On the Estimation of Mixing Ratio of Two Distributions

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1. Introduction

P. R. Rider has estimated the parameters of the mixture of two exponential distributions [6], A. C. Cohen, the parameters of the mixture of two normal distributions [4], W. R. Blischke, the parameters of the mixture of two binomial distributions [1] and those of the mixture of r binomial distributions [2]. They have defined estimators for the parameters of the mixtures of the univariate distributions. We will pay our attention to the mixing ratio of the mixture.

A sample from a mixed population gives us many kinds of information for the mixing ratio. For example, if a population is a mixed population consisting of males and females, then the mixing ratio is a common parameter for the stature, the weight, the girth of the chest, and etc. If we can take independent random samples in succession, then we need only make the sample size large. However, in general, this is not the case. Suppose we have excavated some skeletons in some area, and we want to estimate the mixing ratio of males and females, then the only independent samples avairable are the skeletons which have been excavated. So, we consider the case when the number of the independent random samples is limited.

Let the population π be a mixed population of a population π_1 and a population π_2. Let the distribution function of the random vector $(X_1, X_2, \cdots, X_k)$ under π be

1) $$f(x_1, x_2, \cdots, x_k) = \theta f_1(x_1, x_2, \cdots, x_k) + (1-\theta) f_2(x_1, x_2, \cdots, x_k), \quad 0 \leqq \theta \leqq 1,$$

where $f_1(x_1, x_2 \cdots, x_k)$ and $f_2(x_1, x_2, \cdots, x_k)$ are the distribution functions of $(X_1, X_2, \cdots, X_k)$ under π_1 and π_2, respectively. In this note, we consider the estimation of the mixing ratio θ of π_1 and π_2.

In Section 2, we will introduce some notations which will be used in Section 3, 4, 5. In Section 3, we will study a basic theorem and will investigate an estimator when the means are known. In Section 4, we will investigate some estimators for the mixture of two exponential distributions. In Section 5, we will investigate some estimators in other cases.

2. Notation

Let $W_1, W_2, \cdots\cdots, W_n$ be the independent random samples from π. Let

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$X_1, X_2, \cdots\cdots, X_k$ be some measurements on W_λ, Let W_{λ_α}, $\alpha=1, 2, \cdots\cdots, n_i$ $\lambda_1 < \lambda_2 < \cdots < \lambda_{n_i}$ be the samples which have the measurement X_i. We define $X_i^{(\alpha)}$ as the measurement X_i on W_{λ_α}. Then $X_i^{(\alpha)}$ and $X_j^{(\alpha)}$ $(i \neq j)$ are not necessarily the measurements on W_{λ_α}. Hence they are not necessarily dependent each other. From now on, for convenience sake, let $X_i^{(1)}, X_i^{(2)}, \cdots\cdots, X_i^{(n_i)}$ be an independent random sample from the marginal distribution $f(x_i)$ of X_i.

Next we will give some equalities which will be used in Section 3, 4, 5. Let the mean and the variance of X_i and the covariance between X_i and X_j be

(2) $E(X_i) = m_i, \quad Var(X_i) = \sigma^2_{m_i} \quad (i=1, 2, \cdots\cdots, k),$

(3) $Cov(X_i, X_j) = \sigma_{m_i m_j} \quad (i=2, 1, \cdots\cdots, k, \; i \neq j)$

under π_1, and

(4) $E(X_i) = \ell_i, \quad Var(X_i) = \sigma^2_{\ell_i} \quad (i=1, 2, \cdots\cdots, k),$

(5) $Cov(X_i, X_j) = \sigma_{\ell_i \ell_j} \quad (i=1, 2, \cdots\cdots, k, \; i \neq j),$

under π_2. Then we have the equalities

(6) $E(X_i) = \theta m_i + (1-\theta)\ell_i \quad (i=1, 2, \cdots\cdots, k),$

(7) $Var(X_i) = \theta\sigma^2_{m_i} + (1-\theta)\sigma^2_{\ell_i} + \theta(1-\theta)(m_i - \ell_i)^2 \quad (i=1, 2, \cdots, k),$

(8) $Cov(X_i, X_j) = \theta\sigma_{m_i m_j} + (1-\theta)\sigma_{\ell_i \ell_j} + \theta(1-\theta)(m_i - \ell_i)(m_j - \ell_j) \quad (i, j = 1, 2, \cdots\cdots, k, \quad i \neq j).$

Next we will assume some conditions for the correlation coefficient. Let $\rho(X, Y)$ denote the correlation coefficient between X and Y.

