An Example of Blowing Down

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1.

Let $\tilde{X}$ be a complex manifold of dimension n and S a submanifold of $\tilde{X}$ of codimension 1. We denote by $[S]$ the complex line bundle over $\tilde{X}$ defined by the divisor S of $\tilde{X}$. We assume that S has a structure of an analytic fibre bundle over a complex manifold M with fibre P^{r-1} $(r \geqq 2)$, an $(r-1)$—dimensional complex projective space. We denote by La the fibre over $a \in M$ in the bundle $S \rightarrow M$, by $[e]$ the complex line bundle over $La \approx P^{r-1}$ defined by the hyperplane, and by $[S]_{La}$ the restriction of $[S]$ to La.

Nakano [2] and Fujiki-Nakano [1] have shown the following result; If the condition $[S]_{La} = [e]^{-1}$ for any $a \in M$ is satisfied, then there are an n-dimensional complex manifold X containing M and a holomorphic map π: $\tilde{X} \rightarrow X$ such that $(\tilde{X}, \pi)$ is the monoidal transform of X with centre M and $S = \pi^{-1}(M)$. This is obtained by an application of the cohomology vanishing theorem for a weakly 1-complete manifold with a positive line bundle.

In this paper we show the following result, which is a variant of Nakano's theorem.

Theorem. *If the condition* $[S]_{La} = [e]^{-k}$ $(k \geqq 2)$ *for any* $a \in M$ *is satisfied, then we can construct an n-dimensional complex space* X *containing* M *as the singular locus and a holomorphic map* $\pi : \tilde{X} \rightarrow X$ *so that* $(\tilde{X}, \pi)$ is the monoidal transform of X *with centre* M *and* $S = \pi^{-1}(M)$.

We show this theorem by an analogous argument to that of Nakano.

I understand that Akira Fujiki has a more general result on blowing down. Ours is more concrete in its construction.

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2.

First we give an example of a monoidal transformation in which the assumption of our theorem holds.

Let $j : P^{r-1} \rightarrow P^N$ be the Veronese transformation of degree k i.e. j is given by $j(\eta^1 : \ldots : \eta^r) = (M^1(\eta) : \ldots : M^{N+1}(\eta))$ for a system of homogeneous coordinates $(\eta^1 : \ldots : \eta^r)$ of P^{r-1}, where $M^i(\eta)$ is a monomial in $\eta^1, \ldots, \eta^r$ of degree k ($i = 1, \ldots, N+1 = {}_rH_k$). $j(P^{r-1}) = V$ is an $(r-1)$-dimensional submanifold of P^N. The cone K of V is a normal analytic set of C^{N+1} and has the vertex (O) as its

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unique singularity.

We form $\tilde{C}^{N+1}$ by blowing up C^{N+1} at the origin O. $\tilde{C}^{N+1}$ can be considered as the total space of the complex line bundle over P^N which is defined with respect to the open covering $\{U_i \mid i=1, \dots, N+1\}$, where $U_i = \{(z^1 : \dots : z^{N+1}) \in P^N \mid z^i \neq O\}$, by a system of transition functions $\left\{\frac{z^i}{z^j}\right\}$. Let $\tilde{K}$ be the inverse image of K by the projection $\tilde{p} : \tilde{C}^{N+1} \to C^{N+1}$, then $\tilde{K}$ is the line bundle over V, which is the restriction of the bundle $(\tilde{C}^{N+1} \to P^N)$ to V.

$\tilde{p}^{-1}(O) \subset \tilde{C}^{N+1}$ is the zero-section of the bundle $(\tilde{C}^{N+1} \to P^N)$ and $T = \tilde{p}^{-1}(O) \cap K$ is the zero-section of the bundle $(\tilde{K} \to V)$. By restricting $\tilde{p} : \tilde{C}^{N+1} \to C^{N+1}$ to $\tilde{K}$, we have a holomorphic map p from $\tilde{K}$ onto K which gives a biholomorphic homeomorphism $\tilde{K} - T \approx K - O$.

The divisor T of $\tilde{K}$ defines a complex line bundle $[T]$. Then

$[T]$ = the normal bundle of T in $\tilde{K}$
= the line bundle $(\tilde{K} \to V)$
= the restriction $(\tilde{C}^{N+1} \to P^N) \mid_V$.

