The Ground State Energy of the Charged-Boson and -Fermion System at the High-Density Limit

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Abstract

In the neutural system consisting of the charged-bosons and -nonrelativistic fermions. the contribution ΔE of the interaction between components to the ground state energy at the high-density limit is calculated in the density parameter expansion by the method of canonical transformation.

The bosons which are dressed in the interaction cloud specified by a canonical transformation parameter are introduced. They have acoustic phonon-like dispersion relation for our choice of the parameter.

The energy ΔE in the unit of $n_2 m_2 e_2^4/2$ is calculated exactly to the second term when the sound velocity is smaller than the Fermi velocity and is given by

$$\frac{\Delta E}{(n_2 m_2 e_2^4/2)} = -\frac{4Q}{3^{1/4}\pi^2\alpha} \frac{m_1 |e_1|^{5/3}}{m_2 |e_2|^{5/3} r_2^{1/4}} + \left(\frac{m_1 e_1 \log r_2}{2\pi m_2 e_2}\right)^2 + O(\log r_2),$$

where n_2 is the number density of the bosons with mass m_2 and charge e_2, $r_2 = (3/4\pi n_2)^{1/3} m_2 e_2^2$, $Q=2.6220\cdots\cdots$, $\alpha=(4/9\pi)^{1/3}$, and m_1 and e_1 are mass and charge of fermion, respectively.

1. Introduction

The investigations of the low energy states in many charged-particle systems (many electron system, many charged-boson system, electron-ion system, etc.) have been carried out by many workers.

Above all, the many electron system with background of positive charge which is fixed and distributed uniformly has been studied especially due to both its applicability to many electron system in metals and the interest related with many body problems dealing with Coulomb interaction. Since the potential has no intrinsic range for the case of Coulomb interaction, the state of such system is characterized by a single parameter:

$$r_1 = (3/4\pi n_1)^{1/3} m_1 e_1^2,$$

where n_1 is the number density of particles with mass m_1 and charge e_1. The electron correlations in the low density region, the metallic density region, and the high-density region have been especially of great interest to many workers. The correlation energy per particle of degenerate electron gas at the high-density limit has already been studied enough to be given by $0.0622\log r_1 - 0.096 + 0(r_1 \log r_1)$, where the energy is measured in the unit of $m_1 e_1^4/2$.[1)]

Received November 20, 1972

*The outline of this paper was read at the meeting of the Physical Society of Japan held in Tokyo on April 6, 1971.

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The many charged-boson system with background of fixed and uniformly distributed charge of opposite sign provides many interesting problems in the theory of many body problems, though it is not easy to find any simple physical system in our vicinity corresponding to such model. In other words, the above model provides two interesting problems: one is the existence of the condensate and the other is the appearance of the divergence in a conventional perturbational calculation which results from both Coulomb interaction and the Bose statistics. Since the kinetic and the first order exchange energies of the system are zero because of the existence of the condensate, the energy of the system is simply that of correlation. Foldy[2] was the first to calculate this energy basing on the Bogoliubov's approximation[3], whose validity criterion is satisfied in the high-density region. Even though there were some controversies over the form of the term following Foldy's one in the expansion of the ground state energy in terms of the density parameter[4],[5] the ground state energy per particle at the high-density limit has been sufficiently investigated by now and is given by[5],[6] $-0.803r_2^{-3/4}+0.0280+O(r_2^{3/4})$, where

$$r_2=(3/4\pi n_2)^{1/3}m_2e_2^2$$

and n_2 is the number density of the particles with mass m_2 and charge e_2. The energy is in the unit of $m_2e_2^4/2$. It should be noted that the energy of the boson gas (and hence, the pressure) is negative in the high-density region.

The role of the background charge of the system described above is simply to ensure the neutrality of the whole system. Nevertheless, the interaction between particles and the elementary excitations which will arise when the dynamics of the background is considered will also be of great interest. Considering that the energy and the pressure of the charged-boson gas are negative and that the energy per particle of the system has no minimum in the high-density region, the investigation of the background seems to be essential for the physical behaviors of this system.

The model electron-phonon system has given an excellent basis to the study of metals, but this is not unrelated with the facts that the disparity in mass between the electron and the ion is large and that number density of the particles lies in metallic region[7],[8],[9].

Now our concern will be with the neutral two-component system consisting of nonrelativistic charged-fermion and charged-boson. From the neutrality condition,

$$n_1\ |e_1|=n_2\ |e_2| \quad or \quad r_1=(m_1/m_2)(|e_1|/|e_2|)^{7/3}r_2$$

and therefore our system can also be characterized by a single parameter r_1 or r_2 for fixed m_1/m_2 and e_1/e_2. In this paper we will calculate the contribution of Coulomb interaction between boson and fermion to the ground state energy in the form of expansion in terms of the density parameter at the high-density limit.

In the next section we will discuss the canonical transformations to the Hamiltonian in the high-density region. First of all, we will take for our Hamiltonian the Bogoliubov's approximation and then canonically transform the free boson operators to the bogolon operators which are dressed in the boson-formion interaction cloud specified by a transformation parameter, ε_p. Then, after introducing another canonical transformation in the low momentum region and expanding the transformed Hamiltonian in the effective boson-fermion interactions, we choose ε_p such that the long-range coupling between boson and fermion are eliminated in the lowest order. As the result of these transformations, we obtain the shift in the zero-point motions of the bogolon fields, the interaction between fermions through the boson fields, and the higher-order interactions between particles. Finally, we will show that the above expansion of the Hamiltonian converges rapidly in the high-density region.

