

琉球大学学術リポジトリ

The Second Homology Group of the Quotient Space of a Periodic Transformation

メタデータ	言語: 出版者: 琉球大学工学部 公開日: 2012-02-28 キーワード (Ja): キーワード (En): 作成者: 前原, 潤 メールアドレス: 所属:
URL	http://hdl.handle.net/20.500.12000/23488

The Second Homology Group of the Quotient Space of a Periodic Transformation

by HIROSHI MAEHARA*

Throughout this note M is a compact 2-connected triangulated manifold, $T: M \rightarrow M$ a simplicial periodic transformation of prime period p , F the fixed point set of T and M^* the quotient space of M by T . We shall prove

Theorem. $H_2(M^*, Z) \cong \tilde{H}_0(F, Z_p)$, where \tilde{H}_0 is the reduced homology group.

1. The set F

Since T is simplicial, F is a subcomplex of M . Moreover, by P. A. Smith(1), F is an orientable homology manifold over Z_p .

Proposition. If $\dim F \geq \dim M - 2$, F is connected.

Proof. We consider only the case $\dim F = \dim M - 2$. Let $\dim M = n$ and F_0 be an $(n-2)$ -dimensional component of F and let σ be an $(n-2)$ -simplex of F_0 . Denote by $\hat{\sigma}$ the dual 2-cell of σ in the dual cellular decomposition of M , i. e. $\hat{\sigma}$ is the union of 2-simplexes in the first barycentric subdivision of M which meet σ only at the barycenter of σ . The boundary $\partial \hat{\sigma}$ of $\hat{\sigma}$ is invariant under T and does not meet F .

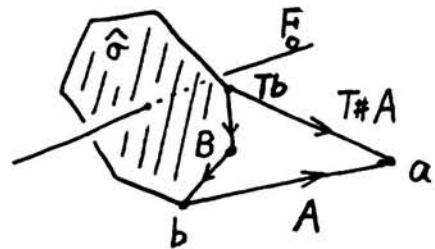
Now assume $F - F_0 \neq \emptyset$. Let a, b be vertices of $F - F_0$, $\partial \hat{\sigma}$, respectively. Let A be a 1-chain in $M - F_0$ such that $\partial A = a - b$, and B a minimal (relative to the number of 1-simplexes) 1-chain in $\partial \hat{\sigma}$ such that $\partial B = b - Tb$. Then in some orientation of $\hat{\sigma}$,

$$\sum_{i=1}^p T_i^i B = \partial \hat{\sigma}$$

where T_i is the chain map induced by T . Since $\partial(A - T_i A + B) = 0$ and $H_1(M, Z) = 0$, there is a 2-chain C in M such that $\partial C = A - T_i A + B$.

Then

$$\begin{aligned} \partial \sum_{i=1}^p T_i^i C &= \sum_{i=1}^p (T_i^i A - T_i^{i+1} A + T_i^i B) \\ &= \sum T_i^i B = \partial \hat{\sigma}. \end{aligned}$$



Hence $\sum T_i^i C - \hat{\sigma}$ is a 2-cycle of M .

Furthermore, F_0 is an $(n-2)$ -cycle (mod p) in M mod ∂M .

Consider now the intersection number $I(\sum T_i^i C - \hat{\sigma}, F_0)$ of two cycles mod p .

Since $\sum T_i^i C - \hat{\sigma}$ is homologous to zero (for $H_2(M, Z) = 0$), $I(\sum T_i^i C - \hat{\sigma}, F_0) = 0$ mod p .

Received November 20, 1972

*Dept. of Math., Div. of Sci. & Eng., Univ. of the Ryukyus

Therefore $I(\sum T_{\#}^i C, F_o) = I(\hat{\sigma}, F_o) \pmod{p}$.

But $I(\hat{\sigma}, F_o) = 1$ and $I(\sum T_{\#}^i C, F_o) = \sum_{i=1}^p I(T_{\#}^i C, F_o) = pI(C, F_o) = 0 \pmod{p}$.