If we take i_α so that $M^{i_\alpha}(\eta) = (\eta^\alpha)^k$ $(\alpha=1, \dots, r)$, then V is covered by r open sets $U_{i_\alpha} \supset \{(M^i(\eta)) \in P^N \mid (\eta^\alpha)^k \neq O\}$ $(\alpha=1, \dots, r)$. So the transition functions become $\frac{z^{i_\alpha}}{z^{i_\beta}} = \frac{(\eta^\alpha)^k}{(\eta^\beta)^k} = \left(\frac{\eta^\alpha}{\eta^\beta}\right)^{-k}$. While the line bundle $[e]$ on P^{r-1} is defined by a system of transition functions $\left\{\frac{\eta^\alpha}{\eta^\beta}\right\}$ with respect to the open covering $\{(\eta) \in P^{r-1} \mid \eta^\alpha \neq O\}$ $(\alpha=1, \dots, r)$. Hence we have $(\tilde{C}^{N+1} \to P^N) \mid_V = [e]^{-k}$.

Thus we have obtained a situation such that the mapping $p : \tilde{K} \to K$ is holomorphic onto, $p : \tilde{K} - T \approx K - O$, $T = p^{-1}(O) \approx P^{r-1}$ and $[T]_T = [e]^{-k}$.

3.

We use the following two theorems.

Theorem 1 (Nakano). *Let V be a weakly 1-complete manifold, B a complex line bundle over V and K_V the canonical line bundle of V. If $K_V^{-1} \otimes B$ is positive, then we have $H^q(V, O(B)) = O$ $(q \geqq 1)$.*

This is a special case of theorem 1 [3].

Theorem 2(k). *If the condition $[S]_{La} = [e]^{-k}$ for any $a \in M$ is satisfied, then there is a neighbourhood V of La in $\tilde{X}$ such that*

1) if $V \cap L_b \neq \phi$, then $L_b \subset V$,

2) V is a weakly 1-complete manifold and $K_V^{-1} \otimes [S]_V^{-\varepsilon}$ is positive for $\varepsilon=1, 2$.

Proof. We take a system of local coordinates $\{\zeta^1, \dots, \zeta^m\}$ $(m=n-r)$ with centre $a \in M$ so that the bundle $S \to M$ is trivial on the coordinate neighbourhood $D = \{\zeta \in C^m \mid \phi(\zeta) < 1\}$ of a i.e. $\Pi^{-1}(D) \approx D \times P^{r-1}$, where $\phi(\zeta) = \sum_{j=1}^{m} |\zeta^j|^2$. Then we have $[S]_{D\times P}=[e]^{-k}$, where we also use the notation $[e]$ to denote the pull-back of $[e]$ on P^{r-1} by the natural projection $D\times P\rightarrow P$.

We fix a system of homogeneous coordinates $(\eta^1 : \dots : \eta^r)$ of P^{r-1}. $\left\{ \xi^{\beta}_{\alpha} = \frac{\eta^{\beta}}{\eta^{\alpha}} \mid \beta = 1, \dots, \hat{\alpha}, \dots, r \right\}$ is a system of inhomogeneous coordinates on the coordinate neighbourhood $U_{\alpha} = \{(\eta) \in P^{r-1} \mid \eta^{\alpha} \neq 0\}$ $(\alpha = 1, \dots, r)$. $[e]$ on $D\times P^{r-1}$ is defined by a system of transition functions $\varepsilon_{\alpha\beta} = \frac{\eta^{\beta}}{\eta^{\alpha}} = \xi^{\beta}_{\alpha}$ with respect to the open covering $\{D\times U_{\alpha} \mid \alpha = 1, \dots, r\}$. The bundle $[e]$ is positive by taking a "metric" $a_{\alpha} = e^{\phi(\cdot)} \sum_{\beta=1}^{m} |\xi^{\beta}_{\alpha}|^2$ on the fibres of $[e]$.

We can choose a finite open covering $\{U'_{\lambda} \mid \lambda \in \Lambda\}$ of P^{r-1} and open subset V'_{λ} of $\tilde{X}$ such that $\{U'_{\lambda}\}$ is a refinement of $\{U_{\alpha}\}$ and $V'_{\lambda} \cap S = D\times U'_{\lambda}$ for each $\lambda \in \Lambda$. We denote the refining map by $\sigma : \Lambda \rightarrow \{1, \dots, r\}$ with $U'_{\lambda} \subset U_{\sigma(\lambda)}$.