In section 3, the contribution ΔE of the boson-fermion interaction to the ground state energy at the high-density limit is shown to be the sum of the shift in the zero-point frequencies of the bogolon fields, of the first order perturbation energy in the interaction between fermions through the long-range part of the boson fields, and of the second order perturbation energy in the short-range interaction between boson and fermion. This energy in the unit of $(n_2 m_2 e_2^4)/2$ can be calculated exactly to the second term when the sound velocity is smaller than the Fermi velocity and is given by

$$\frac{\Delta E}{(n_2 m_2 e_2^4/2)} = -\frac{4Q}{3^{1/4}\pi^2\alpha}\frac{m_1|e_1|^{5/3}}{m_2|e_2|^{5/3}r_2^{1/4}} + \left(\frac{m_1 e_2 \log r_2}{2\pi m_2 e_2}\right)^2 + O(\log r_2).$$

Some discussions are given in section 4.

2. Canonical Transformations

We consider the neutral system consisting of charged-spinless bosons and charged-fermions with spin $\frac{1}{2}$ in a unit volume with periodic boundary condition. The creation and annihilation operators for the fermion with momentm $\bar{k}$ and spin σ (for the boson with momentum $\bar{k}$) are denoted by $\alpha^+_{\bar{k}\sigma}$ and $\alpha_{\bar{k}\sigma}$ (by $b^+_{\bar{k}}$ and $b_{\bar{k}}$), respectively. The kinetic energy of the particle with the momentum $\bar{k}$, $\epsilon_i(k) = k^2/2m_i$ with the suffix 1 for fermion and 2 for boson. The Fourier transforms of the interactions between partices in this system are given by

$$V_{ij}(p) = 4\pi e_i e_j p^{-2} \qquad p \neq O$$
$$= O \qquad p = O \quad ,$$

where the last equality is the consiquence of the neutrality of the system. It is easy to write down the Hamiltonian by the quantities defined above.

Foldy[2] found that the validity criterion for the Bogoliubov's method applied to a charged-boson gas is satisfied when the density is so large that there are many particles within a sphere of the Bohr radius $(m_2 e_2^2)^{-1}$. This criterion was discussed by Nosal and Grandy[6] from the standpoint of quantum statistics. The average potential energy of a boson in our system is inversely proportional to the average inter-particle spacing, which in turn is proportional to the one-third of the average particle number density, while the boson condensation temperature in the ideal gas is proportional to the two-thirds of the average boson number density. Therefore, if the boson number density in our neutral system is very high ($r_2 \ll 1$), the energy of the condensation is much larger than the energy associated with Coulomb interaction. In this case, it is quite natural to expect the validity criterion for the Bogoliubov's method applied to our system is satisfied. The

Bogoliubov's approximation consists of

(1) dropping those terms in the Hamiltonian which contain more than two creation or destruction oprators for the bosons not in condensate, and

(2) replacing b_0^+ and b_0 in the remaining terms by the c-number $n_0^{\frac{1}{2}}$, where n_0 is the mean occupation number of the condensate.

In the charged-boson gas with a neutralizing, uniformly distributed background, the fraction of the condensate is[2])

$$f=1-0.2114r_2^{3/4}+\cdots\cdots \quad r_2<<1. \tag{2.1}$$

For the Hamiltonian obtained with these approximations, we consider the following canonical transformation:

$$b_{\vec{p}}=C_p\beta_{\vec{p}}-S_p\beta^+_{-\vec{p}} \quad , \tag{2.2}$$

where the coefficients C_p and S_p are defined as follows:

$$C_p^2=\frac{\epsilon_2(p)+n_0V_{22}(p)/\varepsilon_p}{2\omega(p)}+\tfrac{1}{2}, \tag{2.3}$$

$$S_p^2=\frac{\epsilon_2(p)+n_0V_{22}(p)/\varepsilon_p}{2\omega(p)}-\tfrac{1}{2}, \tag{2.4}$$

$$\omega(p)=\left[\epsilon_2^2(p)+2n_0\ \epsilon_2(p)\ V_{22}(p)/\varepsilon_p\right]^{1/2}, \tag{2.5}$$

and we assume that $C_p=C_p^*$ and $S_p=S_p^*$.

The canonicality of the above transfomation is understood by showing the conservation of the commutation relations under this transformation. ε_p is here unspecified. If we choose $\varepsilon_p=1$ for all p, the transformation (2.2) is the well-known Bogoliubov transformation which diagonalizes the pair part of the Hamiltonian and introduce bogolons in the charged boson gas theory. But in the system in consideration, fermions, which are regarded as the neutralizing, uniformly distributed background in the charged boson gas theory, can move in boson fields and have Coulomb interaction with them with finite momentum transfers. Therefore, we set ε_p as follows:

ε_p is here unspecifed for $p<p_0$

and $\varepsilon_p=1$ for $p>p_0$, (2.6)

where $p_0=(4\pi m_2e_2^2n_2)^{\frac{1}{4}}$. The reason for setting the equations as above will become clear as our calculation proceeds.

Under the transformation described above, the Hamiltonian obtained with the Bogoliubov's approximation then take the following form:

$$H=E_0+\Delta E_0+H_0+H_{11}+H'_{12}+H''_{12}+H_I\ , \tag{2.7}$$

where

$$E_0=\sum_{\vec{p}}\frac{\omega_B(p)-\epsilon_2(p)-n_0V_{22}(p)}{2}\quad , \tag{2.7a}$$

$$\Delta E_0=\sum_{\vec{p}}\frac{\omega(p)-\omega_B(p)}{2}\quad , \tag{2.7b}$$

$$H_0 = \sum_{k\sigma} \epsilon_1(k)\, \alpha^+_{k\sigma}\, \alpha_{k\sigma} + \sum_p \omega(p)\, \beta^+_p \beta_p \quad , \tag{2.7c}$$

$$H_{11} = \frac{1}{2} \sum_p \sum_{\substack{k\sigma \\ k'\sigma'}} V_{11}(p)\, \alpha^+_{k+p\sigma}\, \alpha^+_{k'-p\sigma'} \alpha_{k'\sigma'} \alpha_{k\sigma} \quad , \tag{2.7d}$$

