Tangent Cones and Multiplicities of Analytic Spaces

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Introduction. Let X be au analytic set in an open set in C^n and Y an analytic subset of X. In Whitney [6] various tangent cones of X at a point x of X and a tangent cone relative to Y are defined. In the case when X is equidimentional and Y smooth, Hironaka [3] states that for p: $C_{X,Y} \to Y$, the normal cone of Y in X, the equality $dim\ p^{-1}(y) = dim\ X$ implies the equimultipleness of X along Y near y.

In § 1 we characterize, for an equidimentional analytic set X, the equimultipleness of X along Y in tangent cone $K(X, Y)$ relative to Y (defined as an analytic cycle in $C^n \times C^n$). That is, if $K(Y, X)_y$ denotes the cone in C^n defined by $((y) \times C^n) \cap |K,(Y,X)| = (y) \times K(Y,X)_y$ where $|K(Y, X)|$ denotes the support of $K(Y, X)$, then the equimultipleness of X is equivalent to the fact that $dim\ K(Y, X)_y = dim\ X$ for all $y \in Y$. In this case we have, for $y \in Y$, $K(Y,X)_y = |C(X)_y|$ where $C(X)_y$ is the tangent cone of X at y (defined as an analytic cycle in C^n) (1.6).

In § 2 we make use of the concept of normal cones in order to give another proof of a theorem on the analytic integral dependence which is stated and proved in Hironaka [2] and in Scheja [5] (see (2.3)). We also show that, given an ideal I in O_n, there is a well defined natural number m such that if $f \in O_n$ is integral over I, then f satisfies an integral equation over I of degree m.

§ 1

(1.1). We recall some notations in the analytic intersection theory (cf. Draper [1]). C^n denotes the complex number space of dimention n, On denotes the local ring of local holomorphic functions at the origin o of C^n. Let X be an equidimentional analytic set of dimention d in an open set U in C_n with $o \in X$ and L a linear subspace of C^n defined by linearly independendent linear equations $p_1(x)=0, \cdots, p_d(x)=0$. For $x \in C^n$, we let $x+L$ denote the affine

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subspace $\{x+x' : x' \in L\}$. If, for $x \in X$, $X \cap (x+L) = \{x\}$ locally around x, then the *intersection number* $i(X, x+L; x)$ is defined to be the covering multiplicity at x of the map $p = (p_1, \cdots, p_d) : X \to C^d$, that is, the number of points in $U' \cap p^{-1}(a)$, where U' is a neighbourhood of x in C^n with $\overline{U'} \cap (X \cap (x+L)) = \{x\}$ and $a = (a_1, \cdots, a_d)$ is a generic point close to $p(x)$. Put $m(X,x) = inf\{i(X,x+L;x) : L\}$, which is called *the multiplicity of X at x.*

Let A be an affine subspace of C^n and Y a proper compornent of $X \cap A$, that is, Y is an irreducible compornent of $X \cup A$ with $dim\ Y = d - \text{codim}\ A$. Let $x \in Y$ be a regular point of $X \cap A$, B an affine subrpace of *codimenion=dim* Y, which intersects with Y transversally at x (then $X \cap A \cap B = \{x\}$ locally). Then $i(X,A;Y)$ is defined to be $i(X, A \cap B; x)$ which is independent of the choices of x and B. If Y is not a proper compornent of $X \cap A$, then we define $i(X,A;Y) = 0$.

An element (a_V) of the direct product $\Pi_V Z_V$, where Π ranges all the irreducible analytic sets V in U and Z_V is the group of the integers for each V, is called an *analytic cycle* if $\{V : a_V \neq 0\}$ is locally finite in U, and in this case we denote this cycle as $\Sigma a_V V$ and in this case the analytic set $\cup \{V : a_V \neq 0\}$ is called *the support of* $\Sigma a_V V$ which is denoted by $|\Sigma a_V V|$. We identify an analytic set A with the analytic cycle ΣV, where Σ ranges all the irreducible compornents of A.