We can take a system of local coordinates $\{ z^1_{\lambda}, \dots, z^m_{\lambda}, y_{\lambda}, x^1_{\lambda}, \dots, x^{\hat{\alpha}(\lambda)}_{\lambda}, \dots, x^r_{\lambda} \}$ on V'_{λ} such that

$$(※)\begin{cases} S \text{ is defined in } V'_{\lambda} \text{ by the local equation } y_{\lambda} = 0, \\ z^j_{\lambda} \mid_S = \zeta^j \ (j = 1, \dots, m),\ x^{\alpha}_{\lambda} \mid_S = \xi^{\alpha}_{\sigma(\lambda)} \quad (\alpha = 1, \dots, r), \\ \text{and } x^{\sigma(\lambda)}_{\lambda} \equiv 1. \end{cases}$$

$V' = \bigcup_{\lambda} V'_{\lambda}$ is a neighbourhood of La and $V' \cup S = D\times P^{r-1}$. The bundle $[S]\,V'$ is defined by a system of transition functions $e_{\lambda\mu} = \frac{y_{\lambda}}{y_{\mu}}$ with respect to the open covering $\mathfrak{B} = \{V'_{\lambda}\}$ of V'. Moreover we take y_{λ} so that $e_{\lambda\mu} \mid S = \varepsilon_{\sigma(\lambda)\sigma(\mu)}{}^{-k}$, and hereafter we write $\varepsilon_{\lambda\mu}$ for $\varepsilon_{\sigma(\lambda)\sigma(\mu)}$.

By the adjunction formula for canonical bundles: $K_{V'\cap S} = K_{V'} \mid_{V'\cap S} \otimes [S]_{V'\cap S'}$ we have $K_{V'} \mid_{V'\cap S} = K_{D\times P^{r-1}} \otimes [S]_{D\times P^{-1}} = [e]^{-r} \otimes [e]^{k} = [e]^{-(r-k)}$, so we may assume that the bundle $K_{V'}$ is represented with respect to the open covering $\mathfrak{B}$ by a system of transition functions $\{k_{\lambda\mu}\}$ with $k_{\lambda\mu} \mid_S = \varepsilon_{\lambda\mu}^{-(r-k)}$.

Using the fact $H^1(D\times P, O([e]^{\ell})) = 0$ $(\ell > 1)$, we can extend holomorphic functions ζ^j on $D\times U'_{\lambda}$ $(j = 1, \dots, m)$ approximately to holomorphic functions on V'_{λ} ([2] p. 495), extend holomorphic cross-sections $\{\tau^1{}_{\lambda}\}_{\lambda}$ of $[e]^k$ on $V\cap S$ defined by $\tau^i_{\lambda} = \frac{M^i(\eta)}{(\eta^{\sigma(\lambda)})^k}$ $(i = 1, \dots, N+1)$ approximately to holomorphic cross-sections of $[S]^{-1}$ on V. Similarly we can extend holomorphic cross-sections $\{\omega^{\rho}_{\lambda}(\varepsilon)\}$ of $[e]^{r+(\varepsilon-1)k}$ on $V\cap S$ $(\rho = 1, \dots, r\ H\ r+(\varepsilon-1)k)$ approximately to holomorphic cross-sections of $K^{-1} \otimes [S]^{-\varepsilon}$ on V ([1] p. 641), where $\omega^{\rho}_{\lambda}(\varepsilon) = \frac{N^{\rho}(\eta)}{(\eta^{(\sigma\lambda)})^{r+(\varepsilon-1)k}}$

and $N^{\rho}(\eta)$ is a monomial in $\eta^1, \dots, \eta^r$ of degree $r+(\varepsilon-1)k$ $(\varepsilon=1, 2)$. As is stated in proposition 6 [2], we obtain holomorphic functions z^j_λ, t^i_λ, $w^\rho_\lambda(\varepsilon)$ on V'_λ which satisfy the following relations in addition to (*).