$$H'_{12} = \sum_p \sum_{k\sigma} V_{12}(p)\, [n_0\, \epsilon_2(p) / \omega(p)]^{1/2} (\beta_p + \beta^+_{-p})\, \alpha^+_{k+p\sigma}\, \alpha_{k\sigma} \quad , \tag{2.7e}$$

$$H''_{12} = \sum_p \sum_{\substack{k\sigma \\ p'}} V_{12}(p) (C_{|p'-p|} \beta^+_{p'-p} - S_{|p'-p|} \beta_{-p'+p}) (C_{p'} \beta_{p'} - S_{p'} \beta^+_{-p'}) \tag{2.7f}$$

$$\cdot\, \alpha^+_{k+p\sigma} \alpha_{k\sigma} \quad ,$$

$$H_1 = \frac{1}{2} \sum_p V_{22}(p)\, [1 - \varepsilon_p^{-1}]\, [n_0\, \epsilon_2(p) / \omega(p)]\, (\beta_p + \beta^+_{-p})(\beta_{-p} + \beta^+_p)\, , \tag{2.7g}$$

$$\omega_B = [\epsilon_2^2(p) + \omega_2^2 f]^{1/2} \quad , \tag{2.8}$$

and

$$\omega_i^2 = 4\pi n_i e_i^2 / m_i \quad . \tag{2.9}$$

E_0 and $\omega_B(p)$ given above are the ground state energy and elementary excitation of the charged boson gas with the background in the Bogoliubov's approximation, respectively, and were calculated first by Foldy[2] and the energy is given as

$$E_0 / (n_2 m_2 e_2^4 / 2) = -0.803 r_2^{-3/4} + O(r_2^0) \quad .$$

As is well-known in the charged boson gas theory[5],[6], the correction term to be added to the energy in the Bogoliubov's approximation, that is, the correction produced by considering the triad and quartet parts of the Hamiltonian and the constancy in boson number is of order of unity in the density parameter expansion of the ground state energy per particle. Since our treatment is based on the Bogoliubov's approximation, the error of order of unity as described above will still remain in our calculation. Therefore, it will be meaningless to carry out any further the calculation of the energies per particle of order of unity or less in the following.

Note that ΔE_0 and H_1 do not vanish unless $\varepsilon_p = 1$ for all p. ΔE_0, which arises from the shift of the frequencies of the bogolon field, is the first contribution of the boson-fermion interaction to the ground state energy.

The interaction (2.7e) is closely analogous to the electron-phonon interaction in metals. Nakajima[9] has given a simple field theoretical treatment of the electron-phonon interaction in the presence of the electron-electron interaction. Also, Bardeen and Pines[8] have investigated the role of the electron-electron interaction in determining the electron-phonon interaction in metals dy extending the Bohm-Pines collective description to take into account the ionic motion. In the Bardeen and Pines theory, collective coordinates are introduced to describe the long-range electron-ion correlation and it is shown by a series of canonical transformations that these give rise to plasma waves and to coupled electron-ion waves which correspond to the longitudinal sound waves. Now, if we introduce plasmon coordinates in obtaining the Hamiltonian (2.7) in the same manners as Bardeen and Pines[8) 9)], we may obtain the energy due to the shift in the zero-point motion of the plasma oscillations which is approximately

$$\sum_{p<k_s}\left[\sqrt{\omega_1^2+\omega_2^2}-\omega_1\right]\times\tfrac{1}{2}\sim O\,(r_2^0)\,n_2\,m_2\,e_2^4\ ,$$

where

$$k_s=\left(4m_1\,e_1^2\,k_f\,/\,\pi\right)^{1/2}\ , \qquad (3.10)$$

and

$$k_f=\left[3\pi^2 n_1\right]^{1/3}\ .$$

As was discussed in the previous paragraph, this energy need not be calculated.

We now consider another canonical transformation from our operators $(\alpha_{k\sigma},\ \alpha^+_{k\sigma},\ \beta_p\ ,\ \beta^+_p)$ to a new set of operators $(A_{k\sigma},\ A^+_{k\sigma},\ B_p,\ B^+_p)$. The relations between these two sets may be written as

$$\alpha_{k\sigma}=e^{-iS}A_{k\sigma}\,e^{iS} \qquad (2.11)$$

with similar equations for $\alpha^+_{k\sigma}$, β_p , and β^+_p . We may take S, the generating function of our canonical transformation, to be a function of the new operators $(A_{k\sigma}\ ,\ A^+_{k\sigma}\ ,\ B_p,\ B^+_p)$ only. The operator relationship between the old and new Hamiltonians is

$$H\,(\alpha,\ \alpha^+,\beta,\beta^+)=e^{-iS}H\,(A,A^+,\ B,B^+)\ e^{iS}\equiv H_{\rm new}\ , \qquad (2.12)$$

where $H(A,\ A^+,\ B,\ B^+)$ represents that Hamiltonian which is the same function of the new coordinates as $H\ (\alpha,\ \alpha^+,\ \beta,\ \beta^+)$ is of the old ones. When we use the same symbols as the energy operators defined by (2.7c)…(2.7g) in the following discussions, it should be understood that their symbols represent the operators obtained by only replacing the old coordinates with the new ones in their corresponding expressions.

Our canonical transformation is generated by

$$S=\sum_{p<p_0}\sum_{k\sigma}i\,\frac{V_{12}(p)}{\varepsilon_p}\left(\frac{n_0\,\epsilon_2(p)}{\omega\,(p)}\right)^{1/2}\frac{A^+_{k+p\sigma}A_{k\sigma}B_p}{\epsilon_1(k+p)-\epsilon_1(k)-\omega\,(p)}+h.c., \qquad (2.13)$$

where the principal parts are to be taken in the sums over the energy denominators. We shall then demonstrate that this leads to the desired results.

