# The Second Homology Group of the Quotient Space of a Periodic Transformation

# The Second Homology Group of the Quotient Space of a Periodic Transformation

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Throughout this note $M$ is a compact 2-connected triangulated manifold, $T$: $M \to M$ a simplicial periodic transformation of prime period $p$, $F$ the fixed point set of $T$ and $M^*$ the quotient space of $M$ by $T$. We shall prove

*Theorem.* $H_2(M^*, Z) \cong \widetilde{H}_o(F, Z_p)$, *where* $\widetilde{H}_o$ *is the reduced homology group.*

## 1. The set F

Since $T$ is simplicial, $F$ is a subcomplex of $M$. Moreover, by P. A. Smith[1], $F$ is an orientable homology manifold over $Z_p$.

***Proposition.*** *If* $dim\ F \geqq dim\ M-2$, *F is connected.*

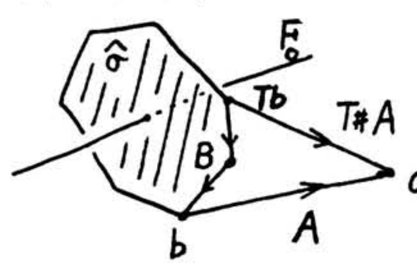
***Proof.*** We consider only the case $dim\ F = dim\ M - 2$. Let $dim\ M=n$ and $F_o$ be an $(n-2)$-dimensional component of $F$ and let $\sigma$ be an $(n-2)$-simplex of $F_o$. Denote by $\hat{\sigma}$ the dual $2-$cell of $\sigma$ in the dual cellular decomposition of $M$, i. e. $\hat{\sigma}$ is the union of $2-$simplexes in the first barycentric subdivision of $M$ which meet $\sigma$ only at the barycenter of $\sigma$. The boundary $\partial\hat{\sigma}$ of $\hat{\sigma}$ is invariant under $T$ and does not meet $F$.

Now assume $F-F_o \neq \emptyset$. Let $a$, $b$ be vertices of $F-F_o$, $\partial\hat{\sigma}$, respectively. Let $A$ be a 1-chain in $M-F_o$ such that $\partial A=a-b$, and $B$ a minimal (relative to the number of 1-simplexes) 1-chain in $\partial\hat{\sigma}$ such that $\partial B=b-Tb$. Then in some orientation of $\hat{\sigma}$,

$$\sum_{i=1}^{p} T_{\#}^{i} B = \partial\hat{\sigma}$$

where $T_\#$ is the chain map induced by $T$. Since $\partial(A-T_\# A+B)=O$ and $H_1(M,Z)=0$, there is a 2-chain $C$ in $M$ such that $\partial C=A-T_\# A+B$.

Then




$$\partial \sum_{i=1}^{p} T_{\#}^{i} C = \sum_{i=1}^{p} (T_{\#}^{i} A - T_{\#}^{i+1} A + T_{\#}^{i} B)$$
$$= \Sigma\, T_{\#}^{i} B = \partial\,\hat{\sigma}.$$

Hence $\Sigma\, T_{\#}^{i} C-\hat{\sigma}$ is a 2-cycle of $M$.

Furthermore, $F_o$ is an $(n-2)$ -cycle (mod$p$) in $M$ mod $\partial$M.

Consider now the intersection number $I\ (\Sigma\ T_{\#}^{i} C-\hat{\sigma},\ F_o)$ of two cycles mod $p$.

Since $\Sigma\, T_{\#}^{i} C-\hat{\sigma}$ is homologous to zero(for $H_2(M,Z)=0$), $I\ (\Sigma T_{\#}^{i} C-\hat{\sigma},\ F_o)=0$ mod $p$.

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Therefore $I(\Sigma\ T^i_{\#} C, F_o) = I(\hat{\sigma}, F_o) \bmod p$.

But $I(\hat{\sigma}, F_o) = 1$ and $I(\Sigma\ T^i_{\#} C, F_o) = \sum_{i=1}^{p} I(T^i_{\#} C, F_o) = pI(C, F_o) = 0 \bmod p$.
This is contradiction.