We assume

(9) $|\rho(X_i, X_j)| \leqq A_1 r_1^{|i-j|},$

(10) $|\rho(X_i^2, X_j)| \leqq A_2 r_2^{|i-j|},$

(11) $|\rho(X_i^2, X_j^2)| \leqq A_3 r_3^{|i-j|},$

(12) $|\rho(X_i^{(1)}, X_j^{(1)} X_j^{(2)})| \leqq A_4 r_4^{|i-j|},$

(13) $|\rho(X_i^{(1)^2}, X_j^{(1)} X_j^{(2)})| \leqq A_5 r_5^{|i-j|},$

(14) $|\rho(X_i^{(1)} X_i^{(2)}, X_j^{(1)} X_j^{(2)})| \leqq A_6 r_6^{|i-j|},$

(15) $|\rho(X_i^{(1)} X_i^{(2)}, X_j^{(1)} X_j^{(3)})| \leqq A_7 r_7^{|i-j|},$

where A_δ, $0 \leqq r_\delta < 1$ are constant numbers and unrelated to (i, j),

3. Basic theorem and θ^* when $\{m_i\}$ and $\{\ell_i\}$ are known

Let θ be a common parameter for each component of a random vector $(X_1, X_2, \cdots\cdots, X_k)$, and the marginal density of X_i be $f(x_i)$. Let $X_i^{(1)}, X_i^{(2)}, \cdots\cdots, X_i^{(n_i)}$ $(i=1, 2, \cdots\cdots, k)$ be independent random variables distributed according to $f(x_i)$. Now, let $s_i \leqq s$, $t_i \leqq t$, $(i=1, 2, \cdots\cdots, k)$ for some positive intigers s, t, and

$$g_i(x_i^{(1)}, x_i^{(2)}, \cdots\cdots, x_i^{(n_i)}) = \sum_{\delta=1}^{s_i} K_i^{(\delta)} g_i^{(\delta)}(x_i^{(1)}, x_i^{(2)}, \cdots\cdots, x_i^{(n_i)}), \tag{16}$$

$$q_i(x_i^{(1)}, x_i^{(2)}, \cdots\cdots, x_i^{(n_i)}) = \sum_{\nu=1}^{t_i} L_i^{(\nu)} q_i^{(\nu)}(x_i^{(1)}, x_i^{(2)}, \cdots\cdots, x_i^{(n_i)}), \tag{17}$$

where $|K_i^{(\delta)}| \leqq K$, $|L_i^{(\nu)}| \leqq K$ and K is a positive number unrelated to (i, δ, ν).

Let g_i and q_i be the statistics satisfing the condition

$$E\{g_i(X_i^{(1)}, X_i^{(2)}, \cdots\cdots, X_i^{(n_i)})\} = h(\theta) E\{q_i(X_i^{(1)}, X_i^{(2)}, \cdots\cdots, X_i^{(n_i)})\} + d \tag{18}$$

$$(i=1, 2, \cdots, k),$$

where $h(\theta)$ is a function of θ only and d is constant. Then we have the equality

$$E\{\sum_{i=1}^{k} c_i g_i\} = h(\theta) E\{\sum_{i=1}^{k} c_i q_i\} + d, \tag{19}$$

where $\sum_{i=1}^{k} c_i = 1$. Now, as an estimator of θ, we define θ^* by a root of the equation

$$\sum_{i=1}^{k} c_i g_i = h(\theta^*) \sum_{i=1}^{k} c_i q_i + d. \tag{20}$$

The next theorem shows that θ^* is a consistent estimator of θ under some conditions.