$$\begin{cases} z^j_\lambda - z^j_\mu = (y_\lambda)^\ell f_{\lambda\mu}{}^j \\ t^i_\lambda - e_{\lambda\mu}{}^{-1} t^i_\mu = (y_\lambda)^\ell g_{\lambda\mu}{}^i \qquad \text{on } V'_\lambda \cap V'_\lambda, \\ w^\rho_\lambda(\varepsilon) - k_{\lambda\mu}{}^{-1} e_{\lambda\mu}{}^{-\varepsilon} w^\rho_\mu(\varepsilon) = (y_\lambda)^\ell h_{\lambda\mu}^\rho(\varepsilon) \end{cases}$$

and $\{f_{\lambda\mu}{}^j\} \in Z^1(\mathfrak{B}, O([S]^{-\ell}))$, $\{g_{\lambda\mu}{}^i\} \in Z^1(\mathfrak{B}, O([S]^{-\ell-1}))$, $\{h_{\lambda\mu}{}^\rho(\varepsilon)\} \in Z^1(\mathfrak{B}, O(K^{-1} \otimes [S]^{-\ell-\varepsilon}))$ $(\varepsilon=1, 2)$. These 1-cocycles are coboundaries of C^∞ sections, so we can find C^∞ functions $F_\lambda{}^j$, $G_\lambda{}^i$ $H_\lambda{}^\rho(\varepsilon)$ on V'_λ such that

$$f_{\lambda\mu}{}^j = F_\lambda{}^j - e_{\lambda\mu}{}^{-\ell} F_\mu{}^j$$
$$g_{\lambda\mu}{}^i = G_\lambda{}^i - e_{\lambda\mu}{}^{-\ell-1} G_\mu{}^i$$
$$h_{\lambda\mu}{}^\rho(\varepsilon) = H_\lambda{}^\rho(\varepsilon) - k_{\lambda\mu}{}^{-1} c_{\lambda\mu}{}^{-\ell-\varepsilon} H_\mu{}^\rho(\varepsilon) \quad (\varepsilon=1, 2)$$

hold on $V'^\lambda \cap V'_\mu$.

We set

$$\begin{cases} Z^j = z_\lambda{}^j - (y_\lambda)^\ell F_\lambda{}^j \\ T_\lambda{}^i = t_\lambda{}^i - (y_\lambda)^\ell G_\lambda{}^i \\ W_\lambda{}^\rho(\varepsilon) = w_\lambda{}^\rho(\varepsilon) - (y_\lambda)^\ell H_\lambda{}^\rho(\varepsilon), \end{cases}$$

then Z^j is a global C^∞ function on V', $T_\lambda{}^i$ and $W_\lambda{}^\rho(\varepsilon)$ are C^∞ functions on V'_λ such that $T_\lambda{}^i = e_{\lambda\mu}{}^{-1} T_\mu{}^i$, $W_\lambda{}^\rho(\varepsilon) = k_{\lambda\mu}{}^{-1} e_{\lambda\mu}{}^{-\varepsilon} W_\mu{}^\rho(\varepsilon)$ on $V'_\lambda \cap V'_\mu$.

We set $A_\lambda = \sum_{i=1}^{N+1} |T_\lambda{}^i|^2$ and $B_\lambda(\varepsilon) = \sum_\rho |W_\lambda{}^\rho(\varepsilon)|^2$.

Since $A_\lambda |y_\lambda|^2 = A_\mu |y_\mu|^2$ on $V'_\lambda \cap V'_\mu$,

$$\psi = \phi(Z) + e^{\phi(Z)} A_\lambda |y_\lambda|^2$$

is a golbal C^∞ function on V'.

Since $\tau_\lambda{}^1, \cdots, \widehat{\tau^{i\sigma(\lambda)}_\lambda \equiv 1}, \cdots, \tau^{N+1}_\lambda$ are all the monomials in $\xi_\lambda{}^1, \dots, \widehat{\xi^{\sigma(\lambda)}_\lambda \equiv 1}, \dots, \xi^r_\lambda$ of degree $\leqslant k$ we can take $t_\lambda{}^1, \dots, \widehat{t^{i\sigma(\lambda)}_\lambda \equiv 1}, \dots, t^{N+1}_\lambda$ to be all the monomials in $x_\lambda{}^1, \dots, \widehat{x^{\sigma(\lambda)}_\lambda \equiv 1}, \dots, x_\lambda{}^r$ of degree $\leqslant k$. Then we have (omitting the lower index λ)