The first order commutators arising from the operator H_{11} is

$$\begin{aligned}i\,[H_{11},S]=&-\sum_{\substack{p<p_0\\ p'}}\ \sum_{\substack{k\sigma\\ k'\sigma'}}\frac{V_{12}(p)}{\varepsilon_p}\left(\frac{n_0\,\epsilon_2(p)}{\omega\,(p)}\right)^{1/2}\frac{V_{11}(p')}{\epsilon_1(k+p)-\epsilon_1(k)-\omega\,(p)}\\ &\times\tfrac{1}{2}\ \Big\{A^+_{k'+p'\sigma'}\ A_{k'\sigma'}\ (A^+_{k+p-p'\sigma}A_{k\sigma}-\ A^+_{k+p\sigma}\ A_{k+p'\sigma})\\ &+(\ A^+_{k+p-p'\sigma}A_{k\sigma}\ -A^+_{k+p\sigma}A_{k+p'\sigma})\,A^+_{k'+p'\sigma'}\ A_{k'\sigma'}\Big\}\ \ B_p\\ &+h.c.\qquad .\end{aligned} \qquad (2.14)$$

The approximation of retaining only the diagonal terms in the round brakets in (2.14) gives essentially the random phase approximation (R.P.A.). There are a number of exchange terms in (2.14) which are diagonal parts of $A^{+}_{k+p\sigma}\ A_{k\sigma}\ B_{p}$. The contribution of these terms may be neglected when $r_2 p_0/m_2 e_2^2 \ll 1$. [8] This condition is sufficiently satisfied at the high-density limit. In the R. P. A., then, we obtain

$$i\ [H_0 + H_{11}\ \ , S] = \Delta H_b - H'_{12} \quad , \tag{2.15}$$

where

$$\Delta H_b = \sum_{p(>p_0)} \sum_{k\sigma} V_{12}(p)\ [n_0\ \epsilon_2(p) /\ \omega_B(p)]^{1/2} (B_p + B^{+}_{-p}) A^{+}_{k+p\sigma} A_{k\sigma} \ ,$$

if we choose

$$\varepsilon_p = 1 + V_{11}(p)\, \Pi^{(0)}(p) \quad , \quad p<p_0 \quad , \tag{2.16}$$

where

$$\Pi^{(0)}(p) = \sum_{k\sigma} \frac{n_{k\sigma} - n_{k+p\sigma}}{\epsilon_1(k+p) - \epsilon_1(k) - \omega(p)}$$

and $n_{k\sigma}$ is the occupation number of the state specified by k and σ . By calculating the first order commutator arising from the operator H'_{12} , the following relation is also obtained:

$$\tfrac{1}{2} i\ [H'_{12}\ \ , S] + H_1 = \Delta H_a + \Delta U \quad , \tag{2.17}$$

where

$$\Delta H_a = \tfrac{1}{2} \sum_{p(<p_0)} \sum_{\substack{k\sigma \\ k'\sigma'}} V_{11}(p)\, \lambda(p, k)\, A^{+}_{k+p\sigma} A_{k\sigma}\ A^{+}_{k'-p\sigma'} A_{k'\sigma'} \ , \tag{2.18}$$

$$\lambda(p,k) = \frac{1}{\varepsilon_p} \frac{\omega_2^2}{[\epsilon_1(k+p) - \epsilon_1(k)]^2 - \omega^2(p)} \quad ,$$

$$\Delta U = \sum_{p(<p_0)} \sum_{\substack{p'(<p_0) \\ (\neq p)}} \sum_{k\sigma} \frac{V_{12}(p)\cdot V_{12}(p')}{\varepsilon_p} \left[\frac{n_0^2\, \epsilon_2(p)\, \epsilon_2(p')}{\omega(p)\, \omega(p')} \right]^{1/2}$$

$$\times \frac{[A^{+}_{k+p+p'\sigma} A_{k\sigma} - A^{+}_{k+p\sigma}\ A_{k-p'\sigma}]\ \ [B_{p'} B_{p} + B^{+}_{-p'} B_{p}]}{\epsilon_1(k+p) - \epsilon_1(k) - \omega(p)}$$

$$+ h.c. \quad . \tag{2.19}$$

Expanding formally (2.12) in the power series of S, putting the definition of $H(A,A^{+},B,B^{+})$ obtained from (2.7), collecting terms appropriately, and making use of the relation (2.15) and (2.17), we obtain

$$H_{new} = E_0 + \Delta E_0 + H_0 + H_{11} + \Delta H_a + \Delta H_b \tag{2.20}$$

$$+ \Delta U + h + e^{-iS} H''_{12}\, e^{iS} \quad ,$$

where

$$h = \sum_{n=1}^{\infty} \frac{i^n}{n!} \frac{1}{n+2} [2\,\Delta H_a + 2\,\Delta U + n\, H_1\ ,\ S]_n \quad . \tag{2.21}$$

In (2.21), $[A,B]_n$ denotes the n-th order commutation of A with B.