Theorem. *Suppose*

$$Var\{g_i^{(\delta)}\} \leqq M, \qquad Var\{q_i^{(\nu)}\} \leqq M \tag{21}$$

$$(i = 1, 2, \cdots\cdots, k; \quad \delta = 1, 2, \cdots\cdots, s_i; \quad \nu = 1, 2, \cdots\cdots, t_i),$$

$$|\rho(g_i^{(\delta 1)}, g_j^{(\delta 2)})| \leqq A_{\delta_1 \delta_2} r_{\delta_1 \delta_2}^{|i-j|} \tag{22}$$

$$(0 \leqq r_{\delta_1 \delta_2} < 1; \ \delta_1 = 1, 2, \cdots, s_i; \ \delta_2 = 1, 2, \cdots, s_j; \ i \neq j),$$

$$|\rho(q_i^{(\nu 1)}, q_j^{(\nu 2)})| \leqq A_{\nu_1 \nu_2} r_{\nu_1 \nu}^{|i-j|}. \tag{23}$$

$(0 \leqq r_{\nu_1 \nu_2} < 1 \,;\ \nu_1 = 1, 2, \cdots, t_i \,;\ \nu_2 = 1, 2, \cdots, t_j \,;\ j \neq i)$,

(24) $C^* = \max_{1 \leqq i \leqq k} | c_i | = o\left(\frac{1}{\sqrt{k}}\right)$

where M, $A_{\delta_1 \delta_2} = A_{\delta_2 \delta_1}$, $A_{\nu_1 \nu_2} = A_{\nu_2 \nu_1}$, $r_{\delta_1 \delta_2} = r_{\delta_2 \delta_1}$, $r_{\nu_1 \nu_2} = r_{\nu_2 \nu_1}$ *are constant numbers unrelated to* (i, j).

If h^{-1} *is continuous, then one of the roots of the equation* (20) *converges to* θ *in probability as* $k \longrightarrow \infty$.

Proof. From (21), (22), we have the inequalities

(25) $$Var\{g_i\} = \sum_{\delta=1}^{s_i} K_i^{(\delta)2}\ Var\{g_i^{(\delta)}\} + \sum_{\delta 1 \neq \delta 2}^{s_i} K_i^{(\delta 1)} K_i^{(\delta 2)} Cov\left(g_i^{(\delta 1)}, g_i^{(\delta 2)}\right)$$
$$\leqq s^2 K^2 M,$$

(26) $$Cov(g_i, g_j) = \sum_{\delta 1=1}^{s_i} \sum_{\delta 2=1}^{s_j} K_i^{(\delta 1)} K_j^{(\delta 2)} Cov\,(g_i^{(\delta 1)}, g_j^{(\delta 2)})$$
$$\leqq K^2 M \sum_{\delta 1=1}^{s_i} \sum_{\delta 2=1}^{s_j} A_{\delta_1 \delta_2} r_{\delta 1 \delta 2}^{|i-j|}.$$

So, we have

(27) $$Var\left\{\sum_{i=1}^{k} c_i g_i\right\} = \sum_{i=1}^{k} c_i^2\, Var\,\{g_i\} + \sum_{i \neq j}^{k} c_i c_j Cov\,(g_i, g_j)$$
$$\leqq s^2 K^2 M \sum_{i=1}^{k} c_i^2 + K^2 M \sum_{i \neq j}^{k} | c_i c_j | \left\{ \sum_{\delta 1=1}^{s_i} \sum_{\delta 2=1}^{s_j} A_{\delta 1 \delta 2} r_{\delta 1 \delta 2}^{|i-j|} \right\}$$
$$\leqq s^2 K^2 MkC^{*2} + K^2 MC^{*2} \sum_{\delta 1=1}^{s_i} \sum_{\delta 2=1}^{s_j} A_{\delta 1 \delta 2} \left\{ \sum_{i \neq j}^{k} r_{\delta 1 \delta 2}^{|i-j|} \right\}$$
$$= s^2 K^2 MkC^{*2}$$
$$+ 2K^2 MC^{*2} \sum_{\delta 1=1}^{s_i} \sum_{\delta 2=1}^{s_j} A_{\delta 1 \delta 2} \left\{ (k-1) \frac{r_{\delta 1 \delta 2}}{1-r_{\delta 1 \delta 2}} - + \left(\frac{r_{\delta 1 \delta 2}}{1-r_{\delta 1 \delta 2}} \right)^2 (1-r_{\delta_1 \delta_2})^{k-1} \right\}.$$

