Abnormal Subgroups, p-Normality and Solvability of Finite Groups

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The purpose of the present paper is to prove two theorems which demonstrate that an adequate combination of the concepts of abnormal subgroups and p-normality (or strong p-normality) of a finite group, together with the deep and fundamental theorem of Feit and Thompson [5] on the solvability of odd order groups, gives us some sufficient conditions for even order groups to be solvable.

Theorem 1. *Let G be a finite group, M a maximal subgroup, H a nilpotent Hall π-subgroup, and N the maximal normal π-subgroup of G. Suppose that $M=H\times A$, where A is a subgroup of G. Then:*

(i) *If $2\in\pi$ and G/N is 2-normal, then G is π-separable of π-length at most 2. G is solvable.*

(ii) *If $2\in\pi$, then G is π-separable of π-length at most two.*

Theorem 2. *Let G be a finite group, H a nilpotent Hall π-subgroup, whose normalizer in G is the direct product of H and a subgroup A, and N the maximal normal π-subgroup of G. If G/N is strongly p-normal for all $p\in\pi$, then G is π-separable of π-length atmost two. In particular, if $2\in\pi$, then G is solvable.*

As immediate consequences of Theorem 1 and Theorem 2 we have the following solvability conditions for finte groups of even order.

Theorem 3. *Let G be a finite group of even order, P a Sylow 2-subgroup, and N the maximal normal 2-subgroup of G. If the normalizer of P is the direct product of P and a subgroup A and is maximal in G, and if G/N is 2-normal, then G is solvable.*

Theorem 4. *Let G be a finite group of even order, P a Sylow 2-subgroup, and N the maximal normal 2-subgroup of G. If the normalizer of P is the direct product of P and a subgroup A, and if G/N is strongly 2-normal, then G is solvable.*

We only remark that finite groups, all of whose proper subgroups or proper self-normalizing (abnormal) subgroups have a nice property, such as nilpotency (Schmidt-Iwasawa [11] and Rose [10]), supersolvability (Huppert [7] and Rose [10]), and p-nilpotency (Ito [8] and Rose [10]) are special cases of Theorem 1 and Theorem 2.

2. All groups considered are finite. Let G be a group. G is *p-normal* if the center $Z(P)$ of a Sylow p-subgroup P is the center of any other Sylow p-subgroup which contains it. Following Bauman [1], G is *strongly p-normal* if the higher centers $Z_i(P)$, or $i+1$ terms in the upper central series of P are normal in any

other Sylow p-subgroup which contains them. Thus strong p-normality is a natural extension of the concept of p-normality to the higher centers of P. However p-normality does not imply strong p-normality, see [1].

π denotes a set of prime numbers and π' the comlementary set. π-group is a group G, every prime divisor of whose order, $|G|$, belongs to π. *If* $\pi=\{p\}$, then we have p-groups. A subgroup H of G is a Hall π-subgroup if it is a π-group and the index, $|G: H|$, of H in G is not divisible by any prime in π. A normal π-complement is a normal π'-subgroup which is complementary to a Hall π-subgroup. Let

$$1=H_0,\ N_0,\ H_1,\ N_1,\ \cdots,H_k,\ N_k,\ \cdots$$

be a lower π-series of G, i. e., a series in which N_i/H_i is the maximal normal π'-subgroup of G/H_i and H_i/N_{i-1} is the maximal normal π-subgroup of G/N_{i-1}. If $N_k=G$ for some k, then G is called π-separable of π-length, $l_\pi(G)$, k. See [7].

Let S be a subgroup of G. $N(S)$ and $C(S)$ denote the normalizer and centralizer of S, respectively, in G. If $N(S)=S$, then S is a self-normalizing subgroup. S is abnormal in G if $g\in\{S,g^{-1}Sg\}$, a subgroup generated by S and $g^{-1}Sg$, for every element $g\in G$. S is abnormal in G if and only if it satisfies the following two conditions (see [2]):

(i) Every subgroup of G containing S is self-normalizing.