$$\frac{\partial}{\partial \bar{x}^\beta} \Big(\sum_i |T_\lambda{}^i|^2\Big) = x^\beta (1 + P_\beta(x)) + y^\ell (\quad) + \bar{y}^\ell (\quad) + |y|^{2\ell} (\quad),$$

$$\frac{\partial^2}{\partial x^\alpha \partial \bar{x}^\beta} \Big(\sum_i |T_\lambda{}^i|^2\Big) = \delta_{\alpha\beta} + Q_{\alpha\beta}(x) + y^\ell (\quad) + \bar{y}^\ell (\quad) + |y|^{2\ell} (\quad),$$

where $x^\beta(1 + P_\beta(x)) = \sum_i t^i \dfrac{\partial \bar{t}^i}{\partial \bar{x}^\beta}$ $(\beta = 1, \dots, r)$ and

$Q_{\alpha\beta}(x) = \dfrac{\partial}{\partial x^\alpha}(x^\beta P_\beta(x))$ $(\alpha, \beta = 1, \ldots, x)$. $P_\beta(x)$ and $Q_{\alpha\beta}(x)$ are polynomials in $x_\lambda{}^1, \ldots, \widehat{x_\lambda{}^{(\sigma\lambda)}}, \ldots, x_\lambda{}^r$ of degree $2k-2$.

Now we estimate the Levi form of $\psi = \phi(Z) + e^{\phi(Z)}(\sum_i |T_\lambda{}^i|^2)|y_\lambda|^2$. We consider in a fixed V'_λ and omit the index λ. Since we can take V'_λ smaller, we may assume $\sum_\alpha{}' |x^\alpha|^2 < G$ for a large $G > 0$, here α runs over $1, \ldots, \widehat{\sigma(\lambda)}, \ldots, r$ in the sum $\sum_\alpha{}'$. We take $\eta > 0$ so that $\eta G < \frac{1}{8}$. If $l \geqq 3$, then we have

$$\frac{\partial^2\psi}{\partial z^j \partial \bar z^k} = \delta_{jk} + 0(|y|^2),$$

$$\frac{\partial^2\psi}{\partial z^j \partial \bar y} = e^{\phi(Z)}(1+\sum_i{}'|t^i|^2)\{y\bar z^j + 0(|y|^2)\},$$

$$\frac{\partial^2\psi}{\partial y \partial \bar y} = e^{\phi(Z)}(1+\sum_i{}'|t^i|^2) + 0(|y|^2),$$

$$\frac{\partial^2\psi}{\partial z^j \partial \bar x^\beta} = e^{\phi(Z)}|y|^2\{\bar z^j x^\beta(1+P_\beta(x)) + 0(|y|)\},$$

$$\frac{\partial^2\psi}{\partial y \partial \bar x^\beta} = e^{\phi(Z)}\bar y\{x^\beta(1+P_\beta(x)) + 0(|y|)\},$$

$$\frac{\partial^2\psi}{\partial x^\alpha \partial \bar x^\beta} = e^{\phi(Z)}|y|^2\{\delta_{\alpha\beta} + Q_{\alpha\beta}(x) + 0(|y|)).$$

If $y = 0$ then the hermitian matrix

$$\begin{pmatrix} \dfrac{\partial^2\psi}{\partial z^j\,\partial\bar z^k} & \dfrac{\partial^2\psi}{\partial z^j\,\partial\bar y} & \dfrac{\partial^2\psi}{\partial z^j\,\partial\bar x^\beta} \\ * & \dfrac{\partial^2\psi}{\partial y\,\partial\bar y} & \dfrac{\partial^2\psi}{\partial y\,\partial\bar x^\beta} \\ * & * & \dfrac{\partial^2\psi}{\partial x^\alpha\,\partial\bar x^\beta} \end{pmatrix}$$

has the form

$$\begin{pmatrix} \delta_{jk} & & \\ & e^{\phi(\zeta)}(1+\sum_i{}'|\tau^i|^2) & \\ & & 0 \end{pmatrix},$$

so this is positive semi-definite.