In the remaining of this section we will consider the problem of the convergency of the expansion of the canonical transformation generated by (2.13) in a power series of S, that is, $[V_{12}(p)/\varepsilon_p]\ [n_0\epsilon_2(p)/\omega(p)]^{\frac{1}{2}}$. The transformation is such as to introduce new individual fermion wave functions which depend on the boson operators. One may expect then that the expansion will converge rapidly if

$$1 \gg \sum_{p<p_0} \left(\frac{V_{12}(p)}{\varepsilon_p}\right)^2 \frac{n_0\ \epsilon_2(p)}{\omega(p)} \frac{1}{[\epsilon_1(\bar{k}+\bar{p})-\epsilon_1(k)+\omega(p)]^2}\ . \tag{2.22}$$

The numerators are small and difficulty arises only from the terms for which the energy denominators are correspondingly small. Let us introduce here the idea of Fröhlich[7)8)] who suggested omitting from the transformation operator (2.13) those terms which have small energy denominators, i.e., those for which

$$|\ \epsilon_1(\bar{k}+\bar{p}) - \epsilon_1(k) \pm \omega(p)\ | < \Gamma \quad .$$

When the same is done in our transformation, we find that we can choose Γ large enough so that (2.22) is satisfied and small enough to have a negligible effect on the self-consistent field. The equation (2.22) is roughly equivalent to

$$\frac{\omega_2}{(2\pi k_f/m_1e_1^2)} \ll \Gamma \quad .$$

The relative error involved in taking the principal part summation is of order of

$$\frac{\Gamma}{\epsilon_f} \quad , \qquad \text{(see Bardeen and Pines}^{8)}.)$$

where ϵ_f is the Fermi energy. Now, as we will see in next section, the leading term of the contribution from the interaction between the components to the energy is of order of r_2^{-1} . Therfore, the error in this energy is of order of $r_2^{-1}\ \Gamma/\epsilon_f$. If we choose $\Gamma \sim \omega_2$, then the error in this energy per particle is of order of $r_2^{\frac{1}{2}}$ and the condition of rapid convergence is $(m_1/m_2)r_2 \ll 1$. Hence, we can treat these two conditions exactly at the high-density limit ($r_2 \longrightarrow$ o).

3. The energy at the high-density limit

The Hamiltonian (2.20) can be treated by perturbation theory in which H_0 is the unperturbed Hamiltonian. The contribution of the interactions ΔU, h, and $e^{-iS} H''_{12} e^{iS}$ to the energy per particle vanish at the high-density limit (see Appendix C), so that the high-density limit Hamiltonian is

$$H_{new} = E_0 + \Delta E_0 + H_0 + H_{11} + \Delta H_a + \Delta H_b \quad . \tag{3.1}$$

ΔH_a is the interaction between fermions through the long-range part of boson fields and ΔH_b is the short-range interaction between boson and fermion.

The first-order perturbation energy is

$$E^{1)} = E_{ex} + \Delta E_a \qquad ,$$

where

$$E_{ex} = -0.916 r_1^{-1} (n_1 m_1 e_1^4/2) \qquad ,$$

$$\Delta E_a = \frac{1}{2} \sum_{p<p_0} \sum_{k\sigma} V_{11}(p)\ \lambda(\vec{p},\vec{k})\ n_{k\sigma}(1-n_{k+p\sigma}) \quad . \tag{3.2}$$

The second-order perturbation energy is

$$E^{(2)} = \Delta E_b + \text{(Second Order in } H_{11} + \Delta H_a)$$

where

$$\Delta E_b = \frac{1}{2} \sum_{p<p_0} \sum_{k\sigma} V_{11}(p) \frac{\omega_2^2}{\omega_B(p)} \frac{n_{k\sigma}(1-n_{k+p\sigma})}{\epsilon_1(k) - \epsilon_1(\vec{k}+\vec{p}) - \omega_B(p)} \quad . \tag{3.3}$$

In order to estimate the second and higher order contributions of the interaction $H_{11} + \Delta H_a$, we approximate $\lambda(\vec{p},\vec{k})\,\theta(p_0 - p)$ with the averaged constant λ which is defined as

$$\lambda = \frac{\sum_{p<p_0} \sum_{k\sigma} \lambda(\vec{p},\vec{k})\ n_{k\sigma}(1-n_{k+p\sigma})}{\sum_{p<p_0} \sum_{k\sigma} n_{k\sigma}(1-n_{k+p\sigma})} \sim O(r_1^{1/2} \log r_1) \quad . \tag{3.4}$$

By following the Gell-Mann and Bruckner's calculation[1] of the correlation energy of an electron gas at the high-density limit, we can estimate the contribution of the second and higher order in the interaction given by $H_{11} + \Delta H_a$, which is

$$\frac{1}{2} \sum_{p>k_s} \sum_{k\sigma} \sum_{k'\sigma'} \frac{n_{k\sigma}(1-n_{k+p\sigma})\, n_{k'\sigma'}(1-n_{k'-p\sigma'})\, V_{11}^2(p)\,(1+\lambda)^2}{\epsilon_1(\vec{k}+\vec{p}) + \epsilon_1(\vec{k}'-\vec{p}) - \epsilon_1(k) - \epsilon_1(k')}$$

$$= \big(0.0622 \log r_1 + O(r_1^0)\big) \cdot (1+\lambda)^2\ n_1 m_1 e_1^4/2 \ . \tag{3.5}$$

From (3.4) and (3.5), it is obvious that the contribution of ΔH_a to this energy per particle vanishes at the high-density limit.

Since ΔH_b represents the short-range interaction, the effect of screening by fermions may be neglected. The fourth-order perturbation energy of ΔH_b, $\Delta E_b^{(4)}$, is of order of $r_2^{\frac{1}{2}} n_2 m_2 e_2^4$ · Therefore, in our calculation, the contributions from ΔH_b to the ground state energy per particle is only ΔE_b given by (3.3).