**2. A homomorohism $\Phi$: $H_1(M, F; Z) \otimes Z_p \to H_2(M^*, Z)$**

Let $Z_p = \{1, T, \dots, T^{p-1}\}$ and $\pi: M \to M^*$ the natural projection. The triangulation of $M$ (subdivide $M$ if necessary) introduces in a natural way a triangulation into $M^*$ and $\pi$ is simplicial. The chain maps induced by $T$, $\pi$ are denoted by $T_{\#}, \pi_{\#}$, respectively.

Let $\Sigma\ a_i \otimes T^{k_i}$ be an element of $H_1(M, F; Z) \otimes Z_p$ and let $A_i$ be $1$-chains representing $a_i$. Then $\partial A_i$ are in $F$, so $\partial(A_i - T^{k_i}_{\#} A_i) = 0$. Thus $\Sigma\ (A_i - T^{k_i}_{\#} A_i)$ is a $1$-cycle of $M$. Since $H_1(M,Z) = 0$, we can choose a $2$-chain $B$ of $M$ satisfying $\partial B = \Sigma\ (A_i - T^{k_i}_{\#} A_i)$.

Then

$$\partial \pi_{\#} B = \Sigma\ (\pi_{\#} A_i - \pi_{\#} T^{k_i}_{\#} A_i) = 0.$$

Hence $\pi_{\#} B$ is a $2$-cycle of $M^*$.

Define $\Phi(\Sigma a_i \otimes T^{k_i}) = [\pi_{\#} B]$, where $[\quad]$ means the homology class.

*$\Phi$ is well defined.*

*Proof.* (*i*) Fix $A_i$ and let $B$, $B'$ be $2$-chains satisfying

$$\partial B = \partial B' = \Sigma\ (A_i - T^{k_i}_{\#} A_i).$$

Then $B - B'$ is a $2$-cycle of $M$. Since $H_2(M,Z) = 0$, there is a $3$-chain $C$ such that $\partial C = B - B'$. Thus $\partial \pi_{\#} C = \pi_{\#} B - \pi_{\#} B'$.

(*ii*) Now let $A_i$ and $A'_i$ represent $a_i$. Then there are $2$-chains $D_i$ of $M$ and $1$-chains $E_i$ of $F$ such that $A_i - A'_i = \partial D_i + E_i$. Let $B$, $B'$ be $2$-chains such that

$$\partial B = \Sigma(A_i - T^{k_i}_{\#} A_i), \partial B' = \Sigma(A'_i - T^{k_i}_{\#} A'_i).$$

Then, by simple caluculation,

$$\partial\ (\Sigma(D_i - T^{k_i}_{\#} D_i) + B') = \partial B.$$

Hence by (*i*),

$$[\pi_{\#}(\Sigma\ (D_i - T^{k_i}_{\#} D_i) + B')] = [\pi_{\#} B].$$

On the other hand,

$$\pi_{\#}(\ \Sigma\ (D_i - T^{k_i}_{\#} D_i) + B') = \Sigma(\pi_{\#} D_i - \pi_{\#} T^{k_i}_{\#} D_i) + \pi_{\#} B' = \pi_{\#} B'.$$

Thus $[\pi_{\#} B] = [\pi_{\#} B']$.

*$\Phi$ is a homomorphism.*

*Proof.* we show only $\Phi\ (a \otimes T^{k+l}) = \Phi\ (a \otimes T^{k}) + \Phi\ (a \otimes T^{l})$.

Let $a = [A]$ and $B$, $C$ be 2-chains such that $\partial B = A - T_{\#}^{k} A$, $\partial C = A - T_{\#}^{l} A$.
Since

$$\partial\,(B + T_{\#}^{k} C) = A - T_{\#}^{k} A + T_{\#}^{k+l} A = A - T_{\#}^{k+l} A,$$

$$\Phi\,(a \otimes T^{k+l}) = [\pi_{\#}\,(B + T_{\#}^{k} C)] = [\pi_{\#} B] + [\pi_{\#} C] = \Phi\,(a \otimes T^{k}) + \Phi (a \otimes T^{l}).$$

### 3. Two lemmas

Let $\Sigma^2$ be a triangulated 2-sphere and $f^{\bullet} : \Sigma^2 \to M^{\bullet}$ a simplicial map. In general the subcomplex $f^{\bullet\,-1} F$ of $\Sigma^2$ is not connected. We can choose polygonal arcs $K_1, \ldots, K_m$ such that

( i ) only the end-points of $K_i$ are in $F$,

( ii) interior of $K_i$ are mutually disjoint,

(iii) $f^{\bullet\,-1} F \cup \bigcup_i K_i$ is connected.