For an irreducible analytic set V in U and an affine subspace L of C^n, the *intersection product* $V \cdot L$ is defined to be the analytic cycle $\Sigma i(V, L; Y)Y$ in U where Σ ranges all the proper components Y of $V \cap L$. For an analytic cycle $\Sigma a_V V$, an affine subspace L and an irreducible analytic set Y in U, we define
$i(\Sigma a_V V, L;Y) = \Sigma a_V i(V, L;Y)$,
$(\Sigma a_V V) \cdot L = \Sigma a_V V \cdot L = \Sigma_W i(\Sigma a_V V, L; W)W$. For equidimentional analytic set X, this definition of the intersection number coincides with the previous one. For an analytic cycle $\Sigma a_V V$ and affine subspaces L, M, we have $(\Sigma a_V V) \cdot (L \cdot M) = ((\Sigma a_V V) \cdot L) \cdot M$.

(1.2) Let X be an equidimenetional analytic set of dimention d, $o \in X \subset U \subset C^n$ as in (1.1), and $o \in Y$ an analytic subset of X, Let $K^*(Y, X)$ denote the closure of $\{(y,x,z) \in U \times C^n \times C : y \in Y,\ z \neq 0,\ y+zx \in X\}$ in $U \times C^n \times C$ The analytic cycle $K(Y,X)$ in $U \times C^n$ defined by $K^*(Y, X) \cdot (C^{2n} \times (o)) = K(Y,X) \times (o)$ is called *the thangent cone relative to* Y. It is clear that $|K(Y, X)| \subset Y \times C^n$.

For $x \in X$, the analytic cycle $C(X)_x$ in C^n defined by $K(\{x\}, X)=(x)\times C(X)_x$ is called *the tangent cone of X at x*. For $y \in Y$, we define the cone $K(Y,X)_y$ in C_n by $((y)\times C^n) \cap |K(Y,X)| = (y)\times K(Y,X)_y$. Then $dim\ K(Y,X)_y \geq d$, since $K(Y,X)_y \supset K(\{y\}, X)_y = |C(X)_y|$ and $dim\ |C(X)_y| = d$ (Whitney [6]).

The defininition of $C(X)_x$ is given in Draper [1]. The set-theoretical definitions (not as cycles) of thc above cones are given in [6]. It is proved in [1] that, for $x \in X$ and a linear subspace M of C^n of codimention d such that X and $x+M$ meet properly at x (hence $X \cap (x+M)=\{x\}$), the equality $i(X, x+M; x)=m(X,x)$ holds if and only if $|C(X)_x|$ and M meet properly at o (i.e., $|C(X)_x| \cap M=\{o\}$), and in this case we have $i(C(X)_x, M; o)=m(X,x)$.

By the curve selection lemma (Milnor [4])applied to $K^*(Y, X)$, we know that $x \in C^n$ is contained in $K(Y, X)_y$ for $y \in Y$ if and only there exist real analytic curves $y(t)$, $x(t)$, $z(t)$, $-1<t<1$, on Y, X, C, respectively, such that $y(0)=x(0)=y$, $z(0)=0$, $z(t)\neq 0$ for $t\neq 0$, and the ratio $(x(t)-y(t))/z(t)$ converges to x as t goes to O.

(1.2). Let X, Y be as in (1.2). We denote the coordinates of $C^n\times C^n\times C$ by (y,x,z). As usual, we let $O_n\{x\} \subset O_{2n}\{y,x\}$, and $O_{2n}\{y,x\} \subset O_{2n+1}\{y,x,z\}$. For $f(y,x) \in O_{2n}/I(Y\times C_n)$, expanding $f(y, y+x)$ as $\Sigma_i f_i(y,x)$, where f_i is homogeneous in x of degree i, we define the non-negative integer $r(f)$ so that $f_{r(f)} \in I(Y\times C^n)$ and $f_i \in I(Y\times C^n)$ for $i<r(f)$. We let $[f]=f_{r(f)}$. For $f \in I(Y\times C^n)$, we put $r(f)=\infty$, $[f]=0$.