From the discussions thus far, the contribution of the boson-fermion interaction to the ground state energy at the high-density limit is obtained as

$$\Delta E = \Delta E_0 + \Delta E_a + \Delta E_b \qquad .$$

The dispersion relation for the introduced bosons can be expanded as follows:

$$\omega(p) = \omega_B(p)\Big(1 + \frac{\omega_2^2}{2\omega_B^2} \frac{1-\varepsilon_p}{\varepsilon_p} + \cdots\cdots\Big) \quad .$$

Using this expansion in the expression (2.7) and refering to the Appendix A, we obtain

$$\Delta E_0 = -\frac{1}{8} \sum_{p<p_0} [\omega_2^2 k_s^2 / \omega_B p^2 \varepsilon_p] \; [g_0(\vec{p}, \omega) + g_0(\vec{p}, -\omega)]$$
$$+ O(r_2^0) n_2 m_2 e_2^4 \quad . \tag{3.6}$$

In the order estimation of the second term in (3.6), we used the (A.8) as ε_p. By the definition of g_0, we also get

$$\Delta E_a = -\frac{1}{8} \sum_{p<p_0} [\omega_2^2 k_s^2 / \omega p^2 \varepsilon_p] \; [g_0(\vec{p}, \omega) - g_0(\vec{p}, -\omega)] , \tag{3.7}$$

$$\Delta E_b = -\frac{1}{4} \sum_{p>p_0} [\omega_2^2 k_s^2 / \omega_B p^2] \; g_0(\vec{p}, \omega_B) \quad . \tag{3.8}$$

Using the expression (A.2), we can calculate (3.6), (3.7), and (3.8), and then obtain the contribution from the boson-fermion interaction to the ground state energy which is calculated exactly to the second term in the density parameter expansion. The result is (see Appendix B),

$$\frac{\Delta E_0 + \Delta E_a + \Delta E_b}{(n_2 m_2 e_2^4 / 2)} = -\frac{4Q\, m_1 e_1^{3/2}}{3^{1/2} \pi^2 \alpha\, m_2 e_2^{3/2} r_2^{1/2}} + \left(\frac{m_1 e_1 \log r_2}{2\pi m_2 e_2}\right)^2 + O(\log r_2), \tag{3.9}$$

where

$$Q = \int_0^\infty dx \frac{1}{[1 + \frac{1}{4} x^4]^{1/2}} = 2.6220\cdots\cdots$$

$$\alpha = (4/9\pi)^{1/3} \quad .$$

4. Discussion and conclusions

Taking the Bogoliubov's approximation for the given Hamiltonian and performing a series of canonical transformations, we obtain the effective Hamiltonian at the high-density limit given by (3.1). In the Hamiltonian obtained this way, there is no long-range interaction between boson and fermion.

It is shown exactly that the dispersion relation of the introduced boson is acoustic phonon-like in the low momentum region for $s/v_f \longrightarrow O$, where s and v_f are the sound velocity and Fermi velocity, respectively. It is shown also for the momentum much smaller than the Thomas-Fermi screening constant that the solution of this type satisfies the dispersion relation approximately for $s/v_f \lesssim 1$.

The contribution of the boson-fermion interaction to the ground state energy is calculated exactly to the second term in the expansion in the density parameter r_2 and is given by (3.9).

In the discussions above, the order estimation of the terms which contribute to the ground state energy were carried out by making use of (A.8) as the transformation parameter, ε_p. The expression (A.8) for ε_p is exact for $s/v_f \to O$ and approximate for $s/v_f \lesssim 1$. That ε_p is given by (A.8) is equivalent to saying that the dispersion relation of the introduced boson is acoustic phonon-like.

The dispersion relation (2.5) describes such boson mode that has infinite life-time for any value of s/v_f since the solution of (2.5) corresponding to the boson mode is real due to our choice of the expressions (2.6) and (2.16) as ε_p. The life-time of an actual mode, however, will be finite in general for finite value of s/v_f. For our calculation, however, the boson mode described above is introduced. Therefore, it is not surprising if there exists such an unphysical case that the solution of acoustic phonon-like disappeares for $s/v_f>1$ in our dispersion relation as shown in Appendix B. It is essential in our treatment whether or not we can calculate the exact result, but there is no reason in our calculation why we can not conclude that the result given by (3.9) is exact to the second term of the expansion in the density parameter r_s for $s/v_f \lesssim 1$.

Appendix A. The dispersion relation of the introduced bosons

From the equation (2.6) and (2.16), the parameter ε_p of our canoical transforma tion can be expressed as follows:

$$\varepsilon_p = 1+\tfrac{1}{2}\,[k_s\,/\,p]^2\,[\,g_0(\vec{p},\omega(p))+g_0(\vec{p},-\omega(p))]\;\theta(p_0-p)\,, \qquad \text{(A.1)}$$

where,

$$g_0(\vec{p},\omega(p))=\frac{4\pi e_1^2}{k_s^2}\sum_{k\sigma}\frac{n_{k\sigma}(1-n_{k+p\sigma})}{\epsilon_1(\vec{k}+\vec{p})-\epsilon_1(k)+\omega(p)}$$

$$=\frac{\theta(2-x)}{2x}\left\{x\,(1+u_p)+(1-S_+^2)\log\left|\frac{1+S_+}{u_p}\right|-(1-S_-^2)\log\left|\frac{1-S_-}{u_p}\right|\right\}$$

$$+\frac{\theta(x-2)}{2x}\left\{x+2u_p+(1-S_+^2)\log\left|\frac{1+S_+}{1-S_+}\right|\right\}\,, \qquad \text{(A.2)}$$

$$x=p/k_f\,,$$

$$u_p=\omega(p)/2x\epsilon_f\,, \qquad \text{(A.3)}$$

and

$$S_\pm = x/2\pm u_p\,.$$

Since $x<p_0/k_f\sim r_s^{\frac{1}{2}}$ for $p<p_0$, $x\ll 1$ for $p<p_0$ in the high-density region. When (A.2) is expanded for fixed u_p in a power series of x, we get

$$g_0(\vec{p},\omega(p))+g_0(\vec{p},-\omega(p))=2\,[a_0(u_p)+a_2(u_p)x^2+\cdots\cdots]\,, \qquad \text{(A.4)}$$

where

$$a_0(u_p)=1+\tfrac{1}{2}\,u_p\,\log\left|\frac{1-u_p}{1+u_p}\right|\,, \qquad \text{(A.5)}$$

$$a_2(u_p)=-\frac{1}{2}\,\frac{1}{1-u_p^2}\,.$$

Since the terms higher than the second in the expansion of (A.4) can be neglected in the case of high-density limt, (A.1) and (2.5) become as follows:

$$\varepsilon_p = 1 + [k_s/p]^2 a_0(u_p)\,\theta(p_0 - p) \quad , \tag{A.6}$$

$$\omega(p) = sp\,[a_0(u_p) + (p/k_s)^2]^{-\frac{1}{2}} \qquad p<p_0$$

$$= \omega_B(p) \qquad p>p_0 \tag{A.7}$$

where s is the velocity defined by $s = \omega_2/k_s$.

