The Field Theoretical Treatment of the Charged-Boson and –Fermion System : Ground State Energy and Elementary Excitations

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Abstract

The neutral system consisting of charged-bosons and -fermions has been investigated in terms of density parameter expansion at the high-density limit.

The ground state energy of this system is calculated and accords with the result obtained in our previous work.

The elementary excitations in the system determined from the poles of the boson Green's function are the two collective excitations in the low momentum region : acoustic- and optical-phonon like. The life-time of the acoustic phonon approches infinite as $s/v_f \to O$ where s and v_f are the sound velocity and Fermi velocity, respectively, while that of the optical phonon is infinite.

I. Introduction

The charged-boson and -fermion system offers a great challenge to the theoretical physicists in the many body problems because of the existence of the boson condensate, the difficuties associated with Coulomb interaction, and the effect of the interaction between components.

In the previous work[1] (hereafter refered as I), we calculated the ground state energy of such system in terms of density parameter expansion at the high-density limit. The canonical transformations introduced in this calculation were such that they would transform the bogolon field, which is in the medium of charged boson gas in the case when there is no interaction between components, into the boson field with the acoustic phonon like dispersion relation.

The self-consistency in the calculation guarantees that the result obtained by the treatment adopted in I is exact. However, it will be significant and interesting enough to deepen our understanding of this system basing on an alternative method. In this paper, we will investigate the ground state energy and collective excitations in the system by the field theoretical technique.

In § 2, we will treat the boson condensate using the method of Hugenholtz and Pines,[2] define the density correlation function and the Green's functions, give the expressions of the correlation energy and the boson condensate fraction using the functions defined above, and give the rules of the perturbational calculation of the functions. In § 3, we will obtain the functions defined in the previous section in the approximation allowed in the high-density region, investigate the elementary excitations given by the poles of the

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boson Green's function and calculate the correlation energy. In § 4, some conclusions will be given and the problems for the future work will be pointed out.

2. Formalism

2.1 We consider the neutral system consisting of charged-spinless bosons and -nonrelativistic fermions with spin 1/2 in a unit volume with periodic boundary condition. The creation and destruction operators of fermion with momentum $\bar{k}$ and spin σ (boson with momentum $\bar{k}$) are denoted by $\alpha^{+}_{k\sigma}$ and $\alpha_{k\sigma}$ (by b^{+}_{k} and b_{k}), respectively. The kinetic energy of a particle with momentum $\bar{k}$, $\epsilon_i(k)=k^2/2m_i$. Fourier transforms of Coulomb interaction between particles, $v(p)e_i e_j=4\pi p^{-2}e_i e_j$ when $p\neq 0$ and $v(p)=0$ when $p=0$. $i=1$ for fermion, $i=2$ for boson, m_i and e_i are mass and charge of a particle specified by i. It is easy to write down the Hamiltonian, H, in terms of the quantities defined above.

At the absolute zero of temperature, the ground state of the noninteracting system is characterized by the facts that all fermions fill the Fermi sphere perfectly and that all bosons occupy the zero momentum state forming the condensate . The existence of this condensate prevents us from applying the quantum field theoretical technique directly. A rather simple way of getting around this difficulty is set forth by Hugenholtz and Pines[2] in the case of the boson system. Their method amounts to a generalization of the original method of Bogoliubov[3] concerning the role of b^{+}_{0} and b_0 . According to their theory, providing that a finite fraction of the bosons remain in the condensate after the interaction between particles is turned on, b^{+}_{0} and b_0 are simply c-number. This c-number is wrtten, $n_0^{\frac{1}{2}}$ and it must be imposed that n_o represents the number of bosons in the condensate. By means of this procedure the condensate has been removed from the problem and with it all the difficulties involved in the application of the quantum-field theoretical techniques. The total number of bosons, however, is not conserved.

We introduce the hermitian operator defined by

$$H' = H - \mu_1 \hat{n}_1 - \mu_2 \hat{n}_2 , \tag{2.1}$$

where H is the Hamiltonian of our system defined above, $\hat{n}_1 = \sum_{k\sigma} \alpha^{+}_{k\sigma} \alpha_{k\sigma}$

$\hat{n}_2=\sum_{k} b^{+}_{k} b_{k}$, and μ_1 and μ_2 are the chemical potencials of fermion and boson, respectively. b^{+}_{o} and b_o are still the operators in (2.1). As long as b^{+}_{o} and b_o represent operators, both H and H' provide acceptable descriptions of the interacting system. When we use the Bogoliubov prescription, however, the thermodynamic potencial offers a definite advantage, for it allows a consistent treatment of the non-conservation of particles.

By making the replacement of b^{+}_{o} and b_o by $n_o^{\frac{1}{2}}$, (2.1) becomes

$$H'(n_o) = H'_o(n_o) + H_1(n_o), \tag{2.2}$$

where

$$H'_0 = \sum_{k\sigma} [\epsilon_1(k) - \mu_1] \alpha^{+}_{k\sigma} \alpha_{k\sigma} + \sum_{k} [\epsilon_2(k) - \mu_2] b^{+}_{k} b_{k} - n_0 \mu_2 , \tag{2.3}$$

$$H_I = H_{11} + H'_{12} + H''_{12} + H_P + H_T + H_Q, \qquad (2.4)$$

$$H_{11} = \frac{1}{2}\sum_{p} v(p) e_1^2 \sum_{k\sigma} \sum_{k'\sigma'} \alpha^+_{k+p\sigma}\, \alpha^+_{k'-p\sigma'}\, \alpha_{k\sigma'}\, \alpha_{k\sigma},$$

$$H'_{12} = \sum_{p} \sum_{k\sigma} v(p) e_1 e_2 n_0^{\frac{1}{2}} (b_p + b^+_{-p})\, \alpha^+_{k+p\sigma}\, \alpha_{k\sigma},$$

$$H''_{12} = \sum_{p} \sum_{k\sigma} \sum_{k'}{}' v(p)\, e_1\, e_2\, \alpha^+_{k+p\sigma}\, \alpha_{k\sigma}\, b^+_{k'-p}\, b_{k'}$$

$$H_P = \frac{1}{2}\sum_{p} v(p) e_2^2\, n_0\, [2b^+_p\, b_p + b^+_p\, b^+_{-p} + b_p\, b_{-p}],$$

$$H_T = \sum_{p} \sum_{k}{}' v(p)\, e_2^2\, n_0^{\frac{1}{2}}\, [\, b^+_{k+p}\, b^+_{-p}\, b_k + b^+_k\, b_{-p}\, b_{k+p}],$$

$$H_Q = \frac{1}{2}\sum_{p} \sum_{p'}{}' \sum_{p''}{}' v(p)\, e_2^2\, b^+_{p'+p}\, b^+_{p''-p}\, b_{p''}\, b_{p'},$$

and $\hat{n}_2 = n_0 + \sum_{k}{}' b^+_k b_k \equiv n_0 + \hat{n}'_2$. In the above equations, the prime on the sum indicates that the terms with the suffix of boson operator equal to zero are excluded . Each term in (2.4) corresponds to each graph in Fig. 1. Since all remaining boson