By '*cutting*' $\Sigma^2$ along $\cup K_i$ we obtain in a natural way a compact surface $\Delta^2$.
Let $\varphi$: $\Delta^2 \to \Sigma^2$ be the '*sewing*' map. The first barycentric subdivision of $\Sigma^2$ induces a triangulation of $\Delta^2$, and $\varphi$ : $\Delta^2 - \Sigma^2$ is simplicial.

*Lemma 1. There is a simplicial map $f$ : $\Delta^2 \to M$ such that the diagram*

$$\begin{tikzcd} \Delta^2 \arrow[r, "f"] \arrow[d, "\varphi"'] & M \arrow[d, "\pi"] \\ \Sigma^2 \arrow[r, "f^{\bullet}"] & M^{\bullet} \end{tikzcd}$$

*commutes.*

*proof.* put $L = f^{\bullet\,-1} F \cup \cup\ K_i$ and let $N$ be the regular neighborhood of $L$ in $\Sigma^2$. $N$ is a connected surface with boundary and $L$ is a deformation retract of $N$. Since the boundary of $N$ is the union of disjoint Jordan curves, every component of $\Sigma^2 - N$, and hence, of $\Sigma^2 - L$ is simply connected. In addition since the partial map $\pi \mid M - F$ is a covering projection, there exists the lifting of $f^{\bullet} \mid \Sigma^2 - L$. Adding the fixed point set, we obtain a map $g$: $\Sigma^2 - \cup K_i \to M$ such that $\pi\ g =$

$f^* \mid \Sigma^2 - \bigcup K_i$. Extending $g\varphi \mid int\,\Delta^2$ linearly, we get a desired map $f : \Delta^2 \to M$.

Let $v : [0, 1] \to \Sigma^2$ be a simple closed curve and let $v\,[0, 1] = J$. $J$ divides $\Sigma^2$ into two domains $H^+$ and $H^-$.

*Lemma 2. Suppose $f^* : \Sigma^2 \to M^*$ satisfys the following conditions:*

*(1) $f^* J \subset M^* - F$.*

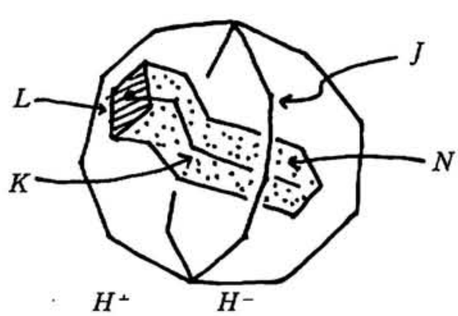
*(2) $f^{*-1} F \cap H^+$ consists of one point $x$.*

*(3) The component of $F$ containing $f^* x$ does not meet $f^* H^-$.*

*(4) $f^*$ is null-homotopic.*

*Then the curve $\tilde{v} : [0, 1] \to M$ which covers $f^* v : [0, 1] \to M^*$ is a closed curve.*

*Proof*. We show that $f^* \mid J : J \to M^* - F$ is null-homotopic in $M^* - F$. In case $f^* H^- \subset M^* - F$, this is trivial. Assume $f^* H^- \cap F \neq \emptyset$. Let $\Delta^3$ be a triangulated *3*-disk with $\Sigma^2$ as the boundary. Since $f^* : \Sigma^2 \to M^*$ is nullhomotopic, there is a map $g : \Delta^3 \to M^*$ such that $g \mid \Sigma^2 = f^*$. We may assume $g$ is simplicial. Let $K$ be the component of $g^{-1} F$ containing $x$. Let $L$ be the regular neighborhood of $x$ in $\Sigma^2$ and $N$ the regular neighborhood of $K$ in $\Delta^3$. $L$ and $N$ are *2* and *3* dimensional manifolds, respectively. Note that $N \cap \Sigma^2 = L$. put $(H^+ - L) \cup \overline{(\partial N - L)} = W$. Then $W$ is a surface and its boundary is just $J$. Since $\partial N$ is orientable, so is $W$. Moreover, $W \subset \Delta^3 - g^{-1} F$. Therefore $J$ bounds an orientable surface in $\Delta^3 - g^{-1} F$, and hence $f^* J$ is homologous to zero in $M^* - F$.