In the case of $y \neq 0$. We have

$$(1+\sum_i{}'|t^i|^2)(dy, d\bar y) + \sum_\alpha{}' y\bar x^\alpha(1+P_\alpha)(dx^\alpha, d\bar y)$$
$$+ \sum_\beta{}' \bar y x^\beta(1+P_\beta)(dy, d\bar x^\beta) + \sum_{\alpha,\beta}{}' |y|^2(\delta_{\alpha\beta}+Q_{\alpha\beta})(dx^\alpha, d\bar x^\beta)$$
$$= |dy|^2 + \sum_\alpha{}' |x^\alpha dy + y dx^\alpha|^2 + \sum_\alpha{}' |x^\alpha|^2 |x^\alpha dy + 2y dx^\alpha|^2 +$$

$$\sum_{\alpha<\beta}{}' \mid x^\alpha x^\beta dy + y x^\beta dx^\alpha + y x^\alpha dx^\beta \mid^2 + \sum{}' \mid \dots \mid^2 + \dots + \dots \sum{}' \mid \dots \mid^2$$
$$\geqslant \mid dy \mid^2 + \sum_\alpha{}' \mid x^\alpha dy + y dx^\alpha \mid^2$$
$$= \mid dy \mid^2 + \sum_\alpha{}' \{ \mid (1+\eta)^{1/2} x^\alpha dy + (1+\eta)^{-1/2} y dx^\alpha \mid^2 - \eta \mid x^\alpha \mid^2 \mid dy \mid^2 +$$
$$(1 - \frac{1}{1+\eta}) \mid y \mid^2 \mid dx \mid^2 \}$$
$$\geqslant \{ 1 - \eta (\sum_\alpha{}' \mid x^\alpha \mid^2) \} \mid dy \mid^2 + \mid y \mid^2 \sum_\alpha{}' \frac{\eta}{1+\eta} \mid dx^\alpha \mid^2.$$

If we take small $\mid y \mid > 0$, then $1 - \eta(\sum_\alpha{}' \mid x^\alpha \mid^2) + 0(\mid y \mid^2) > 2/3$ and the hermitian matrix $(\frac{\eta}{1+\eta} \delta_{\alpha\beta} + 0(\mid y \mid))$ is positive definite, so the minimal eigen value λ of this matrix is positive. For the remaining terms in dy and $\{dx^\alpha\}$ are estimated as follows;

$$\sum_\alpha{}' e^{\phi(z)} y \bar{a}_\alpha (dx^\alpha, dy) + \sum_\beta{}' e^{\phi(z)} \bar{y} a_\beta (dy, d\bar{x}^\beta)$$
$$\geqslant -(\sum_\alpha{}' \mid a_\alpha \mid) \mid dy \mid^2 - e^{\phi(z)} \mid y \mid^2 \sum_\alpha{}' (e^{\phi(z)} \mid a_\alpha \mid) \mid dx^\alpha \mid^2,$$

where a_α is a quantity of $O(\mid y \mid)$ for each α. Hence we have

$$(dy\ dx^\alpha) \begin{pmatrix} \frac{\partial^2 \psi}{\partial y\, \partial \bar{y}} & \frac{\partial^2 \psi}{\partial y\, \partial \bar{x}^\beta} \\ \frac{\partial^2 \psi}{\partial x^\alpha\, \partial \bar{y}} & \frac{\partial^2 \psi}{\partial x^\alpha\, \partial \bar{x}^\beta} \end{pmatrix} \begin{pmatrix} d\bar{y} \\ dx^\beta \end{pmatrix}$$

$$\geqslant [e^{\phi(z)} \{ 1 - \eta (\sum_\alpha{}' \mid x^\alpha \mid^2) + O(\mid y \mid^2) \} - \sum_\alpha{}' \mid a_\alpha \mid] \mid dy \mid^2 +$$
$$e^{\phi(z)} \mid y \mid^2 [\sum_{\alpha\ \beta}{}' (\frac{\eta}{1+\eta} \delta_{\alpha\beta} + O(\mid y \mid))(dx^\alpha, d\bar{x}^\beta) - \sum_\alpha{}' (e^{\phi(z)} \mid a_\alpha \mid) \mid dx^\alpha \mid^2]$$
$$\geqslant \frac{7}{12} \mid dy \mid^2 + e^{\phi(z)} \mid y \mid^2 \sum_\alpha{}' \frac{\lambda}{2} \mid dx^\alpha \mid^2,$$

if $\mid y \mid > 0$ is small enough. Similarly we can estimate the terms containing $\{dz^j, dy\}$ and the terms containing $\{dz^j, dx^\alpha\}$,
Then we can make

the Levi form of $\psi \geqslant \frac{1}{2} (\sum_{j=1}^{m} \mid dz^j \mid^2 + \mid dy \mid^2)$ for small $\mid y \mid > 0$ and for small $\phi(z) = \sum_j \mid z^j \mid^2$.