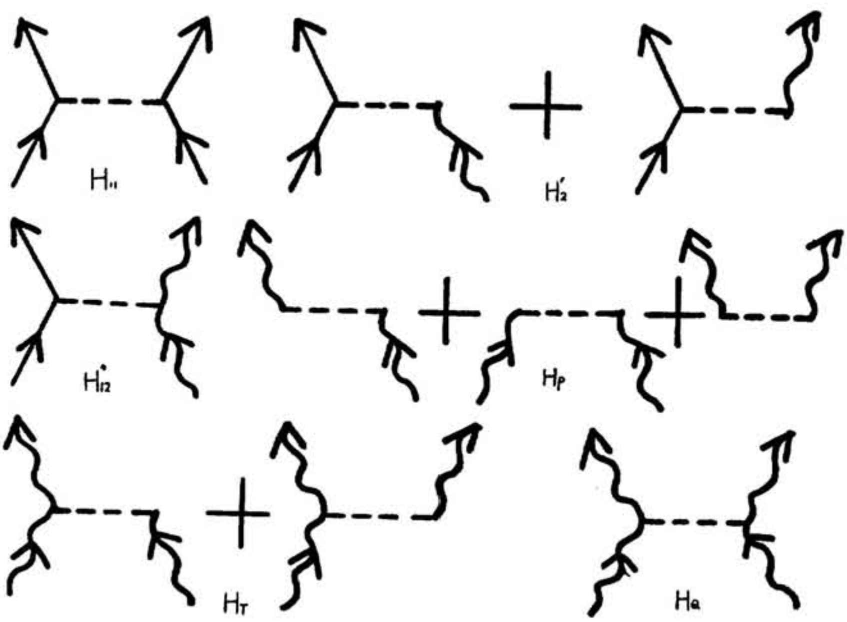


Fig 1. Various interactions in the system. Solid line, wavy line, and dashed line represent fermion, boson with nonzero momentum, and the interaction between the particles, respectively.

destruction operators annihilate the noninteracting ground state, it may be considered to be equivalent to only the Fermi sphere filled perfectly with noninteracting fermions which we denote by $|\,o\rangle$. Wick's theorem is now applicable. Though all the final expressions contain the extra parameter n_0 , since the equilibrium state of any assembly at constant (T, V, μ_1, μ_2) minimizes the thermodynamic potential, the n_0 may be determined from the condition of the thermodynamic equilibrium :

$$\partial / \partial n_0 E'(n_0, \mu_1, \mu_2) \mid_{\mu_1, \mu_2} = 0, \tag{2.5}$$

which is an implicit relation for n_0, where $E' = \langle \psi(n_0, \mu_1, \mu_2) \mid H' \mid \psi \rangle$ and ψ is the ground state of H' given by (2.2). Also, we impose the conditions:

$$\langle \psi \mid \hat{n}_1 \mid \psi \rangle = n_1, \tag{2.6}$$

$$\langle \psi \mid \hat{n}_2' \mid \psi \rangle = n_2 - n_0, \tag{2.7}$$

where n_1 and n_2 are the number of fermions and bosons in our system, respectively. Then, the energy of our system is given by

$$E = E' - \mu_1 n_1 - \mu_2 n_2 . \tag{2.8}$$

We consider the system defined by

$$H'(\lambda) = H_0' + \lambda H_1 , \tag{2.9}$$

where H_0' and H_1 are defined by (2.3) and (2.4) respectively and λ is a coupling constant. The derivative of the ground state energy of this system with respect to λ is

$$dE'(\lambda)/d\lambda = -(d\mu_1/d\lambda)\, n_1 - (d\mu_2/d\lambda)\, n_2 + \lambda^{-1} \langle \psi^\lambda \mid \lambda H_1 \mid \psi^\lambda \rangle + \langle \psi^\lambda \mid \lambda\, (dH_1/d\lambda) \mid \psi^\lambda \rangle - \mu_2\, (dn_0/d\lambda),$$

where ψ^λ is the ground state of the Hamiltonian defined by (2.9) and the relations of (2.6) and (2.7) are used. We can show that the fourth term cancels the fifth one in the above equation : We obtain $\mu_2 = \langle \psi^\lambda \mid \lambda (dH_1/dn_0) \mid \psi^\lambda \rangle$ from (2.5) in the case of the coupling constant λ, and therefore,

$$\langle \psi^\lambda \mid \lambda (dH_1/d\lambda) \mid \psi^\lambda \rangle = \langle \psi^\lambda \mid \lambda (dH_1/dn_0) \mid \psi^\lambda \rangle (dn_0/d\lambda) = \mu_2\, (dn_0/d\lambda).$$

Using the relation (2.8) and integrating with respect to the coupling constant from O to 1, we obtain the well known relation :

$$E = E(\lambda = O) + \int_0^1 d\lambda\ \lambda^{-1} \langle \psi^\lambda \mid \lambda H_1 \mid \psi^\lambda \rangle . \tag{2.10}$$

2.2 Let us define the total density correlation function, D, and the boson Green's function, G_2, as follows:

$$i\, D(\vec{p}, t-t') = \langle \psi \mid T \rho(\vec{p},t)\, \rho(-\vec{p},t') \mid \psi \rangle , \tag{2.11}$$

$$i\, G_2(\vec{p},t-t') = \langle \psi \mid T \tilde{b}_{\vec{p}}(t)\, \tilde{b}_{\vec{p}}^{+}(t') \mid \psi \rangle , \tag{2.12}$$

where $\rho(\vec{p},t)$ and $\tilde{b}_{\vec{p}}(t)$ are the Heisenberg repesntations of $\hat{\rho}(\vec{p})$ and $b_{\vec{p}}$ in the system defined by H', respectively, and

$$\hat{\rho}(\vec{p}) = \sum_{\vec{k}\sigma} e_1\, a_{\vec{k}-\vec{p}\sigma}^{+}\, a_{\vec{k}\sigma} + n_0^{\frac{1}{2}}\, e_2\, (b_{\vec{p}} + b_{-\vec{p}}^{+}) + \sum_{\vec{k}} e_2\, b_{\vec{k}-\vec{p}}^{+}\, b_{\vec{k}} .$$