(ii) S is not contained in two different conjugate subgroups.

3. Throughout this section we assume that G contains a nilpotent Hall π-subgroup H. The reader may find himself familiar with most of our methods of proofs. However we give them for the sake of completeness. The following well-known theorems are used.

(A) **Theorem of Schur and Zassenhaus** [11] : *A normal Hall π-subgroup of G has a complement in G.*

(B) **Theorem of Hall and Grun** [11] : *If G is p-normal and $N(Z(P))$ has a normal p-complement, then G has a normal p-complement.*

(C) **Theorem of Glauberman and Thompson** [6]: *If P is a Sylow p-subgroup, p odd, and if $N(Z(J(P)))$ has a normal p-complement, then so does G, where $J(P)$ is the Thompson subgroup which is characteristic in P.*

Proof of Theorem 1. If $H=N$, then the conclusion is true. Hence H is not normal in G. We may assume $N=1$. Therefore the normalizer of any normal subgroup of the Sylow p-subgroup P, $p\in\pi$, is M and M has a normal p-complement. Thus $N(Z(J(P)))$, p odd, and $N(Z(P))$, $p=2$, have normal p-complement. For p odd by (C) G has a normal p-complement. On the other hand in case $p=2$, since G is 2-normal G has a normal 2-complement by (B). In any case G has normal p-complements for all $p\in\pi$. The intersection of all the p-complements is a normal π-complement. Thus G is π-separable of $l_\pi(G)$ at most 2. In case $2\in\pi$, π'-subgroups are of odd order and hence solvable by [5]. G is solvable.

Proof of Theorem 2. If p is normal in G then $G=H\times A$ and we are done. Hence H is not normal in G. We may assume $N=1$. Therefore we need only show the following:

Theorem 5. *If $N(H)$ is the direct product of H and a subgroup A and G is strongly p-normal for all p,* then G possesses a normal π-complement.

Lemma. *If S is a nilpotent Hall π-subgroup of G then $N(S)$ is abnormal in G.*

Proof. Let $T=\{N(S), g^{-1} N(S) g\}$, $g \in G$. By a theorem of Wielandt [11], S and $g^{-1} Sg$ are conjugate *in* T. Hence there is $t \in T$ such that $t^{-1}St = g^{-1}Sg$. $(gt^{-1})^{-1} S(gt^{1-}) = S$ and hence $gt^{-1} \in N(S) \leq M$. Therefore $g \in T$ and $N(S)$ is abnormal in G.

Proof of Theorem 5. If H is normal in G, then $G = H \times A$ and we are done. Therefore H is not normal in G, i. e., $N(H) = H \times A < G$. We consider two cases.

Case 1. For one p, the center $Z(P)$ of the Sylow p-subgroup of H is normal in G, i. e., $N(Z(P)) = G$. Then $C(Z(P))$ is normal in G. Moreover $C(Z(P))$ contains $N(H)$. Hence by Lemma,

$$C(Z(P)) = G. \qquad (*)$$

Now by induction $G/Z(P)$ has a normal π-complement $L/Z(P)$ of $H/Z(P)$. Since $|Z(P)|$ and $|L:Z(P)|$ are relatively prime by (A) $Z(P)$ has a complement K in L. But by (*) K must be normal, in fact, characteristic in L. Hence K is normal in G. We have

$$G = K \cdot Z(P) \cdot H = K \cdot H,$$

and

$$K \cap H = K \cap Z(P) \cap H = 1.$$

Case 2. $Z(P)$ are not normal for all $p \in \pi$. Thus for each P, $N(H) \leq N(Z(P)) < G$. Hence by induction, $N(Z(P))$ has a normal π-complement. Therefore $N(Z(P))$ has a normal p-complement. Hence, by (B) G has a normal p-complement. The intersection of normal p-complements for all p, then, is the desired π-complement. This completes the proof.

Remark. Theorem 5 is a generalization of Theorem 3 of Certok [4]. For examples of groups, see [9,10].

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