Accordingly, by (24), we have

(28) $$Var\left\{\sum_{i=1}^{k} c_i g_i\right\} \longrightarrow 0 \qquad (k \longrightarrow \infty),$$

namely,

(29) $$\sum_{i=1}^{k} c_i g_i \xrightarrow{p} E\left\{\sum_{i=1}^{k} c_i g_i\right\} \qquad (k \longrightarrow \infty)\ .$$

In the same way, we have

(30) $\sum_{i=1}^{k} c_i q_i \xrightarrow{p} E\left\{\sum_{i=1}^{k} c_i q_i\right\} \quad (k \longrightarrow \infty).$

If h^{-1} is continuous, then one of the roots of the equation (20) coverges to θ in probability.

Now, we define θ^* when $\{m_i\}$ and $\{\ell_i\}$ are known. Let $\overline{X}_i = (1/n_i) \sum_{\alpha=1}^{n_i} X_i^{(\alpha)}$ and $Y_i = (\overline{X}_i - \ell_i)/(m_i - \ell_i)$ and $\theta^* = (1/k) \sum_{i=1}^{k} Y_i$, then we have $E(\theta^*) = \theta$.

Corollary 3. 1. *In addition to (9), we assume that*

(31) $\sigma^2_{m_i} \leqq M_1, \quad \sigma^2_{\ell_i} \leqq M_1, \quad M_2 \leqq |m_i - \ell_i| \quad (i = 1, 2, \cdots, k),$

where M_1, M_2 are the constant numbers unrelated to i. Then we have

$$\theta^* \xrightarrow{p} \theta \quad (k \longrightarrow \infty).$$

Proof. We have from (9), (31)

(32) $Var\left\{\frac{X_i^{(\alpha)} - \ell_i}{m_i - \ell_i}\right\} \leqq \frac{M_1}{M_2^2} + \frac{1}{4} \quad (i = 1, 2, \cdots, k),$

(33) $\left|\rho\left(\frac{X_i^{(\alpha_1)} - \ell_i}{m_i - \ell_i}, \frac{X_j^{(\alpha_2)} - \ell_j}{m_j - \ell_j}\right)\right| = \left|\rho\left(X_i^{(\alpha_1)}, X_j^{(\alpha_2)}\right)\right|$

$$\leqq A_1 r_1^{|i-j|},$$

$$(\alpha_1 = 1, 2, \cdots, n_i; \quad \alpha_2 = 1, 2, \cdots, n_j; \ i \neq j).$$

From Theorem, it will be seen that $\theta^* \xrightarrow{p} \theta$ as $k \longrightarrow \infty$.

4. θ^* when $f(x_i)$ is the mixture of two exponential distrbutions

Let the marginal density of X_i be

(34) $f(x_i) = \frac{\theta}{m_i} e^{-x_i/m_i} + \frac{1-\theta}{\ell_i} e^{-x_i/\ell_i} \quad (i = 1, 2, \cdots, k),$

then we have

(35) $E(X_i) = \theta m_i + (1-\theta)\ell_i,$

(36) $E(X_i^2) = 2\theta m_i^2 + 2(1-\theta)\ell_i^2,$

(37) $Var(X_i) = \theta m_i^2 + (1-\theta)\ell_i^2 + \theta(1-\theta)(m_i - \ell_i)^2 \quad (i=1, 2, \cdots, k).$