From (2.4), (2.10), and (2.11), the expression of the correlation energy, E_{corr}, is given in terms of D by

$$E_{corr} = \tfrac{1}{2} \sum_{\vec{p}} v(p) \int_0^1 d\lambda \int_{-\infty}^{\infty} d\omega\ (2\pi)^{-1}\ \mathrm{Im}\ [D_{HF}^{\lambda}(\vec{p},\omega) - D^{\lambda}(\vec{p},\omega)] , \tag{2.13}$$

where $D(\vec{p},\omega)$ is the Fourier transform of $D(\vec{p},t)$, D_{HF} is the total density correlation function in the Hartree-Fock approximation, and λ on these functions shows the dependence of these on the coupling constant, explicitly. It is also shown from (2.7) and (2.12) that the fraction of bosons in the condensate, $f(\equiv n_0/n_2)$, is calculated from the following expression :

$$f=1-\lim_{\eta\to+0} n_2^{-1}\sum_{\vec{p}}\int_{-\infty}^{\infty} id\omega\ (2\pi)^{-1}G_2(\vec{p},\ \omega)e^{i\eta\omega}\ , \tag{2.14}$$

where $G_2(\vec{p},\ \omega)$ is the Fourier transform of $G_2(\vec{p},t)$. In addition to G_2 , it is advantageous to introduce two similar functions

$$i\ \hat{G}_2(\vec{p},t-t')=\langle\Psi\mid T\tilde{b}_{\vec{p}}(t)\tilde{b}_{-\vec{p}}(t')\mid\Psi\rangle\quad ,$$

$$i\tilde{G}_2\ (\vec{p},t-t')=\langle\Psi\mid T\tilde{b}_{\vec{p}}^{+}(t)\tilde{b}_{-\vec{p}}^{+}(t')\mid\Psi\rangle\ .$$

These functions obviously have no counterpart in the unperturbed system.

A perturbation expansion for the functions defined above are obtained by going over to the interaction representation[4]. The energy-momentum representations of fermion bare Green's function, boson bare Green's function, and instantoneous Coulomb interaction are obtained from their definions, respectively, as follows

$$G_1^{(0)}(\vec{k}\alpha,\omega)=[\omega-\epsilon_1(k)+\mu_1+i\delta\mathrm{sgn}(\epsilon_1(k)-\mu_1)]^{-1}\ , \tag{2.15}$$

$$G_2^{(0)}(\vec{k},\omega)=[\omega-\epsilon_2(k)+\mu_2+i\delta]^{-1} \tag{2.16}$$

and $v(p,\omega)e_i e_j=v(p)e_i e_j$. Each term in the perturbation expansion is functional of $G_1^{(0)}$, $G_2^{(0)}$, and $ve_i e_j$ and may be represented by a Feynman diagram with appropriate rules. A Feynman diagram of the density correlation function (called polarization diagram) is defined to be a diagram with two externl Coulomb lines and no other external line (see Fig.2). $i\ G_2$, $i\ \hat{G}_2$, and $i\ \tilde{G}_2$ are repretented by diagrams with one ingoing boson line and one outgoing boson line, two ingoing boson lines, and two outgoing boson lines, respectively (see Fig. 3). The analytical representation of each Feynman diagram is obtained from a set of rules given below . Rules for $i\ G_2\ (\vec{p},\ \omega)$, $i\ \hat{G}_2(\vec{p},\omega)$, $i\tilde{G}_2(\vec{p},\ \omega)$, and $i\ D(\vec{p},\omega)$:

(i) Write down all possible topologically non-equivalent connected Feynman diagrams construtecd from the ten elementary interactions given in Fig. 1.

(ii) In the diagram thus obtained, flow energy and momentum in each line where energy and momentum are conserved in each vertex. Also assign spin index to the fermion line.

(iii) Assign the following quantities to each line and vertex :

$\vec{p}\alpha,\omega$ ———— ; $i\ G_1^{(0)}\ (\vec{p}\alpha,\omega)$given by (2.15) ,

$\vec{p},\ \omega$ ～～～→ ; $i\,G_2^{(0)}\ (\vec{p},\omega)$ given by (2.16) ,

$\vec{p},\ \omega$ ....; $-i\ v\ (p,\omega)$,

vertex ....  ; e_1

vertex ....  ; e_2

vertex ········· ~ ; $n_0^{\frac{1}{2}}\ e_2$

(iv) When ⟶ and ----→ are equal time propergators, they correspond to $i\ G_1^{(0)}(\vec{p}\alpha,\omega)$ $e^{i\eta\omega}$ and $iG_2^{(0)}(\vec{p},\omega)$ $e^{i\eta\omega}$, respectively, with η denoting a positive infinitesimal.

(v) The analitical contribution of the diagram is the one obtained by multipling all quantities assigned to all the lines and vertices in the diagam following the rules (ii), (iii), and (iv), integrating over the energies and momenta which are not determined by the conservation law ($\prod_i \sum_{k_i}\int\frac{d\omega_i}{2\pi}$), summing over all spin variables, and finally multiplying the resulting expression by $(-1)^\ell$ where ℓ is the number of the fermion closed loops.

In order to calculate $D(\vec{p},\omega)$, it is convenient to introduce a proper polarization diagram defined as a polarization diagram which can not be broken up into two simpler polarization diagrams by cutting a single interaction line. The sum of the contributions from all such diagrams is denoted by $i\overline{D}(\vec{p},\omega)$. The relation between D and $\overline{D}$ is given by Fig. 2, diagramatically .

From the rules, it is easy to write down the expression of

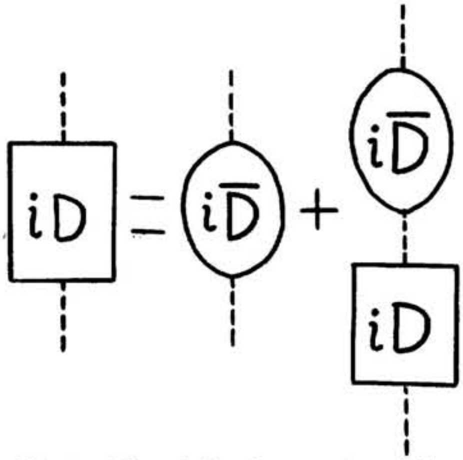


Fig. 2. The relation between D and $\overline{D}$.

