^2}} \begin{pmatrix} \cos k_F R + \frac{\omega \sin k_F R}{\sqrt{\Delta^2-\omega^2}} & -\frac{\Delta \sin k_F R}{\sqrt{\Delta^2-\omega^2}} \\ -\frac{\Delta \sin k_F R}{\sqrt{\Delta^2-\omega^2}} & -\cos k_F R + \frac{\omega \sin k_F R}{\sqrt{\Delta^2-\omega^2}} \end{pmatrix}$$

for $|\omega| < \Delta$, (4—5)

and $\hat{F}_o^{\pm}(\omega)$ is given by Equation (2—10), which holds independent of R. Using the explicit forms of $\hat{F}_o$ and $\hat{F}$, we can obtain the bound state in the energy gap by seeking poles of four inverse matrices $(a+b+c+d)^{-1}$, $(a+b-c-d)^{-1}$, etc. The independent six formal solutions which we obtained are as follows;

$$\omega = \pm \frac{\Delta}{\sqrt{1+\left\{\dfrac{\frac{J\pi\rho}{2N}(1+K(\omega)\sin k_F R)}{1-\frac{J^2\pi^2\rho^2}{16N^2}(K^2(\omega)+2K(\omega)\sin k_F R+1)}\right\}^2}}$$

$$= \pm \frac{\Delta}{\sqrt{1+\left\{\dfrac{\frac{J\pi\rho}{2N}(1-K(\omega)\sin k_F R)}{1-\frac{J^2\pi^2\rho^2}{16N^2}(K^2(\omega)-2K(\omega)\sin k_F R+1)}\right\}^2}}$$

$$= \pm \frac{\sqrt{1-4\dfrac{\left(\frac{J\pi\rho}{4N}\right)^2 K^2(\omega)\cos^2 k_F R}{1+\left(\frac{J\pi\rho}{4N}\right)^2 K^2(\omega)}}}{\sqrt{1+4\dfrac{\left(\frac{J\pi\rho}{4N}\right)^2(1-K(\omega)\sin k_F R)^2+\left(\frac{J\pi\rho}{4N}\right)^4 K^2(\omega)\cos^2 k_F R}{1+\left(\frac{J\pi\rho}{4N}\right)^2 K^2(\omega)}}}\,\Delta \,.$$

(4—6)

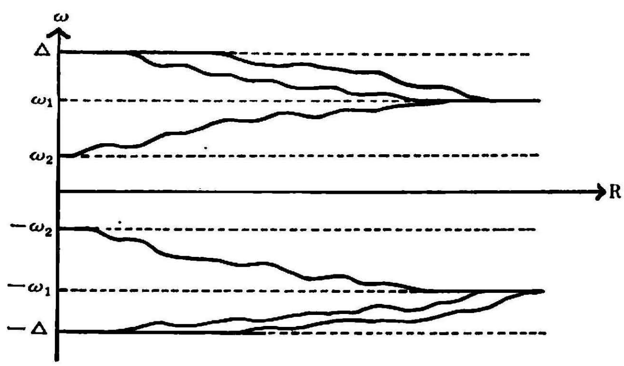
where

$$K(\omega) = \frac{1}{k_F R} e^{-\frac{R}{v_F}\sqrt{\Delta^2-\omega^2}} \,. \qquad (4-7)$$

It can be easily seen that in the limiting case, $R\to\infty$, these results reasonably coincide with the results (2—11) of the one impurity system.

Combining the results obtained above, ω-R curves are given qualitatively in Fig. 2. Regarding R as the inverse of impurity concentration, we can get some suggestions about the growth of impurity band. This will be discussed at § 5.

Fig. 2. $\omega-R$ curve for the case of two impurities.




## § 5. Discussion

In the case of one Ising impurity, it can be shown that the order parameter $\Delta$ has not impurity effects. Therefore, existence of an impurity does not shift transition-temperature $T_C$ and other physical quantities such as entropy and specific heat. This impurity will not destroy the Cooper pairs, as non-magnetic potential does not either, which was first pointed out by P. W. Anderson.[14] We have assumed $\Delta$ to be constant in space. By the same reason mentioned above, the local spatial variation will not exist around the impurity. These results are completely different from those of many authors for the Kondo's Hamiltonian. The differences are quite natural.

J. Zittartz[15] calculated the magnetic susceptibility using the exact solution of the Nagaoka's equation with approximation in the normal state, and pointed the unfavorable result that the susceptibility was negative at $T=0$ for spin $S=\frac{1}{2}$. Does this fault from Nagaoka's approximation for higher order Green function, or from the model Hamiltonian? Our exact solution of the equations without approximation might partly answer this question.

In the case of two Ising impurities, we have obtained the excited bound states dependent on the distance R between impurities, which have reasonably coincided, at limit of R infinity, with the case of one impurity and, at limit of R zero, with the case of one impurity of resultant spin 1. Fig. 2 has given the position and number of impurity levels for an arbitrary R. Taking R as the averaged distance among impurities, we will be able to use inverse of R instead of the impurity concentration. Fig. 2 suggests the growth of impurity band begins with the localized levels, $\omega=\pm\omega_1$, at low concentration, which coincides with Shiba's[12] results. In addition, the growth of band begins also with the energy gap edge and with suitable levels, near $\omega=\pm\omega_2$, at high concentration. These conjectures seem to be quite reasonable.

We consider that around an impurity the magnetic field will rather easily penetrate into superconductor under suitable conditions. Therefore, it will be expected that the local spatial variation of the order parameter exists also around the impurity.[16] Taking this effect, it will be interesting to investigate the bound state. This problem is left for future studies.

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