Let $X_i^{(1)}, X_i^{(2)}, \cdots, X_i^{(n_i)}$ be the independent random varibles according to $f(x_i)$, and

(38) $\bar{X}_i = \frac{1}{n_i} \sum_{\alpha=1}^{n_i} X_i^{(\alpha)} \qquad (i=1, 2, \cdots, k),$

(39) $\bar{Y}_i = \frac{1}{n_i} \sum_{\alpha=1}^{n_i} X_i^{(\alpha)^2} \qquad (i=1, 2, \cdots, k),$

(40) $\bar{Z}_i = \frac{1}{n_i - 1} \sum_{\alpha=1}^{n_i} (X_i^{(\alpha)} - \bar{X}_i)^2 \qquad (i=1, 2, \cdots, k),$

then we have

(41) $E(\bar{X}_i) = \theta m_i + (1-\theta) \ell_i \qquad (i=1, 2, \cdots, k),$

(42) $E(\bar{Y}_i) = 2\theta m_i^2 + 2(1-\theta) \ell_i^2 \qquad (i=1, 2, \cdots, k),$

(43) $E(\bar{Z}_i) = \theta m_i^2 + (1-\theta) \ell_i^2 + \theta (1-\theta)(m_i - \ell_i)^2 \ (i=1, 2, \cdots, k).$

So, we have the equalities

(44) $E(\bar{Z}_i - \frac{1}{2}\bar{Y}_i) = \theta(1-\theta)(m_i - \ell_i)^2 \qquad (i=1, 2, \cdots, k),$

(45) $E(\sum_{i=1}^{k} \bar{Z}_i - \frac{1}{2}\sum_{i=1}^{k} \bar{Y}_i) = \theta(1-\theta)\sum_{i=1}^{k} (m_i - \ell_i)^2 \qquad (i=1, 2, \cdots, k).$

We will define θ^* in two cases

Case I. $\{m_i - \ell_i\}$ is known and $m_i - \ell_i \neq 0$ for some i.

In this case, as an estimator of θ, we define θ^* by a root of the equation

(46) $\sum_{i=1}^{k} (\bar{Z}_i - \frac{1}{2}\bar{Y}_i) = \theta^* (1-\theta^*) \sum_{i=1}^{k} (m_i - \ell_i)^2,$

and the other as an estimator of $1-\theta$.

Corollary 4. 1. *In addition to* (11)~(14), *we assume that*

(47) $|m_i| \leqq M, \quad |\ell_i| \leqq M \qquad (i=1, 2, \cdots, k),$

where M *is a constant number unrelated to* i, *then* $\theta^* \xrightarrow{p} \theta \ (k \longrightarrow \infty)$.

Proof. As it can easily be seen, we have

(48) $Var(X_i) \leqq 2M^2, \ Var(X_i^2) \leqq 48M^2 \qquad (i=1, 2, \cdots, k),$

(49) $Var\{X_i^{(\alpha)} X_i^{(\beta)}\} \leqq 5M^4 \qquad (i=1, 2, \cdots, k; \ \alpha \neq \beta),$

(50) $\bar{Z}_i - \frac{1}{2}\bar{Y}_i = \frac{1}{2n_i} \sum_{\alpha=1}^{n_i} X_i^{(\alpha)^2} - \frac{1}{n_i(n_i-1)} \sum_{\alpha \neq \beta}^{n_i} X_i^{(\alpha)} X_i^{(\beta)},$

then, by Theorem, $\theta^* \xrightarrow{p} \theta$ as $k \longrightarrow \infty$.

Case II. $\{m_i\}$ is unknown and $\{\ell_i\}$ is known.