In Whitney [6], it is shown that $|C(X)_o|$ is the set of common zeros of initial forms f^* of all $f \in I(X)$, and that, for a hypersurface $X=\{x : f(x)=0\}$, we have $|C(X)_o|=\{x : f^*(x)=0\}$. We will generalize this result as follows.

Proposition. *Let $I(Y, X)$ denote the image of $I(K^*(Y,X))$ by the natural epimorqhism of O_{2n+1} onto O_{2n} with the kernel $(z)O_{2n+1}$. Then*

1) *$I(Y,X)$ is generated in O_{2n} by $I(Y\times C^n)$ and $\{[f] : f \in I(Y\times X)\}$.*

2) *If Y is irreducible, $I(Y,X)$ is generated by $I(Y\times C^n)$ and $\{[f] : f \in I(X)\}$.*

3) *If Y is irreducible and X is a hypersurface with $I(X)=(f)\,O_n$, then $I(Y, X)$ is generated by $I(Y\times C^n)$ and $[f]$.*

4) *If* Y *is irreducible and* $X=\{x: f(x)=0\}$ *is a hypersurface, then we have* $|K(Y,X)| = \{(y,x) : y\in Y, [f](y,x)=0\}$.

Proof: If is clear that $I(Y\times C^n) \subset I(K^*(Y, X))$. Let $f(y,x)$ be in $I(Y\times X)/I(Y\times C^n)$, then $g(y,x,z)=f(y,y+zx)$ is in $I(K^*(Y,X))$ and is congruent to $z^{r(f)}([f]+zg'(y,x,z))$ modulo $I(Y\times C^n)$, $g'\in O_{2n+1}$. Since z is not in any prime divisor of $I(K^*(Y,X))$, $[f]+zg'$ is in $I(K^*(Y,X))$. Hence $[f]$ is in $I(K^*(Y,X)) + (z)O_{2n+1}$.

Suppose now that $h(y,x,z)$ is in $I(K^*(Y,X))$. Expand h as $\Sigma a_{IJK}y^Ix^Jz^K$, where $I=(i_1,\cdots,i_n)$, $J=(j_1,\cdots,j_n)$ are multi-indices. Put $h_m(y,x,z) = \Sigma_{|J|=k+m}a_{IJK}y^Ix^Jz^K$, where $|J|=j^1+\cdots+j^n$, and $h^{(m)}(y,x)=h^m(y,x,1)$, then $ord_x h^{(m)} \geq m$ and $h_m(y,x,z)=h^{(m)}(y,zx)z^{-m}$. For $0\neq a\in C$, $h(y,ax,a^{-1}z)=\Sigma a^m h_m(y,x,z)$ and we have that $h_m\in I(K^*(Y,X))$. Put $h'_m(y,x)=h^{(m)}(y,x-y)$ then $h'_m\in I(Y\times X)$ and $h^{(m)}(y,x)=h'_m(y,y+x)$. If $ord_x h^{(m)}>m$ then $h_m\in(z)O_{2n+1}$. If $ord_x h^{(m)}=m$then $h_m(y,x,0)$ is the initial form of $h^{(m)}(y,x)$ with respect to x. And if the initial form is not in $I(Y\times C^n)$, then it is equal to $[h'_m]$. We thus conclude that, for each m, h_m is in the ideal of O_{2n+1} generated by $I(Y\times C^n)$, $\{[f]: f\in I(Y\times X)\}$ and $(z)O_{2n+1}$, and so is h since an ideal is closed by the uniform convergence (or the maximal ideal adic topology). This proves 1).