$D(\vec{p},\omega)$ in terms of $\overline{D}(\vec{p},\omega)$:

$$D(\vec{p},\omega)=\overline{D}(\vec{p},\omega)/K(\vec{p},\omega), \tag{2,17}$$

where we define the dynamical dielectric constant $K(\vec{p},\omega)$ as

$$K(\vec{p},\omega)=1-v(p)\overline{D}(\vec{p},\omega). \tag{2.18}$$

As for the calculation of the boson Green's functions, it is useful to define a proper boson self-energy diagram as a diagram not consisting of two or more parts connected only by one wavy line . Such diagrams are classified into three groups : diagrams having one incoming and one outgoing wavy lines, two outgoing wavy lines , and two incoming wavy lines . The sum of the contributions from all diagrams belonging to each group is denoted by $-i\Sigma_{io}(\vec{p},\omega)$, $-i\Sigma_{oo}(\vec{p},\omega)$, or $-i\Sigma_{ii}(\vec{p},\omega)$, where Σ_{io} , Σ_{oo} , and

Σ_{11} are the proper self-energies (see Fig. 3). The general structure of the perturbation expansion of the boson Green's functions leads, if we follow the discussion first given by Beliaev[5] in the case of boson system, to the diagramatical represention of the algebraic relations among $G_2\ (\vec{p},\omega)$, $\hat{G}_2\ (\vec{p},\omega)$, and $\tilde{G}_2\ (\vec{p},\omega)$ shown in Fig. 3. When these

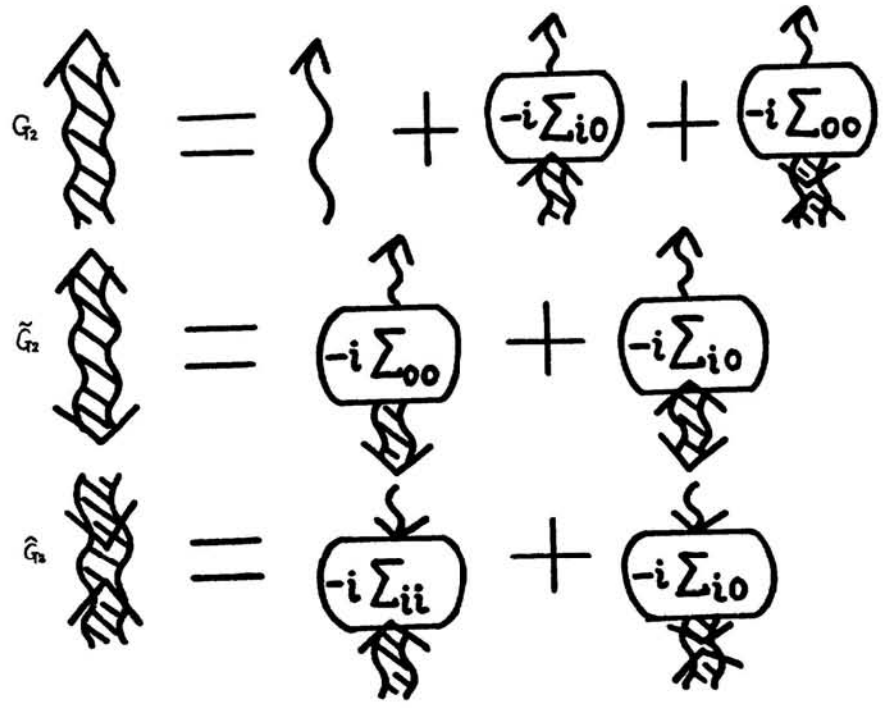


Fig. 3. The algebraic relations among $G_2\ (\vec{p},\omega)$, $\hat{G}_2\ (\vec{p},\omega)$, and $\tilde{G}_2(\vec{p},\omega)$.

equations are expressed analytically, they can be solved and one finds the following expressions:

$$G_2(\vec{p},\omega)=[\omega+\epsilon_2(p)-\mu_2+\Sigma_{10}(-\vec{p},-\omega)]\cdot\mathfrak{D}^{-1}(\vec{p},\omega), \tag{2.19}$$

$$\hat{G}_2(\vec{p},\omega)=\tilde{G}_2(\vec{p},\omega)=-\Sigma_{11}(\vec{p},\omega)\cdot\mathfrak{D}^{-1}(\vec{p},\omega),$$

where

$$\mathfrak{D}(\vec{p},\omega)=\{\omega-\tfrac{1}{2}[\Sigma_{10}(\vec{p},\omega)-\Sigma_{10}(-\vec{p},-\omega)]\}^2 - \{\epsilon_2(p)-\mu_2+\tfrac{1}{2}[\Sigma_{10}(\vec{p},\omega)-\Sigma_{10}(-\vec{p},\omega)]\}^2+\Sigma_{11}^2(\vec{p},\omega), \tag{2.20}$$

and we used the relation $\Sigma_{11}(\vec{p},\omega)=\Sigma_{00}(\vec{p},\omega)$.

The usefulness of eqs. (2.17) and (2.19) is that, given approximate expressions for the proper polarization and the proper self-energies, these equations contain the result summing the contributions from an infinite number of diagrams so that the divergency in the expression for the ground state energy which appears in a finite order perturbation theory is eliminated.

3. Calculations at the high-density limit

In this section, we will calculate the "proper" quantities defined above at the high-density limit, investigate the collective excitations and calculate the correlation energy.

The effective interaction is defined by the diagramatical equation given by Fig. 4

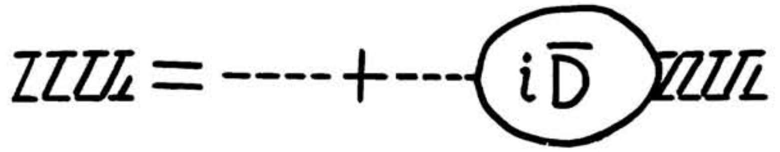
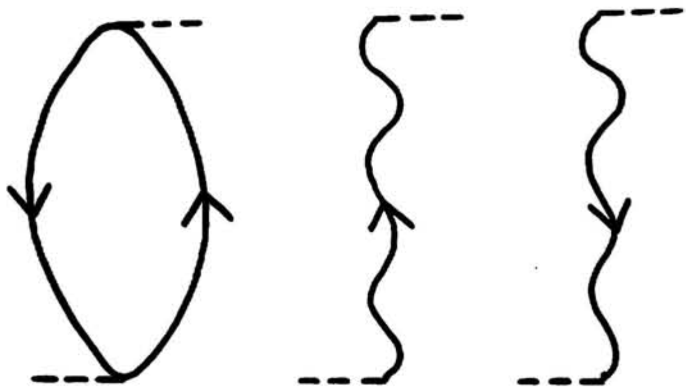