This is contradiction.

2. A homomorphism $\Phi: H_1(M, F; Z) \otimes_{Z_p} \rightarrow H_2(M', Z)$

Let $Z_p = \{1, T, \dots, T^{p-1}\}$ and $\pi: M \rightarrow M'$ the natural projection. The triangulation of M (subdivide M if necessary) introduces in a natural way a triangulation into M' and π is simplicial. The chain maps induced by T, π are denoted by $T_{\#}, \pi_{\#}$, respectively.

Let $\sum a_i \otimes T^{h_i}$ be an element of $H_1(M, F; Z) \otimes_{Z_p}$ and let A_i be 1-chains representing a_i . Then ∂A_i are in F , so $\partial(A_i - T_{\#}^{h_i} A_i) = 0$. Thus $\sum (A_i - T_{\#}^{h_i} A_i)$ is a 1-cycle of M . Since $H_1(M, Z) = 0$, we can choose a 2-chain B of M satisfying $\partial B = \sum (A_i - T_{\#}^{h_i} A_i)$.

Then

$$\partial \pi_{\#} B = \sum (\pi_{\#} A_i - \pi_{\#} T_{\#}^{h_i} A_i) = 0.$$

Hence $\pi_{\#} B$ is a 2-cycle of M' .

Define $\Phi(\sum a_i \otimes T^{h_i}) = [\pi_{\#} B]$, where $[\quad]$ means the homology class. Φ is well defined.

Proof. (i) Fix A_i and let B, B' be 2-chains satisfying

$$\partial B = \partial B' = \sum (A_i - T_{\#}^{h_i} A_i).$$

Then $B - B'$ is a 2-cycle of M . Since $H_2(M, Z) = 0$, there is a 3-chain C such that $\partial C = B - B'$. Thus $\partial \pi_{\#} C = \pi_{\#} B - \pi_{\#} B'$.

(ii) Now let A_i and A'_i represent a_i . Then there are 2-chains D_i of M and 1-chains E_i of F such that $A_i - A'_i = \partial D_i + E_i$. Let B, B' be 2-chains such that

$$\partial B = \sum (A_i - T_{\#}^{h_i} A_i), \partial B' = \sum (A'_i - T_{\#}^{h_i} A'_i).$$

Then, by simple calculation,

$$\partial (\sum (D_i - T_{\#}^{h_i} D_i) + B') = \partial B.$$

Hence by (i),

$$[\pi_{\#} (\sum (D_i - T_{\#}^{h_i} D_i) + B')] = [\pi_{\#} B].$$

On the other hand,

$$\pi_{\#} (\sum (D_i - T_{\#}^{h_i} D_i) + B') = \sum (\pi_{\#} D_i - \pi_{\#} T_{\#}^{h_i} D_i) + \pi_{\#} B' = \pi_{\#} B'.$$

Thus $[\pi_{\#} B] = [\pi_{\#} B']$.

Φ is a homomorphism.

Proof. we show only $\Phi(a \otimes T^{k+l}) = \Phi(a \otimes T^k) + \Phi(a \otimes T^l)$.

Let $a = [A]$ and B, C be 2-chains such that $\partial B = A - T_{\#}^k A$, $\partial C = A - T_{\#}^l A$.

Since

$$\partial(B + T_{\#}^k C) = A - T_{\#}^k A + T_{\#}^{k+l} A = A - T_{\#}^{k+l} A,$$

$$\Phi(a \otimes T^{k+l}) = [\pi_{\#}(B + T_{\#}^k C)] = [\pi_{\#} B] + [\pi_{\#} C] = \Phi(a \otimes T^k) + \Phi(a \otimes T^l).$$

3. Two lemmas

Let Σ^2 be a triangulated 2-sphere and $f^* : \Sigma^2 \rightarrow M^*$ a simplicial map.