Any g in $I(Y\times X)$ is of the from $g''(y,x) + g'(y,x)f(x)$ with $g''\in I(Y\times C^n)$, $f\in I(X)$. In this case $r(g)=r(g'f)$. Recall that $r(g)$ is the order of g, considered as an element of the ring $(O_n\{y\}/I(Y))[[x]]$ of formal power series with coefficients in $O_n\{y\}/I(Y)$. So if Y is irreducible, $r(g)=r(g')+r(f)$ and consequently $[g]$ is congruent to $[g'][f]$ modulo $I(Y\times C^n)$. This proves 2) and 3).

In the case of 4) let $g(z)$ be a generator of $I(X)$. Take a natural number k and functions $g'(x)$, $f'(x)$ such that $f=gg'$, $g^k=ff'$, then by the above argument $[f]$ is congruent to $[g][g']$ and $[g]^k$ to $[f][f']$ modulo $I(Y\times C^n)$. This proves 4), since $|K(Y,X)|$ is the set of common zeros of functions in $I(Y,X)$.

(1.4). Let X, Y be as in (1.2). If $dim\ K(Y,X)_o=d$, then X is equimultiple along Y near o.

Proof: Let $Y=\cup Y_i$ be the irreducible decomposition, then $K(Y,X)_o=\cup K(Y_i,X)_o$, hence $dim\ K(Y_i,X)_o=d$ for all i with $o\in Y_i$. Consequently we may assume that Y is irreducible in U (hence equidimentional). Then $K^*(Y,X)=d+r+1$, is also equidimentional. By a coordinate linear tranceformation of C^n, if necessary, we may assume that the linear subspace $L=\{x:x_1=\cdots=x_r=0\}$ intersects with Y and the linear subspace $M=\{x:x_1=\cdots=x_d=0\}$ with $K(Y,X)_o$ properly at o, respectively. By contracting U, if necessary, we may also assume that, for any affine subspace L, parallel to and close to L, we have $Y\cap L'=\{y^1,\cdots,y^k\}$ with

1) $\Sigma i(Y,L';y^j)=i(Y,L;o)$. Suppose

2) $K(Y,X)\cdot(L'\times C^n)=\Sigma_j\ i(Y,L'\ y^j)\ (y^j)\times C(X)_{y^j}$.

Then $K(Y,X)\cdot(L'\times M)=\Sigma_j i(Y,L';y^j)m(X,y^j)((y^j)\times(o))$ because $i(C(X)_{y^j},M;\ o)=m(X,y^j)$ (1.2). Since $\Sigma_j i\ (K(Y,X),L'\times M;\ (y^j,o))$ remains constant when L' moves, we have

3) $\Sigma_j i(Y,L';y^j)m(X,y^j)=i(Y,L;o)m(X,o)$.

The upper semicontinuity of the multiplicity $m(X,y^j)\leqq m\ (X,o)$ together with 1) and 3) shows the equalities $m(X,y^j)=m\,(X,o)$ for $1\leq j\leq k$. Since L' is arbitrary, $m(X,y)=m\ (X,o)$ for all $y\in Y$ close to o.

We now prove 2). Since $(K(Y,X)\cdot(L'\times C^n))\times(o)=(K(Y,X)\times(o))\cdot(L'\times C^n\times C)$, and $K(Y,X)\times(o)=K^*(Y,X)\cdot(C^{2n}\times(o))$, we have

4) $(K(Y,X)\cdot(L')\times C^n))\times(o)=(K^*(Y,X)\cdot(L'\times C^n\times C))\cdot(C^{2n}\times(o))$.

We denote $C^{2n}\times(o)$ by $\{z=o\}$, and $\{(y,x,z):z\neq o\}$ by $\{z\neq o\}$, respectively. Since $K^*(Y,X)\cup(L'\times C^n\times C)\cap\{z\neq o\}=\cup_{j=1}^{k}K^*(\{y^j\},X)\cap\{z\neq o\}$ and $K^*(Y,X)\cap(L'\times C^n\times C)\cap\{z=o\}$ (*resp.* $K^*(\{y^j\},X)\cap\{z=o\}$) is nowhere dense in $K^*(Y,X)\cap(L'\times C^n\times C)$ (*resp.* $K^*(\{y^j\},X)$), we have $K^*(Y,X)\cap(L'\times C^n\times C)=\cup_j K^*(\{y^j\},X)$.