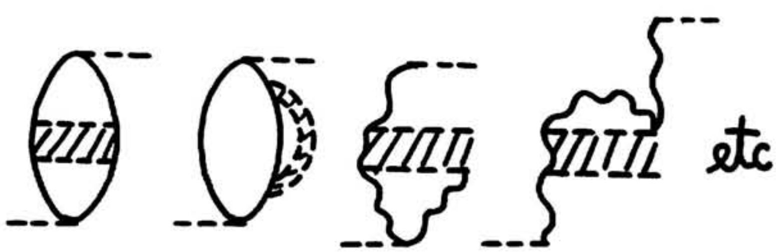
Fig. 4. The definition of the effective interaction.

and is $e_i \; e_j \; v(p,\omega) \; /K(\bar{p},\omega)$, where $K \; (\bar{p},\omega)$ is given by (2.18). The proper polarization diagrams are classified in terms of the order in this effective interaction and the zeroth- and the first-order diagrams are shown in Fig. 5 and Fig. 6, respectively.



(a) (b) (c)

Fig. 5. The zeroth-order proper polarization diagrams.



(a) (b) (c) (d)

Fig. 6. The first-order proper polarization diagrams

According to the theory of an electron gas[6], only the proper polarization diagram corresponding to Fig. 5 (a) contributes to the lowest order term in the density parameter expansion of the correlation energy of the electron gas and the diagrams correspnding to Fig. 6 (a) and 6 (b) contribute to the next order term. On the other hand, as is well known in the charged boson gas theory[7] , the diagrams which contain the interaction H_T or H_Q (see Fig. 1) such as Fig. 6 (c) and 6 (d) contribute to the higher order terms in the density parameter expansion of the correlation energy of the charged boson gas than the diagrams Fig. 5 (b) and (c) by the three-fouths or more power of the density parameter. Therefore, we consider only the zeroth order diagrams as the proper polarization. This yields the error of order of unity in the correlation energy per particle at the high-density limit.

As for the proper boson self-energies, the above discussion is equivalent to considering the contribution from only the diagrams shown in Fig. 7 to them. In Fig. 7, the

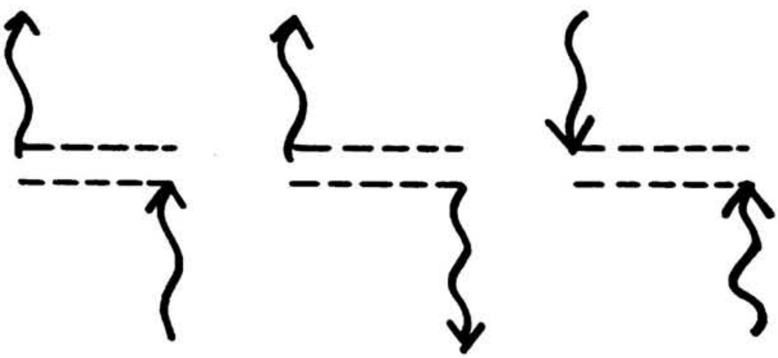


Fig 7. The diagrams contributing to the proper boson self-energies at the high-density limit.

doubly dashed line is the effective interaction in the case when the proper polarization in Fig. 4 is replaced with the one in Fig. 5(a).

As the results, we obtain as follows

$$\bar{D}(\vec{p},\omega) = D_{11}^{(0)}(\vec{p},\omega) + D_{22}^{(0)}(\vec{p},\omega), \tag{3.1}$$

$$\Sigma_{10}(\vec{p},\omega) = \Sigma_{11}(\vec{p},\omega) = \Sigma_{00}(\vec{p},\omega) = 4\pi n_2 e_2^2 p^{-2} \varepsilon^{-1}(\vec{p},\omega), \tag{3.2}$$

where

$$D_{11}^{(0)}(\vec{p},\omega) = e_1^2 \sum_{k\sigma} \left\{ \frac{n_{k\sigma}(1-n_{k+p\sigma})}{\omega - \epsilon_1(\vec{k}+\vec{p}) + \epsilon_1(k) + i\delta} - \frac{n_{k\sigma}(1-n_{k+p\sigma})}{\omega + \epsilon_1(\vec{k}+\vec{p}) - \epsilon_1(k) - i\delta} \right\}, \tag{3.3}$$

$$D_{22}^{(0)}(\vec{p},\omega) = e_2^2\, n_2 \left\{ \frac{1}{\omega - \epsilon_2(p) + i\delta} - \frac{1}{\omega + \epsilon_2(p) - i\delta} \right\}, \tag{3.4}$$

$$\varepsilon(\vec{p},\omega) = 1 - v(p) D_{11}^{(0)}(\vec{p},\omega), \tag{3.5}$$

and $n_{k\sigma}$ is the occupation number of the fermion state specified by $\vec{k}$ and σ. In the equations obtained above, we used $f=1$ and $\mu_2=0$, which are the consequence of the self-consistency of our discussions.

Let us investigate the collective excitations basing on the results obtained above . The dispersion relations of bosons are obtained from the poles of the boson Green's function. From (2.19), (2.20), (3.2), and (3.5), we can obtain the following equation :

$$1-v(p)D_{11}^{(0)}(\vec{p},\omega)-v(p)D_{22}^{(0)}(\vec{p},\omega)=0. \tag{3.6}$$