Therefore we see that the Levi form of ψ is positive semidefinite on an open subset of V' where $\mid y \mid$ and $\mid z \mid$ are small enough.

We set $C_\lambda^{(\varepsilon)} = B_\lambda^{(\varepsilon)} e^{l\psi}$. Then $\{ C_\lambda^{(\varepsilon)} \}_\lambda$ is a "metric" on the fibres of the bundle $K_{V'}^{-1} \otimes [S]_{V'}^{-\varepsilon}$ for $\varepsilon = 1, 2$. If $l \geqslant 3$ the coefficient matrix of the curvature form of this "metric" has the form

$$\begin{pmatrix} l\dfrac{\partial^2}{\partial\zeta^j\,\partial\bar{\zeta}^k}\left(\sum_{j=1}^{m}|\zeta^j|^2\right) & & \\ & l e^{\phi(\zeta)}\left(1+\sum_{i=1}^{N+1}{}'|\tau^i|^2\right) & \\ & & \dfrac{\partial^2}{\partial\xi^\alpha\,\partial\bar{\xi}^\beta}\log\left(\sum_\rho|\omega^\rho(\varepsilon)|^2\right) \end{pmatrix}$$

on S and this matrix is positive definite. Hence the bundle $K_{V'}^{-1}\otimes[S]_{V'}^{-\varepsilon}$ is positive for small $|y|$ $(\varepsilon=1, 2)$. $V=\{p\in V' \mid \psi(p)<\delta\}$ is relatively compact in V' for a small $\delta>0$. So V is weakly 1-complete with respect to the C^∞ function $\bar{\psi}=(1-\frac{\psi}{\delta})^{-1}$ on V and the bundle $K_V^{-1}\otimes[S]_V^{-\varepsilon}$ is positive for $\varepsilon=1, 2$.

4. Construction of X

For each $a\in M$ we take a neighbourhood V of La as in theorem 2 (k). Then we have $V\cap S\approx D\times \mathbf{P}^{r-1}$ and $[S]_{V\cap S}=[e]^{-k}$.

Exact sequences of sheaves : $0\to O([S]^{-1})\to O_V\to O_{V\cap S}\to 0$ and $0\to O([S]^{-2})\to O([S]^{-1})\to O_{V\cap S}([S]_{V\cap S}^{-1})\to 0$ induce long exact sequences of cohomology groups, then the restrictions $\Gamma(V, O_V)\to\Gamma(V\cap S, O_{V\cap S})$ and $\Gamma(V, O([S]^{-1}))\to\Gamma(V\cap S, O([e]^k))$ are surjective since $H^1(V, O([S]^{-\varepsilon}))=0$ $(\varepsilon=1, 2)$ by theorem 1.

Hence there are extensions $z^j\in\Gamma(V, O_V)$ of local cooidinates ζ^j on D (here we regard ζ^j to be an element of $\Gamma(D\times\mathbf{P}, O))$ $(j=1, \ldots, m)$ and extensions $\{f_\lambda{}^i\}_\lambda\in\Gamma(V, O([S]^{-1}))$ of holomorphic cross-sections $\{\tau_\lambda{}^i\}_\lambda\in\Gamma(V\cap S, O([e]^k))$ $(i=1, \ldots, N+1)$ (here we cover V By V'_λ's). We heve $f^i=y_\lambda f_\lambda{}^i\in\Gamma(V, \mathfrak{J}(S))$ $(i=1, \ldots, N+1)$ by an isomorpnism $\Gamma(V, \mathfrak{J}(S))\approx\Gamma(V, O([S]^{-1}))$, where $\mathfrak{J}(S)$ is the ideal sheaf of S.