Let x^o be a regular point of X, then $(y^j,x^o-y^j,1)$ is a regular point of $K^*(\{y^j\},X)$, Note that each component of $K^*(\{y^j\},X)$ contains a regular point of such a form. Choose linearly independent linear functions $p_1(x),\cdots,p_d(x)$ so that the affine subspace $A=\{x:p_i(x-x^o)=0,\ for\ 1\leq i\leq n\}$ meets X transversally at x^o. Then the affine subspace $B=\{(y,x-y,1):y\in C^n,\ x\in A\}$ of C^{2n+1} meets $K^*(\{y^j\},X)$ transversally at $(y^j,x^o-y^j,1)$. we consider the holomorphic map p of $K^*(Y,X)$ into C^{r+d+1} defined by

$p(y, x, z)=(y_1, \cdots, y_r, p_1(x+y), z)$. locally around $(y^j, x^o-y^j, 1)$. For a generic point $(b_1,\cdots,b_r, a_1,\cdots,a_d,c)$ close to $(y_j^1,\cdots,y_r^j, p_1(x^o),\cdots,p_d(x^o),1)$, there are exactly $i=i(Y,L';y^j)$ points $q^1, \cdots, q^i$ of Y such that $(q^s_1,\cdots,q^s_r)=(b_1,\cdots,b_d)$, for $1\leq s\leq i$. Fix s, and put $a'_t=ca_t+(1-c)p_t(q^s)$, then $p_t(x+q^s)=a_t$ is equivalent to $p_t(q^s+cx)=a_t$, for $1\leq t\leq d$. The latter system of equations determines the unique point q^s+cx of X close to x^o, whence the former system determines the unique point x close to x^o-y^j. This concludes that the covering multiplicity of p at $(y^j, x^o-y^j, 1)$, which is equal to $i(K^*(Y,X), (L'\times C^{n+1})\cap B; (y^j, x^o-y^j, 1))$, is $i(Y, L'; y^j)$. Thus we have $K^*(Y,X)\cdot(L'\times C^{n+1})=\Sigma_j i(Y,L';y^j)\, K^*(\{y^j\}, X)$. Taking the intersection products of the both sides with $\{z=0\}$, we have, by 4), $K(Y, X)\cdot(L'\times C^n)\times(o)=\Sigma i(Y, L';y^j)\,(y^j)\times C(X)_y^j\times(o)$. This proves 2).

(1.5) Let X,Y be in (1.2). If X is equimultiple along Y near o, then $K(Y,X)_o=|C(X)_o|$.

Prooj: Assume $K(Y,X)_o\subset|C(X)_o|$ then by a coordinate linear transformation of C^n, we may assume that the linear subspace $M=\{x:x_1=\cdots=x_d=0\}$ intersects with $|C(X)_o|$ properly at o, and $e_n=(0,\cdots,0,1)$ is in $K(Y,X)_o$ but not in $|C(X)_o|$. Then $i(X,M;o)=m(X,o)$. Since the map $y\to i(X,y+M;y)$ is upper semicontinuous, we have $m(X,o)=i(X,M;o)\geqq i(X,y+M;y)\geqq m(X,y)=m(X,o)$. This proves the equality $i(X,y+M;y)=m(X,y)$ which is equivalent to the fact that $|C(X)y|$ and M meet properly at o. Note that $O_d\to O_n/I(X)$ is finite injective. Let $f(x',x_n)=x_n^m+\Sigma_{j=1}^m a_j(x')\,x_n^{m-j}$, with $x'=(x_1,\cdots,x_d)$, be the minimal polynomial of x_n *mod.* $I(X)$ over O_d. Then $m=m(X,o)$. Since e_n is not in $|C(X)_y|$, the order of a_j at y is$\geqq j$, for $1\leq j\leq m$ and $y\in Y$.