In general the subcomplex $f^{*-1} F$ of Σ^2 is not connected. We can choose polygonal arcs K_1, \dots, K_m such that

- (i) only the end-points of K_i are in F ,
- (ii) interior of K_i are mutually disjoint,
- (iii) $f^{*-1} F \cup \bigcup_i K_i$ is connected.

By 'cutting' Σ^2 along $\bigcup K_i$ we obtain in a natural way a compact surface Δ^2 .

Let $\varphi : \Delta^2 \rightarrow \Sigma^2$ be the 'sewing' map. The first barycentric subdivision of Σ^2 induces a triangulation of Δ^2 , and $\varphi : \Delta^2 \rightarrow \Sigma^2$ is simplicial.

Lemma 1. There is a simplicial map $f : \Delta^2 \rightarrow M$ such that the diagram

$$\begin{array}{ccc} \Delta^2 & \xrightarrow{f} & M \\ \varphi \downarrow & & \downarrow \pi \\ \Sigma^2 & \xrightarrow{f^*} & M^* \end{array} \quad \text{commutes.}$$

proof. put $L = f^{*-1} F \cup \bigcup K_i$ and let N be the regular neighborhood of L in Σ^2 . N is a connected surface with boundary and L is a deformation retract of N . Since the boundary of N is the union of disjoint Jordan curves, every component of $\Sigma^2 - N$, and hence, of $\Sigma^2 - L$ is simply connected. In addition since the partial map $\pi |_{M-F}$ is a covering projection, there exists the lifting of $f^* |_{\Sigma^2 - L}$. Adding the fixed point set, we obtain a map $g : \Sigma^2 - \bigcup K_i \rightarrow M$ such that $\pi g =$

$f^* \mid \Sigma^2 - \cup K_i$. Extending $g \mid \text{int } \Delta^2$ linearly, we get a desired map $f : \Delta^2 \rightarrow M$.

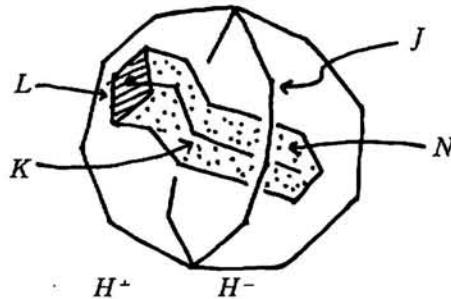
Let $v : [0, 1] \rightarrow \Sigma^2$ be a simple closed curve and let $v [0, 1] = J$. J divides Σ^2 into two domains H^+ and H^- .

Lemma 2. Suppose $f^* : \Sigma^2 \rightarrow M^*$ satisfies the following conditions:

- (1) $f^* J \subset M^* - F$.
- (2) $f^{*-1} F \cap H^+$ consists of one point x .
- (3) The component of F containing $f^* x$ does not meet $f^* H^-$.
- (4) f^* is null-homotopic.

Then the curve $\bar{v} : [0, 1] \rightarrow M$ which covers $f^* v : [0, 1] \rightarrow M^*$ is a closed curve.

Proof. We show that $f^* \mid J : J \rightarrow M^* - F$ is null-homotopic in $M^* - F$. In case $f^* H^- \subset M^* - F$, this is trivial. Assume $f^* H^- \cap F \neq \emptyset$. Let Δ^3 be a triangulated 3-disk with Σ^2 as the boundary. Since $f^* : \Sigma^2 \rightarrow M^*$ is nullhomotopic, there is a map $g : \Delta^3 \rightarrow M^*$ such that $g \mid \Sigma^2 = f^*$. We may assume g is simplicial. Let K be the component of $g^{-1} F$ containing x . Let L be the regular neighborhood of x in Σ^2 and N the regular neighborhood of K in Δ^3 . L and N are 2 and 3 dimensional manifolds, respectively. Note that $N \cap \Sigma^2 = L$. put $(H^+ - L) \cup (\partial N - L) = W$. Then W is a surface and its boundary is just J . Since ∂N is orientable, so is W . Moreover, $W \subset \Delta^3 - g^{-1} F$. Therefore J bounds an orientable surface in $\Delta^3 - g^{-1} F$, and hence $f^* J$ is homologous to zero in $M^* - F$.