From (41),

$$(51)\quad m_i = \frac{E(\overline{X}_i) - (1-\theta)\ell_i}{\theta} \qquad (i=1, 2, \cdots, k).$$

If we substitute (51) for m_i of (44), we have

$$(52)\quad E(\overline{Z}_i - \frac{1}{2}\overline{Y}_i) = (\frac{1}{\theta} - 1)E\Big\{(X_i - \ell_i)^2 + \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2\Big\}.$$

Accordingly,

$$(53)\quad E\sum_{i=1}^{k}(\overline{Z}_i - \frac{1}{2}\overline{Y}_i) = (\frac{1}{\theta}-1)E\sum_{i=1}^{k}\Big\{(X_i - \ell_i)^2 - \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2\Big\}.$$

So, we define θ^* by the root of the equation

$$(54)\quad \sum_{i=1}^{k}(\overline{Z}_i - \frac{1}{2}\overline{Y}_i) = (\frac{1}{\theta^*}-1)\sum_{i=1}^{k}\Big\{(\overline{X}_i - \ell_i)^2 - \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2\Big\},$$

namely,

$$(55)\quad \theta^* = \frac{\sum_{i=1}^{k}\Big\{(X_i - \ell_i)^2 - \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2\Big\}}{\sum_{i=1}^{k}\Big\{\frac{1}{n_i}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2 - \frac{1}{2n_i}\sum_{\alpha=1}^{n_i}X_i^{(\alpha)2} + (\overline{X}_i - \ell\ )^2\Big\}}.$$

Corollary 4.2. *Assume* (9)~(15), (47), *then* $\theta^* \xrightarrow{p} \theta$ *as* $k \longrightarrow \infty$.

Proof.

$$(\overline{X}_i - \ell_i)^2 - \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}(X_i^{(\alpha)} - \overline{X}_i)^2$$

$$= \frac{1}{n_i(n_i-1)}\sum_{\alpha\neq\beta}^{n_i}X_i^{(\alpha)}X_i^{(\beta)} - \frac{1}{n_i(n_i-1)}\sum_{\alpha=1}^{n_i}X_i^{(\alpha)2} - \frac{2\ell_i}{n_i}\sum_{\alpha=1}^{n_i}X_i^{(\alpha)} + \ell_i^2.$$

Accordingly, by Theorem and Corollary 4. 1, θ^* converges to θ in probability as $k \rightarrow \infty$.

5. θ^* when $\sigma^2_{m_i} = \sigma^2_{\ell_i} = \sigma^2$ and σ^2 is known

If $\sigma^2_{m_i} = \sigma^2_{\ell_i} = \sigma^2$ $(i=1, 2, \cdots, k)$, we have

$$(56)\quad E(X_i) = \theta m_i + (1-\theta)\ell_i \qquad (i=1, 2, \cdots, k),$$

(57) $Var\ (X_i) = \sigma^2 + \theta\ (1-\theta)\ (m_i - \ell_i)^2 \qquad (i=1,\ 2,\ \cdots,\ k).$

We will consider two cases as in the last case.

Case I. $\{m_i - \ell_i\}$ is known and $m_i - \ell_i \neq 0$ for some i.

If we define $\bar{Z}_i$ by (40), then we have

(58) $$E\ (\sum_{i=1}^{k} \bar{Z}_i) = k\sigma^2 + \theta\ (1-\theta) \sum_{i=1}^{k} (m_i - \ell_i)^2.$$

So we define θ^* by a root of the equation

(59) $$\sum_{i=1}^{k} \bar{Z}_i = k\ \sigma^2 + \theta^*\ (1-\theta^*) \sum_{i=1}^{k} (m_i - \ell_i)^2,$$

as an estimator of θ, and the other as an estimator of $1-\theta$.

Corollary 5. 1. *In addition to* (11), (13)~(15), *we assume*

(60) $|\ m_i - \ell_i\ | \leqq M_1 \qquad (i = 1,\ 2,\ \cdots,\ k),$

then θ^* *converges to* θ *in probability as* $k \to \infty$.

Proof.