From $f^{*} H^{-} \cap F \neq \emptyset$ and the condition (3), we know that $F$ is not connected. Hence, by proposition in section 1, codimension of $F > 2$, and hence, by general position argument, $\pi_1(M-F) = \pi_1(M) = 1$. Therefore $\pi_1(M^{*}-F) \cong Z_p$ (which is abelian) $\cong H_1(M^{*} - F, Z)$. Hence, a closed curve in $M^{*} - F$ which is homologous to zero in $M^{*}-F$, is necessarily homotopic to zero in $M^{*} - F$.

### 4. $\Phi$ is onto

For any compact oriented triangulated manifold $X$ (with or without boundary), we denote by the same letter $X$, the fundamental cycle of $X$ (relative or absolute). Suppose $\Sigma^2$ be an oriented triangulated 2-sphere.

We prove now the homomorphism defined in section 2 is onto. Since $M^{*}$ is simply connected (M. A. Armstrong [2]), $H_2(M^{*}, Z)$ is isomorphic to $\pi_2(M^{*})$, and hence, any element of $H_2(M^{*}, Z)$ is represented by $f^{*}_{\#} \Sigma^2$ for some simplicial map $f^{*} : \Sigma^2 \to M^{*}$, where $f^{*}_{\#}$ is the chain map induced by $f^{*}$. Let $\Delta^2$ be the oriented surface obtained by 'cutting, $\Sigma^2$ along $\bigcup K_i$ as in section 3 and $\varphi : \Delta^2 \to \Sigma^2$ the 'sewing' map. By Lemma 1. there is a smpliciail map $f : \Delta^2 \to M$ such that $\pi f = f^{*} \varphi$. Let $K_i^{+}$, $K_i^{-}$ be the arcs obtained from $K_i$. Orient them so that $\partial \Delta^2 = \sum_i (K_i^{+} + K_i^{-})$. Since $f^{*} K_i$ meet $F$ at their end-points,

$$f_{\#} K_i^{-} = -T_{\#}^{k_i} f_{\#} K_i^{+} \textit{ for some } T^{k_i} \textit{ of } Z_p.$$

Then

$$\partial f_{\#} \Delta^2 = \Sigma (f_{\#} K_i^{+} + f_{\#} K_i^{-}) = \Sigma (f_{\#} K_i^{+} - T_{\#}^{k_i} f_{\#} K_i^{+}).$$

Hence, by the definition of $\Phi$,

$$\Phi (\Sigma [f_{\#} K_i^{+}] \otimes T^{k_i}) = [\pi_{\#} f_{\#} \Delta^2 .$$

But

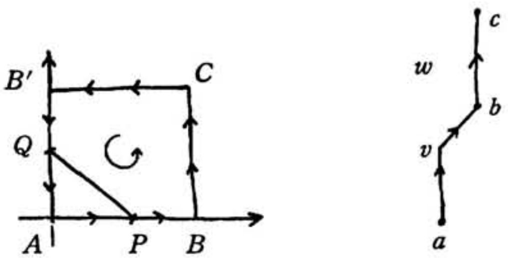
$$\pi_{\#} f_{\#} \Delta^2 = f^{*}_{\#} \varphi_{\#} \Delta^2 = f^{*}_{\#} \Sigma^2.$$

This proves that $\Phi$ is onto.

### 5. $\Phi$ is one to one

We prove now $\Phi$ is one to one. Assume, for simplicity, that $F$ consists of three components.

Let $\Delta^2 = \{ (x,y) \in R^2 ; 0 \leqq x, y \leqq 1 \}$, $R^2$ the Euclidean plane, and $A=(0,0)$, $B=(1, 0)$, $C=(1,1)$ $B'=(0,1)$.Denote by $AB$, $BC$,$\cdots$the line segements connecting $A$ and $B$, $B$ and $C$,$\cdots$. By identifying $(t,0)$ and $(0,t)$, $0 \leqq t \leqq 1$; $(t,1)$ and $(1,t)$ $0 \leqq t \leqq 1$, in $\Delta^2$, we obtain a 2-sphere $\Sigma^2$. Let $\varphi : \Delta^2 \to \Sigma^2$ be the identification map. By triangulating $\Delta^2$, $\Sigma^2$ suitably, we may assume that $\varphi : \Delta^2 \to \Sigma^2$ is simlicial. Fix orientations on $\Delta^2$, $\partial \Delta^2$ coherently and introduce an an orientation on $\Sigma^2$ *by* $\varphi$ .