This equation is also obtained from $K(\vec{p},\omega)=0$ by making use of (2.18) and (3.1). For real ω, the real part and imaginary part of (3.3) are, respectively,

$$\mathrm{R_e}\ D_{11}^{(0)}(\vec{p},\omega)=-(k_s^2/8\pi)\left[g_0(\vec{p},\omega)+g_0(\vec{p},-\omega)\right], \tag{3.7}$$

and

$$\begin{aligned}
\mathrm{Im}D_{11}^{(0)}(\vec{p},\omega) &=0, \quad for\ \omega>p^2/2m_1+k_f p/m_1 \\
&=-\frac{m_1^2 e_1^2 \omega}{2\pi p}, for\ k_f p/m_1-p^2/2m_1>\omega>0\ and\ p<2k_f \\
&=-\frac{m_1 e_1^2 k_f^2}{4\pi p}\left[1-(m_1\omega/k_f p-p/2k_f)^2\right], \\
&\qquad for\ p^2/2m_1+k_f p/m_1>\omega>|k_f p/m_1-p^2/2m_1| \\
&=0,\ for\ \omega<p^2/2m_1-k_f p/m_1\ and\ p<2k_f
\end{aligned} \tag{3.8}$$

where $k_s=(4m_1 e_1^2 k_f/\pi)^{\frac{1}{2}}$, $k_f=(3\pi^2 n_1)^{1/3}$, and

$$g_0(\vec{p},\omega)=\frac{8\pi e_1^2}{k_s^2}\sum_{k\sigma}\frac{n_{k\sigma}(1-n_{k+p\sigma})}{\epsilon_1(\vec{k}+\vec{p})-\epsilon_1(k)-\omega}. \tag{3.9}$$

The $(p,\ \omega)$ space can be divided into four regions according to $\mathrm{Im}D_{11}^{(0)}$ as shown by the dashed lines in Fig. 8.

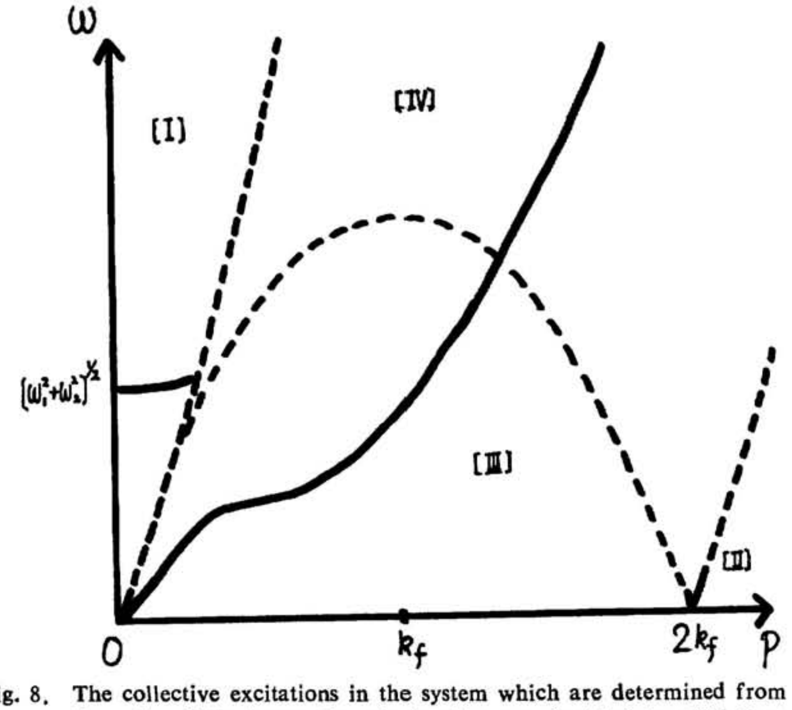


Fig. 8. The collective excitations in the system which are determined from the poles of the boson Green's function are sketched by solid lines.

Let us consider the solutions of the equation given by (3.6). Since $\mathrm{Im}D_{11}^{(0)}(\vec{p},\omega)$ and $\mathrm{Im}D_{22}^{(0)}(\vec{p},\omega)$ are not zero for real ω, the solutions are complex functions of momentum p and the ratio of the imaginary part to the real one in each solution is, in general, finite. Only the solutions especially when these ratios are sufficently small have the meaning as the elementary excitations of this system. Let us write such solutions as $\omega=\omega_R-i\omega_I$ for $\omega_R\gg\omega_I$. From (3.6) making use of (3.4) and (3.5) we get

$$\omega_R^2=\epsilon_2^2(p)+\omega_2^2\,\mathrm{Re}\,\varepsilon^{-1}(\vec{p},\omega_R), \tag{3.10}$$

$$(\omega_I/\omega_R)=-(\omega_2^2/2\omega_R^2)\ \mathrm{Im}\,\varepsilon^{-1}(\vec{p},\omega_R), \tag{3.11}$$

$$\omega_i^2=4\pi n_i e_i^2/m_i$$

which are the classical plasma frequencies of the fermion($i=1$) and the boson ($i=2$) components, repectively.

It is now straightforword to investigate the dispersion relations of the elementary excitation by (3.10) and (3.11) with the use of (3.7), (3.8), and (3.9). As the results we obtain as follows:

(i) optical phonon which lies in the longwavelength region above the continuum.

$$\omega_R\equiv\omega_0(p)=[\epsilon_2^2(p)+\omega_2^2\,\mathrm{Re}\,\varepsilon^{-1}(\vec{p},\omega_0)]^{1/2}\cong[\omega_1^2+\omega_2^2]^{1/2}, \tag{3.12}$$

$$(\omega_I/\omega_R)=0.$$

The boson component, by its polarizability, acts to shift the plasmon frequency ω_1 to $(\omega_1^2+\omega_2^2)^{1/2}$,

(ii) acoustic phonon.

$$\omega_R\equiv\omega_a(p)=[\epsilon_2^2(p)+\mathrm{Re}\,\varepsilon^{-1}(\vec{p},\omega_a)]^{1/2}, \tag{3.13}$$

$$\longrightarrow sp \quad for \quad p\ll k_s\ ,$$
$$\longrightarrow \omega_B(p) \quad for \quad p\gg k_s\ ,$$

$$(\omega_I/\omega_R)\propto(m_1e_2/3m_2e_1)^{1/2}$$

where $s=(m_1e_2/3m_2e_1)^{1/2}(k_f/m_1)$, (3.14)

and $\omega_B(p)$ is the bogolon belonging to the boson component in the case when the interaction between components is neglected as given in the following:

$$\omega_B(p)=[\epsilon_2^2(p)+\omega_2^2]^{1/2}\ . \tag{3.15}$$

It is perhaps interesting that as the momentum of excitation increases there is a continuous change in the energy of the elementary excitation from that appropriate to the sound wave to that appropriate to single particle excitation along the bogolon excitation line.

The sketch of these excitations is given in Fig. 8.