From $f^* H^- \cap F \neq \emptyset$ and the condition (3), we know that F is not connected. Hence, by proposition in section 1, codimension of $F > 2$, and hence, by general position argument, $\pi_1(M-F) = \pi_1(M) = 1$. Therefore $\pi_1(M^* - F) \cong Z_p$

(which is abelian) $\cong H_1(M^* - F, Z)$. Hence, a closed curve in $M^* - F$ which is homologous to zero in $M^* - F$, is necessarily homotopic to zero in $M^* - F$.

4. Φ is onto

For any compact oriented triangulated manifold X (with or without boundary), we denote by the same letter X , the fundamental cycle of X (relative or absolute). Suppose Σ^2 be an oriented triangulated 2-sphere.

We prove now the homomorphism defined in section 2 is onto. Since M^* is simply connected (M. A. Armstrong [2]), $H_2(M^*, Z)$ is isomorphic to $\pi_2(M^*)$, and hence, any element of $H_2(M^*, Z)$ is represented by $f_* \Sigma^2$ for some simplicial map $f^* : \Sigma^2 \rightarrow M^*$, where f_* is the chain map induced by f^* . Let Δ^2 be the oriented surface obtained by 'cutting' Σ^2 along $\cup K_i$ as in section 3 and $\varphi : \Delta^2 \rightarrow \Sigma^2$ the 'sewing' map. By Lemma 1. there is a simplicial map $f : \Delta^2 \rightarrow M$ such that $\pi f = f^* \varphi$. Let K_i^+, K_i^- be the arcs obtained from K_i . Orient them so that $\partial \Delta^2 = \sum_i (K_i^+ + K_i^-)$. Since $f^* K_i$ meet F at their end-points,

$$f_* K_i^- = -T^{*i} f_* K_i^+ \text{ for some } T^{*i} \text{ of } Z_p.$$

Then

$$\partial f_* \Delta^2 = \sum (f_* K_i^+ + f_* K_i^-) = \sum (f_* K_i^+ - T^{*i} f_* K_i^+).$$

Hence, by the definition of Φ ,

$$\Phi (\sum \{ f_* K_i^+ \} \otimes T^{*i}) = \{ \pi_* f_* \Delta^2 \}.$$

But

$$\pi_* f_* \Delta^2 = f_*^* \varphi_* \Delta^2 = f_*^* \Sigma^2.$$

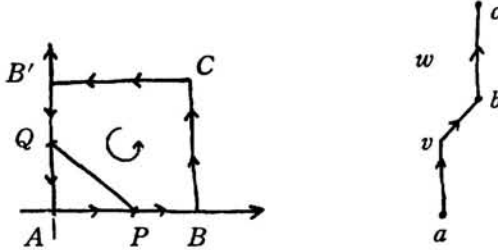
This proves that Φ is onto.

5. Φ is one to one

We prove now Φ is one to one. Assume, for simplicity, that F consists of three components.

Let $\Delta^2 = \{ (x,y) \in R^2 ; 0 \leq x, y \leq 1 \}$, R^2 the Euclidean plane, and $A=(0,0), B=(1,0), C=(1,1) B'=(0,1)$. Denote by AB, BC, \dots the line segments connecting A and B, B and C, \dots . By identifying $(t,0)$ and $(0,t), 0 \leq t \leq 1; (t,1)$ and $(1,t) 0 \leq t \leq 1$, in Δ^2 , we obtain a 2-sphere Σ^2 . Let $\varphi : \Delta^2 \rightarrow \Sigma^2$ be the identification map. By triang-

ulating Δ^2 , Σ^2 suitably, we may assume that $\varphi : \Delta^2 \rightarrow \Sigma^2$ is simplicial. Fix orientations on Δ^2 , $\partial \Delta^2$ coherently and introduce an orientation on Σ^2 by φ .