(61) $$\bar{Z}_i = \frac{1}{n_i} \sum_{\alpha=1}^{n_i} X_i^{(\alpha)2} - \frac{1}{n_i\ (n_i - 1)} \sum_{\alpha \neq \beta}^{n_i} X_i^{(\alpha)} X_i^{(\beta)} \quad (i=1,\ 2,\ \cdots,\ k),$$

and $|\ E(X_i)\ | \leqq E\ |\ X_i\ | \leqq [E(X_i^2)]^{1/2} \leqq \sqrt{M_1}$. Hence, by Theorem, θ^* converges to θ in probability as $k \to \infty$.

Case II. $\{m_i\}$ is unknown and $\{\ell_i\}$ is known.

In the same way as in the case II of the last section, we have

(62) $$E(\sum_{i=1}^{k} \bar{Z}_i) = k\sigma^2 + (\frac{1}{\theta} - 1)\ E \sum_{i=1}^{k} \Big\{(\bar{X}_i - \ell_i)^2 - \frac{1}{n_i\ (n_i - 1)} \sum_{\alpha=1}^{n_i} (X_i^{(\alpha)} - \bar{X}_i)^2\Big\}.$$

So, as an estimator of θ, we define θ^* by the root of the equation

(63) $$\sum_{i=1}^{k} \bar{Z}_i = k\ \sigma^2 + (\frac{1}{\theta^*} - 1) \sum_{i=1}^{k} \Big\{(\bar{X}_i - \ell_i)^2 - \frac{1}{n_i\ (n_i - 1)} \sum_{\alpha=1}^{n_i} (X_i^{(\alpha)} - \bar{X}_i)^2\Big\},$$

namely,

(64) $$\theta^* = \frac{\sum_{i=1}^{k} \Big\{(\bar{X}_i - \ell_i)^2 + \frac{1}{n_i\ (n_i - 1)} \sum_{\alpha=1}^{n_i} (X_i^{(\alpha)} - \bar{X}_i)^2\Big\}}{\sum_{i=1}^{k} \Big\{\frac{1}{n_i} \sum_{\alpha=1}^{n_i} (X_i^{(\alpha)} - \bar{X}_i)^2 + (\bar{X}_i - \ell_i)^2\Big\} - k\ \sigma^2}$$

By Theorem and Corollary 5. 1, we have next corollary.

Corollary 5. 2. *Assume* (9)~(15), (60), *then* $\theta^* \xrightarrow{p} \theta$, *as* $k \longrightarrow \infty$.

In particular, let $f(x_i)$ be a mixture of two normal distributions, i.e.,

$$(65)\quad f(x_i) = \frac{\theta}{\sqrt{2\pi}} e^{-(x_i - m_i)^2/2} + \frac{1-\theta}{\sqrt{2\pi}} e^{-(x_i - \ell_i)^2/2}.$$

We have next corollaries.

Corollary 5. 3. *Assume* (11), (13)~(15), (47), *then* θ^* *of* (59) *converges to* θ *in probability as* $k \to \infty$.

Corollary 5. 4. *Assume* (9)~(15), (47), *then* θ^* *of* (64) *converges to* θ *in probability as* $k \to \infty$.

Peferences

[1] Blischke W. R., Moment estimators for the parameters of a mixture of two binomial distributions, *Ann. Math. Stat.*, Vol. 33 (1962) 444-454.

[2] Blischke W. R., Estimating the parameter of binomial distributions, *Journal o, the Am. Stat. Association*, Vol. 59 (1964), 510-528.

[3] Bowman K. O., Space of solutions for a normal mixture, *Biometrika*, Vol. 60 (1973), 629-636.

[4] Cohen A. C., Estimation in mixture of two normal distributions, *Technometrics*, Vol. 9 (1967), 15-28.

[5] Rao C. R., *Advanced Statistical Methods in Biometric Research*. New York ; John Wiley and Sons, Inc., 1952.

[6] Rider P. R., The method of moments applied to a mixture of two exponential distributions. *Ann. Math. Stat.* Vol. 32 (1961), 143-147.