Pick one vertex from each component of $F$ and denote them by $a$, $b$, $c$.
Let $v : AB \to M$ and $w : BC \to M$ be simplicial maps (subdivide $\Delta^2$ if necessary) such that $vA = a$, $vB = b = wB$, $w\,C = c$. Since *codim* $F > 2$ by proposition in section 1, we may assume that $v$ and $w$ meet $F$ only at their end-points.
Since $[v_\# AB]$, $[w_\# BC]$ are generators of $H_1(M, F; Z)$, any element $h$ of $H_1(M, F; Z) \otimes Z_p$ is written as

$$h = [v_\# AB] \otimes T^k + [w_\# BC] \otimes T^l \ , \quad T^k, \ T^l \ \epsilon \ Z_p.$$

Define two maps $v' : AB' \to M$, $w' : B'C \to M$ by the equations

$$v'(0,t) = T^k v(t,0), \ w'(t,1) = T^l w(1,t), \ 0 \leqq t \leqq 1.$$

Then

$$w'_\# CB' = -T^l_\# w_\# BC, \quad v'_\# B'A = -T^k_\# v_\# AB.$$

Since $M$ is simply connected, there is a simplicial map $f : \Delta^2 \to M$ which coinsdes with $v$, $w$, $w'$, $v'$ on $\partial \Delta^2$, and since *codim* $F > 2$, we may assume that $f(int\ \Delta^2) \cap F$ is empty. Since $\partial \Delta^2 = AB + BC + CB' + B'A$,

$$\partial f_\# \Delta^2 = v_\# AB - T^k_\# v_\# AB + w_\# BC - T^l_\# w_\# BC.$$

Then, by the definition of $\Phi$, $\Phi h = [\pi_\# f_\# \Delta^2]$.
From the definition of $v'$, $w'$, it follows that there is a simplicial map $f^* : \Sigma^2 \to M^*$ such that $\pi f = f^* \varphi$.
Then $\Phi h = [\pi_\# f_\# \Delta_2] = [f^*_\# \varphi_\# \Delta^2] = [f^*_\# \Sigma^2]$.

We now show that if $\Phi h = 0$, $h = 0$. Let $P$, $Q$ be the mid-points of $AB$, $AB'$, respectively. Then $\varphi P = \varphi Q$, and $J = \varphi PQ$ is a Jordan curve in $\Sigma^2$.
$J$ divides $\Sigma^2$ into two domains. Denote by $H^+$ the domain containing $\varphi A$ and by $H^-$ the other. Since $H_2(M^*, Z) = \pi_2(M^*)$, $\Phi h = 0$ means that $f^*$ is null homotopic.
Now we can apply Lemma 2. The curve $f \big| PQ : PQ \to M$ which covers $f^* \varphi : PQ$

$\to M^*$, is closed, and hence, $fP = fQ$. On the other hand, $fQ = v'Q = T^k vP = T^k fP$. Thus $fP = T^k fP$. But $fP$ is in $M - F$, this means $T^k = 1$. Similarly $T^l = 1$. Therefore $h = 0$.

## 6. Proof of Theorem

From the reduced homology exact sequence of $(M, F)$, $\widetilde{H}_0(F,Z) \cong H_1(M,F; Z)$, and since $\widetilde{H}_0(F, Z_p) \cong \widetilde{H}_0(F,Z) \otimes Z_p$, it is sufficient to show that $H_1(M,F; Z) \otimes Z_p \cong H_2(M^*,Z)$. The homomorphism $\Phi$ defined in section 2 is onto and one to one, hence $\Phi$ is an isomorphism. This proves Theorem.

### Reference

[1] P. A. Smith, *Transformations of finite period II*, Annals of Math., *40* (1939), 690-711.

[2] M. A. Armstrong, *On the fundamental group of an orbit space*, Proc. Camb. Phil. Soc., *61* (1965), 639-646.