Pick one vertex from each component of F and denote them by a, b, c . Let $v : AB \rightarrow M$ and $w : BC \rightarrow M$ be simplicial maps (subdivide Δ^2 if necessary) such that $vA = a, vB = b = wB, wC = c$. Since $\text{codim } F > 2$ by proposition in section 1, we may assume that v and w meet F only at their end-points.

Since $[v_* AB], [w_* BC]$ are generators of $H_1(M, F; Z)$, any element h of $H_1(M, F; Z) \otimes Z_p$ is written as

$$h = [v_* AB] \otimes T^k + [w_* BC] \otimes T^l, \quad T^k, T^l \in Z_p.$$

Define two maps $v' : AB' \rightarrow M, w' : B'C \rightarrow M$ by the equations

$$v'(0,t) = T^k v(t,0), \quad w'(t,1) = T^l w(1,t), \quad 0 \leq t \leq 1.$$

Then

$$w'_* CB' = -T^l_* w_* BC, \quad v'_* B'A = -T^k_* v_* AB.$$

Since M is simply connected, there is a simplicial map $f : \Delta^2 \rightarrow M$ which coincides with v, w, w', v' on $\partial \Delta^2$, and since $\text{codim } F > 2$, we may assume that $f(\text{int } \Delta^2) \cap F$ is empty. Since $\partial \Delta^2 = AB + BC + CB' + B'A$,

$$\partial f_* \Delta^2 = v_* AB - T^k_* v_* AB + w_* BC - T^l_* w_* BC.$$

Then, by the definition of $\Phi, \Phi h = [\pi_* f_* \Delta^2]$.

From the definition of v', w' , it follows that there is a simplicial map $f^* : \Sigma^2 \rightarrow M'$ such that $\pi_* f = f^* \varphi$.

$$\text{Then } \Phi h = [\pi_* f_* \Delta^2] = [f^*_* \varphi_* \Delta^2] = [f^*_* \Sigma^2].$$

We now show that if $\Phi h = 0, h = 0$. Let P, Q be the mid-points of AB, AB' , respectively. Then $\varphi P = \varphi Q$, and $J = \varphi PQ$ is a Jordan curve in Σ^2 .

J divides Σ^2 into two domains. Denote by H^+ the domain containing φA and by H^- the other. Since $H_2(M', Z) = \pi_2(M')$, $\Phi h = 0$ means that f^* is null homotopic.

Now we can apply Lemma 2. The curve $f|_{PQ} : PQ \rightarrow M$ which covers $f^* \varphi : PQ$

$\rightarrow M^*$, is closed, and hence, $fP = fQ$. On the other hand, $fQ = v' Q = T^h vP = T^h fP$. Thus $fP = T^h fP$. But fP is in $M - F$, this means $T^h = 1$.

Similarly $T^l = 1$. Therefore $h = 0$.

6. Proof of Theorem

From the reduced homology exact sequence of (M, F) , $\tilde{H}_0(F, Z) \cong H_1(M, F; Z)$, and since $\tilde{H}_0(F, Z_p) \cong \tilde{H}_0(F, Z) \otimes Z_p$, it is sufficient to show that $H_1(M, F; Z) \otimes Z_p \cong H_2(M^*, Z)$. The homomorphism Φ defined in section 2 is onto and one to one, hence Φ is an isomorphism. This proves Theorem.

Reference

- (1) P. A. Smith, *Transformations of finite period II*, *Annals of Math.*, 40 (1939), 690-711.
- (2) M. A. Armstrong, *On the fundamental group of an orbit space*, *Proc. Camb. Phil. Soc.*, 61 (1965), 639-646.