By putting the total density correlation function obtained from (2.17), (2.18), and (3.1) and the function in Hartree-Fock approximation which is given by $D_{HF}(\vec{p},\omega)=D_{11}^{(0)}(\vec{p},\omega)+D_{22}^{(0)}(\vec{p},\omega)$ with the coupling constant λ being written explicitly, into (2.13) we obtain

$$E_{corr} = E^{F}_{corr} + E^{B}_{corr} + \Delta E,$$

where

$$E^{F}_{corr} = \sum_{\vec{p}} \int_0^\infty \frac{d\omega}{2\pi} \int_0^1 d\lambda \, \mathrm{Im} \left[v(p)\, D^{(0)}_{11}(\vec{p},\omega) - \frac{v(p)\, D^{(0)}_{11}(\vec{p},\omega)}{1-\lambda v(p)\, D^{(0)}_{11}(\vec{p},\omega)} \right]$$

$$= [0.0622 \log r_1 + O(r_1^0)]\; n_1 m_1 e_1^4 / 2,$$

$$E^{B}_{corr} = \sum_{\vec{p}} \int_0^\infty \frac{d\omega}{2\pi} \int_0^1 d\lambda \; \mathrm{Im} \; [v(p)\, D^{(0)}_{22}(\vec{p},\omega) - \frac{v(p)\, D^{(0)}_{22}(\vec{p},\omega)}{1-\lambda v(p)\, D^{(0)}_{22}(\vec{p},\omega)}]$$

$$= [-0.803 r_2^{-3/4} + O(r_2^0)]\; n_2 m_2 e_2^4 / 2,$$

$$\Delta E = \sum_{\vec{p}} \int_0^\infty \frac{d\omega}{2\pi} \; \mathrm{Im} \; \log \frac{1 - v(p)\, D^{(0)}_{11}(\vec{p},\omega) - v(p)\, D^{(0)}_{22}(\vec{p},\omega)}{\prod_{i=1}^{2} \left[1 - v(p)\, D^{(0)}_{ii}(\vec{p},\omega) \right]} \qquad (3.16)$$

and the density parameter r_i is defined by $r_i = (3/4\pi n_i)^{1/3} m_i \; e_i^2$.

In the above, E^{F}_{corr} and E^{B}_{corr} are the correlation energies of the fermion and the boson components, respectively, in the case when the interaction between components is neglected. ΔE is the contribution of the boson-fermion interaction to the correlation energy of our system.

Making use of (3. 4) , we can rewrite (3.16) as

$$\Delta E = \sum_{p} \int_0^\infty \frac{d\omega}{2\pi} \; \mathrm{Im} \; \log \; \frac{\omega^2 - \epsilon_2^2(p) - \omega_2^2 \varepsilon^{-1}(\vec{p},\omega) + i\delta}{\omega^2 - \omega_B^2(p) + i\delta} = \Delta E_1 + \Delta E_2 \quad ,$$

where

$$\Delta E_1 = \frac{1}{2} \sum_{\vec{p}} \int_0^\infty d\omega \, [\theta(\epsilon_2^2(p) + \omega_2^2 \, \mathrm{Re}\, \varepsilon^{-1}(\vec{p},\omega) - \omega^2) - \theta(\omega_B^2 - \omega^2)] \qquad (3.17)$$

and

$$\Delta E_2 = \sum_{\vec{p}} \int_0^\infty \frac{d\omega}{2\pi} \; \tan^{-1} \; \frac{-\omega_2^2 \, \mathrm{Im}\, \varepsilon^{-1}(\vec{p},\omega)}{\omega^2 - \epsilon_2^2(p) - \omega_2^2 \, \mathrm{Re}\, \varepsilon^{-1}(\vec{p},\omega).} \qquad (3.18)$$

In obtaining (3.17) and (3.18), we used the relation that $\mathrm{Im} \log z = \pi \theta(-\mathrm{Re} z) + \tan^{-1}(\mathrm{Im} z/\mathrm{Re} z)$. In the case of $s/v_f \leq 1$ where v_f is the Fermi velocity, the calculation of (3.17) and(3.18) will be given in Appendix A. The energy given by (3.17) corresponds to the contribution from the shifts of the zero-point motions of optical phonon and acoustic phonon. Re $\varepsilon^{-1}(\vec{p},\omega)$ in the argument of the θ -function in (3.17) can be written as $\varepsilon_R^{-1}(1+\varepsilon_I^2/\varepsilon_R^2)$ with the real and imaginary parts of $\varepsilon(\vec{p},\omega)$. The contribution from $\varepsilon_I^2/\varepsilon_R^2$ to the energy may be the effect of the finiteness of the life-time of elementary excitation, but it should be emphasized that this contribution is of order of unity in the density parameter expansion of energy per particle. ΔE_1 and ΔE_2 are equal to the first- and second-terms of (3.9) in the paper I, so that our result obtained here accords with the one in the paper I.

4. Conclusions

The main conclusions about the neutral charged-boson and -fermion system at the high-density limit investigated in this paper are the following points:

(i) The contribution of the boson-fermion interaction to the ground state energy is calculated for $s/v_f \lesssim 1$ and accords with the result obtained in the paper I. This result gives the exact terms in the density parameter expansion of the groud state energy.

(ii) The elementary excitations in this system determined from the poles of the boson Green function are the two collective excitations in the low momentum region: acoustic- and optical-phonon like. The lifetime of the acoustic-phonon like elementary excitation approaches infinite as $s/v_f \longrightarrow 0$ where s and v_f are the sound velocity and Fermi velocity, respectively.

Superfluidity and superconductivity have been discussed in connection with astrophysical system[8]. A typical model is a neutral mixture of Alpha particles and electrons. Fetter[9] demonstrated that an interacting charged-boson gas exhibits a Meissner effect. According to our investigation, the two-component system is expected to differ from the charged-boson gas in the physical behaviors as pointed out by Fetter. We are hoping to investigate these points further,

Appendix A. Calculation of (3.17) and (3.18)

Let us calculate ΔE_1 given by (3.17). It should be emphasized that only the values of $\varepsilon(\vec{p},\omega)$ for real ω are used in our calculation since the integration over ω is performed along the real axis. we set

$$\varepsilon(\vec{p},\omega) = \varepsilon_R(\vec{p},\omega) + i\varepsilon_I(\vec{p},\omega), \tag{A.1}$$

where $\varepsilon_R(\vec{p},\omega) = 1 - v(p)\,\mathrm{Re}D_{11}^{(0)}(\vec{p},\omega)$,

$\varepsilon(\vec{p},\omega) = -v(p)\,\mathrm{Im}D_{11}^{(0)}(\vec{p},\omega)$, and $\mathrm{Re}D_{11}^{(0)}(\vec{p},\omega)$ and $\mathrm{Im}\,D_{11}^{(0)}(\vec{p},\omega)$

are given by (3.7) and (3.9), respectively.