Since $u_p < [s/v_t\, a_o^{\frac{1}{2}}]$ owing to (A.7) where $v_t = k_t/m_1$, $u_p \longrightarrow 0$

(hence $a_0=1$) especially in the case of

$$(s/v_t) = [m_1 e_2 / 3m_2 e_1]^{\frac{1}{2}} \longrightarrow 0 \ .$$

Then (A.6) and (A.7) become as follows:

$$\varepsilon_p = 1 + (k_s/p)^2\ \theta(p_0 - p) \quad , \tag{A.8}$$

$$\omega(p) = sp\,[1 + (p/k_s)^2]^{-\frac{1}{2}} \qquad p<p_o$$

$$= \omega_B(p) \qquad p>p_o \quad . \tag{A.9}$$

Therefore, in this case the dispersion relation of the introduced bosons becomes that of acoustic phonon in the low momentum region which is quite different from that of bogolons in the charged boson gas. The sketch of the dispersion relation is shown in Fig. A.I.

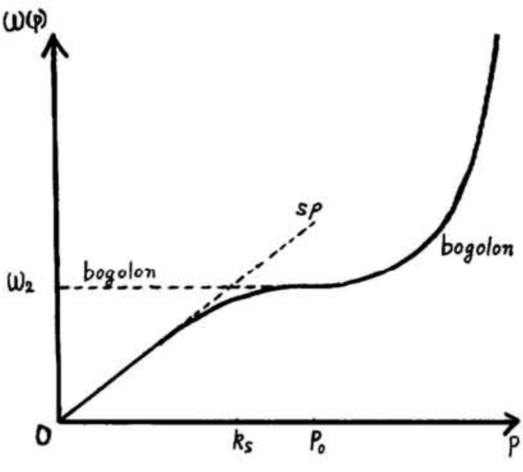

Fig.AI. The sketch of the dispersion relation (solid line) of the introduced boson.

When $p \ll k_s$, we can investigate the dispersion relation given by (A.7) for finite value of s/v_f . Using the definition of u_p given by (A.3), (A.7) can be rewritten as follows:

$$a_o(u_p) = (s/v_f)^2 u_p^{-2} \quad . \tag{A.10}$$

The solutions of this equation are shown in Fig. A. II as the intersectioins of the graphs $a_o(u_p)$ and $(s/v_f)^2 u_p^{-2}$.

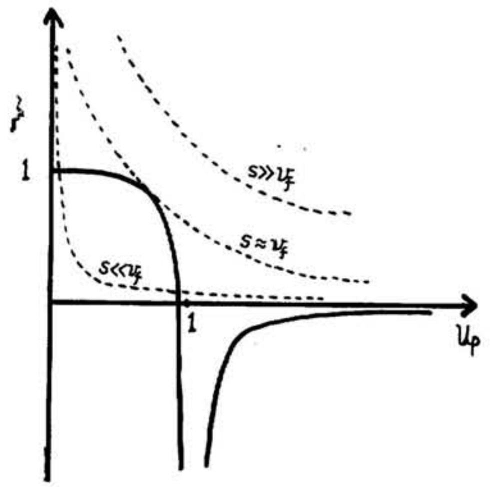

Fig. AII. The acoustic phonon like solutions which are shown as the intersections of the graphs $a_o(u_p)$ shown by solid line and $(s/v_f)^2 u_p^{-2}$ shown by dashed line.

Since we may choose any solutions in specifying the canonical transformation parameter ε_p, we will take the value corresponding to the solution A as the parameter ε_p. Then since $a_o(u_p)\approx 1$ in this choice for $(s/v_f) \lessapprox 1$, the equation (A.8) and (A.9) hold approximately also for $(s/v_f) \lessapprox 1$.

Appendix B. The calculation of (3.6), (3.7), and (3.8)

From (A.2) we obtain the expansion in terms of S_+ and S_-, that is,

for $p < p_o$,

$$g_o(\vec{p}, \omega(p)) + g_o(\vec{p}, -\omega(p)) = 2\left\{1 - \frac{1}{x}\sum_{n=o}^{\infty} \frac{S_+^{2n+3} + S_-^{2n+3}}{(2n+1)(2n+3)}\right\}, \tag{B.1}$$

$$g_o(\vec{p}, \omega(p)) - g_o(\vec{p}, -\omega(p)) = 2\left\{u_p \log u_p - \frac{1}{4x}\sum_{n=1}^{\infty} \frac{S_-^{2n+2} - S_+^{2n+2}}{n(n+1)}\right\}, \tag{B.2}$$

for $p > p_o$,

$$g_o(\vec{p}, \omega(p)) = \theta(2-x)\left\{1 + u_p \log u_p - \frac{1}{x}\sum_{n=0}^{\infty} \frac{S_+^{2n+3} + S_-^{2n+3}}{(2n+1)(2n+3)} - \frac{1}{4x}\sum_{n=1}^{\infty} \frac{S_-^{2n+2} - S_+^{2n+2}}{n(n+1)}\right\}$$

$$+ \theta \ (x-2)\left\{ \frac{1}{2}+\frac{u_p}{x}+\frac{1-S_+^2}{2x} \log \left| \frac{1+S_+}{1-S_+} \right| \right\} . \tag{B.3}$$

The integration after substitution of (B.1) into (3.6), gives

$$\frac{\Delta E_0}{(n_2 m_2 e_2^4/2)} = -\frac{4m_1 e_1^{5/3}}{3^{1/4} \pi^2 \alpha m_2 e_2^{5/3} r_2^{1/4}} \int_0^1 dx \frac{1}{\sqrt{1+\frac{1}{4}x^4}} + O\,(r_2^0)\,, \tag{B.4}$$

where we made use of (A.8) and (A.9) as ε_p and $\omega\,(p)$, respectively, in the calculation of the order estimation in (B.4). The contribution from the second term of (B.1) to (B.4) is of order of unity.