Let $y(t),x(t),z(t)$ be real analytic curves defining e_n as stated in (1.2), and put $f_t(x)=f(x+y(t))$. Since $ord_x f_t(x)=m$, the initial form f_t^* of f_t coverges to $f_o^*=f^*$ as t goes to o. Put $e(t)=(x(t)-y(t))/z(t)$. Then we have $0=f_t(z(t)\,e(t))=z(t)^m(f_t^*(e(t))+z(t)g(t))$ with some real analytic function g. Since $z(t)\neq 0$ for $t\neq 0$, we have $f_t^*(e(t))+z(t)\,g(t)=0$ for all t. But this cannot occur since $f_o^*(e(o))=f^*(e_n)=1$. This contradiction proves that $K(Y,X)_o\subset|C(X)_o|$ and hence $K(Y,X)_o=|C(X)_o|$.

(1.4) and (1.5) may be combined into the following theorem.

(1.6). **Theorem.** *Let* $X \ni o$ *be an equidimentional analytic set in a neighbourhood of* o *in* C^n, $Y \ni o$ *an analytic subset of* X, *then the following statements are equivalent*:

1) X *is equimultiple along* Y.
2) $\dim K(Y,X)_o = \dim X$.
3) $K(Y,X)_o = |C(X)_o|$.

(1.7). Let X, Y be as in (1.6) and we assume Y is smooth. Let $p : C_{X,Y} \to Y$ be the normal cone of Y in X (Hironaka [3]), and $T(Y)_o$ be the tangent space of Y at o. Then $K(Y,X)_0$ is isomorphic to $p^{-1}(o) \times T(Y)_o$ as reduced analytic spaces, and (1.6) says that the equimultipleness of X along Y is equivalent to the normally pseudo-flatness of X along Y in the sense of [3].

(1.8). *Corollary* to (1.6). $K(X,X)_o$ $(=C_5(X)_o[6])$ has the same dimention as X if and only if X is smooth at o.

§ 2.

(2.1). Let $F_1=(f_1,\cdots,f_r)$ be a holomorphic map germ from (C^n,o) to (C^r,o), and $F_o=(f_o, F_1)$ from (C^n,o) to (C^{r+1},o). We let i represent 0 or 1. Let the coordinates of $C^n \times C^{r+1-i} \times C$ be $(x;y,z)$, where y denotes $(y_o,y_1,\cdots,y_r)$ or $(y_1,\cdots,y_r)$ according to the value of i. Put $Y_i=F_i^{-1}(o)$ and let $X_i{}^*$ be the closure of $\{(x, F_i(x)/z, z) : 0 \neq z \in C\}$ in $C^{n+r+2-i}$. The equieimentional (*dimention* $= n$) analytic cycle $N(F_i)$ defined by $X_i{}^* \cdot \{z=0\} = N(F_i) \times (o)$ is called the ***normal cone*** of the (not necessarily) reduced analytic subset $(Y_i,O_n/(F_i)O_n)$ in (C^n,O_n). For $x \in Y_i$, the cone $N(F_i)_x$ in C^{r+1-i} is defined by $|N(F_i)| \cap (x \times C^{r+1-i}) = (x) \times N(F_i)_x$.

Let I_i be the ideal in $O_{n+r+1-i}\{x,y\}$ such that $(I_i, z)\, O_{n+r+2-i} = I(X_i{}^*) + (z)O_{n+r+2-i}$. Then the normal cone of $(Y_i,O_n/(F_i)O_n)$ in (C^n,O_n) defined in [2] or [3] is essentially equal to the analytic space $(|N(F_i)|, O_{n+r+1-i}/I_i)$.