We define a holomorphic map $\Phi: V=\bigcup_\lambda V'_\lambda\to \mathbf{C}^m\times\tilde{\mathbf{C}}^{N+1}$ by $\Phi(p)=((z^1(p), \ldots, z^m(p)), (f^1(p), \ldots, f^{N+1}(p)), (f^1(p):\ldots:f^{N+1}(p))$ for any $p\in V$. Φ is of rank n at any point of La and $\Phi(La)\approx(0)\times P^{r-1}$. Therefore we can take a neighbourhood W of La and a neighbourhood $D'\subset D$ of $0\in\mathbf{C}^m$ such that $\Phi(W)=D'\times\tilde{K}$. $D'\times\tilde{E}$ is obtained by blowing up $\Delta^*=D'\times K$ with centre $\Gamma=D'\times(0)$. By a biholomorphic homeomorphism $p: D'\times\tilde{K}-D'\times T\approx D'\times K-D'\times(0)$ we have $\Pi: W-S\approx\Delta^*-\Gamma$. We can identify Γ with a neighbourhood of a in M.

We construct such Da', Wa, Δa^*, Γa and $\Pi a: Wa\to\Delta a^*$ for each $a\in M$, then we have $\Pi a: Wa-S\approx\Delta a^*-\Gamma a$, $S=\Pi a^{-1}(\Gamma a)$, $\Pi a: S\to\Gamma a$ is a $\mathbf{P}^{r-1}$-bundle, $[S]_{L_b}=[e]^{-k}$ for any $b\in Da'$.

If $Wa\cap Wb\neq\phi$ we have the following commutative diagram:

$$\begin{array}{ccccccccc} & Wa & \supset & Wa \cap Wb & \overset{id}{=} & Wa \cap Wb & \subset & Wb & \\ \pi_a & \downarrow & & \downarrow & & \downarrow & & \downarrow & \pi_b \\ & \Delta^*a & \subset & \pi_a(Wa \cap Wb) & \overset{\phi_{ab}}{\longrightarrow} & \pi_b(Wa \cap Wb) & \subset & \Delta^*b . & \end{array}$$

Then a holomorphic mapping $\phi_{ab} = \pi_b \circ id \circ \pi_a^{-1} : \pi_a(Wa \cap Wb) - \Gamma a \rightarrow \pi_b(Wa \cap Wb) - \Gamma_b$ can be naturally extended to a continuous mapping $\phi_{ab} : \pi_a(Wa \cap Wb) \rightarrow \pi_b(Wa \cap Wb)$ and this becomes a biholomorphic homeomorphism. Since Δ^* is a normal analytic set of C^{m+N+1}, $\phi_{ab} : \pi_a(Wa \cap Wb) \rightarrow \pi_b(Wa \cap Wb)$ gives an isomorphism of complex spaces. Hence $\{\Delta^*a\}$ and $\{\Gamma a\}$ can be patched together and form a complex space X^* and the singular locus of X^* respectively (the latter is an analytic subspace of X^* which is biholomorphic to M). Since the open sets $\bigcup_a Wa - S$ *of* $\tilde{X}$ and $X^* - M$ are biholomorphic by π_a on each $Wa - S$, we can patch $\tilde{X} - S$ and X^*, thus we obtain a complex space X.

$$\pi = \begin{cases} id & \text{on } \tilde{X} - S \\ \pi_a & \text{on } Wa \end{cases}$$ is a holomorphic mapping from $\tilde{X}$ onto X and we have

$\pi : \tilde{X} - S \approx X - M$, $S = \pi^{-1}(M)$, $\pi : S \rightarrow M$ is a $\mathbf{P}^{r-1}$ – bundle, $[S]_{La} = [e]^{-k}$ for any $a \in M \subset X$. This completes the proof of the theorem.

References

[1] A. Fujiki - S. Nakano, *Supplement to "On the inverse of monoidal transformation"*, Publ. RIMS, Kyoto Univ., 7 (1971–72) ; 637–644.

[2] S. Nakano, *On the inverse of monoidal transformation*, ibid., 6 (1970–71), 483–502.

[3] S. Nakano, *Vanishing theorems for weakly 1-complete manifolds* II, (to appear in Publ. RIMS, Kyoto Univ., 10 (1973–74)).