According to the division shown in Fig. 8, (8.17) can be rewritten as

$$\begin{aligned}\Delta E_1 = \frac{1}{2}\sum_{p}\Big\{&\int_{[I]} d\omega\ \theta\,(\epsilon_2^2(p) + \omega_2^2\,\varepsilon^{-1}(\vec{p},\omega) - \omega^2) - \int_0^\infty d\omega\,\theta\,(\omega_B^2 - \omega^2) \\ &+ \int_{[II]} d\omega\ \theta\,(\epsilon_2^2(p) + \omega_2^2\,\varepsilon^{-1}(\vec{p},\omega) - \omega^2) \\ &+ [\int_{[III]} d\omega + \int_{[IV]} d\omega\,]\ \theta\,(\epsilon_2^2(p) + \omega_2^2\,\mathrm{Re}\,\varepsilon^{-1}(\vec{p},\omega) - \omega^2)\Big\},\end{aligned} \tag{A.2}$$

where [I], [II], etc. on the integral signs indicate the integration in each region in Fig. 8.

In the last term in the curly braket in (A.2), it is easy to see that for $p \ll k_f$, $\int_{[IV]} = 0$ if $s < v_f$ where v_f is the Fermi velocity defined by k_f/m_1. We will calculate (A.2) for $s < v_f$. In this case. (A.2) is given as

$$\Delta E_1 = \frac{1}{2}\sum_{p<(k_s')}[\omega_0(p) - \omega_1] + \frac{1}{2}\sum_{p}[\omega_a(p) - \omega_B(p)], \tag{A.3}$$

where ω_0 (p) and ω_s (p) are given by (3.12) and (3.13) ,respectively, and $k_s' = [\omega_1^2 + \omega_2^2]^{\frac{1}{2}}/v_t$. The first term in (A. 3) is the contribution from the shift of the zero-point motion of the plasmon :

$$\Delta E_{\text{plasmon}} \simeq \sum_{p(<k_s')} [\sqrt{\omega_1^2 + \omega_2^2} - \omega_1] \sim (m_1/m_2) O(r_2^0) n_2 m_2 e_2^4 ,$$

where r_2 is the density parameter of our system and is defined by $r_2 = (3/4\pi n_2)^{1/3} m_2 e_2^2$. The second term in (A.3) is the contribution from the shift of the zero-point motion of bogolon to that of the acoustic phonon. We see from (3.10), (3.11), and (A.3) that the contribution from the fact that $\varepsilon_I \neq 0$ may be understood as the effect of the finiteness of the life-time of the acoustic phonon on the ground state energy.

Expanding (3.13) in the power series of $\omega_2^2 \omega_B^{-2}[\mathrm{Re}\,\varepsilon^{-1}(\bar{p}, \omega_s) - 1]$ and putting the expression obtained into the second term in (A.3), we obtain

$$\Delta E_{\text{acoustic}} = \frac{1}{4} \sum_p \frac{\omega_2^2}{\omega_B(p)} \left[\frac{\varepsilon_R^{-1}}{1+(\varepsilon_I/\varepsilon_R)^2} - 1 \right] + O(r_2^0) n_2 m_2 e_2^4 ,$$

where we used $\varepsilon_R(\bar{p}, \omega_s) = 1 + k_s^2/p^2$ and $\varepsilon_I(\bar{p}, \omega_s) = 2m_1^2 e_1^2 \omega_s p^{-3}$ in the order estimation. It is easily shown that the effect of considering the life-time of the acoustic phonon to the ground state energy is

$$\frac{1}{4} \sum_p \frac{\omega_2^2}{\omega_B(p)} \frac{1}{\varepsilon_R} \left[\frac{1}{1+(\varepsilon_I/\varepsilon_R)^2} - 1 \right] \sim O(r_2^0) n_2 m_2 e_2^4 .$$

Therefore,

$$\Delta E_{\text{acoustic}} = \frac{1}{4} \sum_p \frac{\omega_2^2}{\omega_B} [\varepsilon_R^{-1}(\bar{p}, \omega_s) - 1] + O(r_2^0) n_2 m_2 e_2^4$$

$$= \left[-\frac{4Q m_1 e_1^{5/3}}{3^{1/4} \pi^2 \alpha m_2 e_2^{5/3}} \frac{1}{r_2^{1/4}} + O(r_2^0) \right] n_2 m_2 e_2^4/2 \qquad \text{(A.4)}$$

where we used the results given in Appendix B of the paper I.

Next, let us calculate ΔE_2 given by (3.18). By going over to the non-dimensional integration of (3.18), we obtain

$$\frac{\Delta E_2}{(n_1 m_1 e_1^4/2)} = \frac{3}{2\pi\alpha^2 r_1^2} \int_0^2 dx\, x^3 \int_0^{1-x/2} du$$

$$\times \tan^{-1} \left\{ \frac{\dfrac{(4\alpha m_1 e_2 r_1/3\pi m_2 e_1)(2\alpha r_1 u)}{(x^2 + 4\alpha r_1 a_0(u)\pi^{-1})^2 + (2\alpha r_1 u)^2}}{u^2 - (m_1 x/2m_2)^2 - \dfrac{(4\alpha m_1 e_2 r_1/3\pi m_2 e_1)(x^2 + 4\alpha r_1 a_0 \pi^{-1})}{(x^2 + 4\alpha r_1 a_0(u)\pi^{-1})^2 + (2\alpha r_1 u)^2}} \right\} , \qquad \text{(A.5)}$$

where $\alpha = (4/9\pi)^{1/3}$, $r_1 = (m_1/m_2)(e_1/e_2)^{7/3} r_2$, $a_0(u) = 1 + \frac{1}{2} u \log \dfrac{1-u}{1+u}$.

Since

$$\frac{3}{2\pi\alpha^2 r_1^2} \int_0^{k_s/k_t} dx\, x^3 \int_0^{1-x/2} du\, \pi \sim O(r_1^0) ,$$

we may put the lower cut, (k_s/k_t), into the integration with respect to x. Now it is easy to calculate (A.5). Neglecting $4\alpha r_1 a_0(u)/\pi$ and $2\alpha r_1 u$ in comparison with x^2

in the integrand of (A.5) and performing the integration, we obtain

$$\Delta E_2 = \left[\left(\frac{m_1 e_1 \log r_2}{2\pi m_2 e_2} \right)^2 + O(\log r_2) \right] (n_2 m_2 \ e_2^4 /2). \qquad (A.6)$$

The sum of (A.4) and (A.6) gives the (3.9) in the paper I.

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