(2.2). By the curve selection lemma applied to $X_o{}^*$, y_o-axis is not contained in $|C(X_o{}^*)_o|$ (or equivalently y_o-axis is not contained in $N(F_o)_o$) if and only if for any holomorphic curve $x(t) : \{t \in C : |t| < 1\} \to C^n$ with $x(0) = 0$, we have $ord_t\, f_o\, (x(t)) \geq \inf_{1 \leq j \leq r} ord_t\, f_j\, (x(t))$. In this case we have

$m(X_0^*, o) = m(X_1^*, o)$.

Note that if f_0 is integral over $(f_1, \cdots, f_r)$, the above condition is satisfied. The converse is also true as proved in [2] or [5]. We will give here an elementary proof of this. So, assume y_0-axis is not contained in $|C(X_0^*)_0|$.

Choose linearly inedependent linear functions $p_0(x, y', z), \cdots, p_n(x, y', z)$ with $y' = (y_1, \cdots, y_r)$ so that the map $p = (p_0, \cdots, p_n) : X_0^* \to C^{n+1}$ has the covering multiplicity $m = m(X_0^*, o)$ at o.

Then $p^* : O_{n+1} \to O_{n+r+2} / I(X_0^*)$ is a finite injection. Let $g(p_0, \cdots, p_n, y_0) = y_0^m + a_1(p) y_0^{m-1} + \cdots + a_m(p)$ be the minimal polynomial of y_0 *mod.* $I(X_0^*)$ over O_{n+1}. Then $ord\ a_j \geqq j$ for $1 \leq j \leq m$ and $h(x, y, z) = y_0^m + \Sigma_{j=1}^m a_j(p(x, y', z)) y_0^{m-j}$ is in $I(X_0^*)$. Expand $a_j(p(x, y' z))$ as $\Sigma_{JK} a_{jJK}(x) y'^J z^K$ and put $a_j^*(x, y', z) = \Sigma_{|J| = j+K} a_{jJK}(x) y'^J z^K$, where $J = (j_1, \cdots, j_r)$ is multi-indices and $|J|$ denotes $j_1 + \cdots + j_r$. Then $ord_y, a_j^* \geqq j$ and by the similar argument as in (1.3) $h^* = y_0^m + \Sigma_{j=1}^m a_j^* y_0^{m-j}$ is in $I(X_0^*)$. This implies the equality $f_0(x)^m + \Sigma_{j=1}^m a_j^*(x, F_1(x), 1) f_0^{m-j} = 0$ which is an integral equation over $(f_1, \cdots, f_r)$ of degree m.

We summarize this result as follows.

(2.3). *Let $f_1, \cdots, f_r$ be elements of the maximal ideal of O_n, and $m = m(X_1^*, o)$ be as in* (2.2). *Then for $f \in O_n$, the following statements are equivalent*:

1) *f is integral over $(f_1, \cdots, f_r)$.*

2) *f satisfies an integral equation over $(f_1, \cdots, f_r)$ of degree m.*

3) *For any curve $x(t) : \{t \in C : |t| \leq 1\} \to C^n$ with $x(0) = 0$, $ord_t\ f(x(t)) \geqq inf_{1 \leq j \leq r}\ ord_t\ f_j(x(t))$,*

(2.4). *Remark.* In (2.2) the essential condition for $p : X_0^* \to C^{n+1}$ is that p is independent of y_0. If $F_1 = (f_1, \cdots, f_n) : C^n \to C^n$ is a finite (m'-sheeted) holomorphic map germ, then we can choose p as $p_0 = z$, $p_i = y_j$ for $1 \leq j \leq n$. In such a case if f is integral over $(f_1, \cdots, f_n)$ we obtain an integral equation over $(f_1, \cdots, f_n)$ of degree m'. For example, if $f \in O_n$ has an isolated singularity at o, then f satisfies an integral equation over $(\partial f / \partial x_1, \cdots, \partial f / \partial x_n)$ of *degree = the Milnor number of f.*

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