When (B.2) is put into (3.7), the contribution from the first term of (B.2) is

$$-\frac{n_2 m_2 e_2^4}{2} \left(\frac{2m_1 e_1}{\pi m_2 e_2} \right)^2 \int_0^{p_0} dp \frac{\log u_p}{p \varepsilon_p} \quad ,$$

where the leading term can be calculated exactly and is given as follows:

$$\left(\frac{1}{2} \left(\frac{m_1 e_1}{2\pi m_2 e_2} \right)^2 \log^2 r_2 + O\,(\log r_2) \right) \frac{n_2 m_2 e_2^4}{2} \quad .$$

The contribution from the second term of (B. 2) is $O\,(r_2^0)\ (n_2 m_2 e_2^4)\ /2$. Hence,

$$\frac{\Delta E_a}{(n_2 m_2 e_2^4/2)} = \frac{1}{2} \left(\frac{m_1 e_1}{2\pi m_2 e_2} \log r_2 \right)^2 + O\,(\log r_2)\,. \tag{B.5}$$

The integration after substitution of (B.3) into (3.8) yields

$$\frac{\Delta E_b}{(n_2 m_2 e_2^4/2)} = -\frac{4m_1 e_1^{5/3}}{3^{1/4} \pi^2 \alpha m_2 e_2^{5/3} r_2^{1/4}} \int_1^\infty dx \frac{1}{\sqrt{1+\frac{1}{4}x^4}}$$

$$+\frac{1}{2} \left(\frac{m_1 e_1 \log r_2}{2\pi m_2 e_2} \right)^2 + O\,(\log r_2) \quad . \tag{B.6}$$

In these calculations, the first and second terms in the braket of the first term of (B.3) contribute to the first and second terms of (B.6), respectively, and the contribution from the second term of (B.3) to (3.6) is of order of unity.

Finally, (3.9) is obtained by adding (B.4), (B.5), and (B.6) together.

Appendix C. The contribution from $e^{-iS} H''_{12} e^{iS}$, ΔU, and h.

First of all, let us estimate the contribution of $e^{-iS} H''_{12} e^{iS}$ to the ground state energy on the basis of the perturbation theory, where H_0 is regarded as the unperturbed Hamiltonian and the remaining operators in the Hamiltonian (2.20) are the perturbations. Since the energy contribution of each term in the expansion of canonically transformed Hamiltonian generated by (2.13) in the power series of S may converge rapidly as is shown in the last part of section 2, we will estimate only the contribution of the first term H''_{12} to the ground state energy. The lowest-order perturbation energy of H''_{12} is [see (2.7f)] ,

$$\Delta E''=\sum_{\substack{p(>k_0)\\ p'}} V_{12}^2(p)\left(C^2_{|p'-p|}\,S^2_{p'}+C_{|p'-p|}\,S_{|p'-p|}\,C_{p'}\,S_{p'}\right)M(p,p'), \quad (C.1)$$

where C_p and S_p are defined by (2.3) and (2.4), respectively, and

$$M(p,p')=\sum_{k\sigma}\frac{n_{k\sigma}\,(1-n_{k+p\sigma})}{\epsilon_1(k)-\epsilon_1(k+p)-\omega(p')-\omega(|p'-p|)} \quad . \quad (C.2)$$

We can estimate the order of magnitude of $\Delta E''$ by noticing that the dominant contribution in the sum over p in (C. 1) is in the low momentum region. The dependence of $\Delta E''$ on the dentsiy parameter is as follows:

$$\frac{\Delta E''}{(n_2 m_2 e_2^4/2)}\sim O\,(r_2^{1/4}) \quad . \quad (C.3)$$

Next, let us consider the contribution of ΔU given by (2.19). The terms with $p'=-p$, which are excluded in the expression (2.19), are used to eliminate the interaction H_I as is seen from the equation (2.17). Each term in the interaction ΔU is quadratic in the boson variables multiplied by the nondiagonal product of fermion operators such as $A^+_{k+p+p'\sigma}A_{k\sigma}$. This is the same character with that of H''_{12}. Moreover, ΔU consists of terms higher in order than H''_{12} in the power series expansion of S. Therefore, the contribution of ΔU to the ground state energy per particle may vanish.

Finally, it is obvious that the main contribution of h given by (2.21) to the ground state energy arises from $n=1$:

$$h\simeq -\frac{i}{3}\left[2\,\Delta H_a + H_1,\ S\right] \quad . \quad (C.4)$$

Calculating the commutator, we obtain

$$h\simeq \frac{1}{3}\sum_{p(<p_0)}\sum_{k\sigma}\lambda(p,k)\,[1-\varepsilon_p^{-1}]\,V_{12}(p)\,[n_0\,\epsilon_2(p)/\omega(p)]^{1/2}\,(B_p+B^+_{-p})$$

$$A^+_{k+p\sigma}A_{k\sigma} \ .$$

This interaction is of exactly the same character as H'_{12} except that this is smaller than H'_{12} by a factor of $\langle\lambda(p,k)\,[1-\varepsilon_p^{-1}]\rangle_{\substack{p<p_0\\ k\sim k_f}}\sim r_2^{3/4}$.

Therefore, h can be neglected at the high-density limit.

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