FINITE PART OF L (1, x)

FINITE PART OF L $(1, \chi)$

by

Akira TAKAKU*

Abstract. Let k be a positive integer and $\chi(n)$ a real non-principal character $(\bmod k)$. Let ε be any positive number, then we have

$$\sum_{n=1}^{t} \chi(n)\ n^{-1} > 0,$$

provided $k \geqslant k_0(\varepsilon)$ and $t \geqslant [k^{\frac{1}{4}+\varepsilon}]+1$, if k is cubefree and $t \geqslant [k^{\frac{3}{8}+\varepsilon}]+1$, otherwise.

1. Let k be a positive integer and $\chi(n)$ be a real non-principal character $(\bmod\ k)$. S. Chowla [2] proposed the following question: Is it true that

$$\sum_{n=1}^{t} \chi(n)\ n^{-1} > 0,$$

where t is any number ?

Theorem. *Let k be a positive integer and $\chi(n)$ a real non-principal character* $(\bmod\ k)$. *Let ε be any positive number, then we have*

$$\sum_{n=1}^{t} \chi(n)\ n^{-1} > 0$$

provided $k \geqslant k_0(\varepsilon)$ and $t \geqslant [k^{\frac{1}{4}+\varepsilon}]+1$, if k is cubefree and $t \geqslant [k^{\frac{3}{8}+\varepsilon}]+1$, otherwise. Here $k_0(\varepsilon)$ is a positive constant depending on ε.

In § 2 we prove a lemma and in § 3 we prove Theorem.

2. Lemma.

Let k be a positive integer and $\chi(n)$ a real non-principal character $(\bmod\ k)$. D. A. Burgess [1] proved following: *Let δ be a fixed positive number and r be any fixed integer > 0. Then if either k is cubefree or $r=2$, for any integer $H > 0$, we have*

$$\left| \sum_{n=1}^{H} \chi(n) \right| < c_1\ H^{1-1/r}\ k^{(r+1)/(4r^2)+\delta}$$

where $c_1 = c_1(\delta, r)$ is a positive number depending on δ and r.

Lemma. *Let ε be any fixed positive number. Then we have*

$$\left| \sum_{n=t+1}^{\infty} \chi(n)\ n^{-1} \right| < c_2 k^{-\varepsilon_1}$$

for $t \geqslant [k^{\frac{1}{4}+\varepsilon}]+1$, if k is cubefree and $t \geqq [k^{\frac{3}{8}+\varepsilon}]+1$, otherwise. Here $c_2 = c_2(\varepsilon)$ and $\varepsilon_1 = \varepsilon_1(\varepsilon)$ are positive numbers depending on ε.

Recieved: April 30, 1975

* Division of General Education, Ryukyu Univ.

Proof. Let k be cubefree. Let ε be any fixed positive number. We choose at first an integer r such that $(r+1)/(4r) < 1/4 + \varepsilon/2$ and we find a positive number δ such that $\delta r < \varepsilon/2$. Put

$$S(H) = \sum_{n=1}^{H} \chi(n).$$

Then

$$\sum_{t+1}^{\infty} \frac{\chi(n)}{n} = \sum_{t}^{\infty} \{S(n+1) - S(n)\} \frac{1}{n+1}$$

$$= -\frac{S(t)}{t+1} + \sum_{t+1}^{\infty} S(n) \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Hence

$$\left| \sum_{t+1}^{\infty} \right| \leq \frac{|S(t)|}{t+1} + \sum_{t+1}^{\infty} |S(n)| \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$< c_1 t^{-1/r} k^{(r+1)/(4r^2)+\delta} + c_1 k^{(r+1)/(4r^2)+\delta} \sum_{t+1}^{\infty} n^{-1-1/r}$$

$$< c_1 \left(1 + \frac{1}{r}\right) t^{-1/r} k^{(r+1)/(4r^2)+\delta}$$

$$< c_1 \left(1 + \frac{1}{r}\right) k^{-f/r+(r+1)/(4r^2)+\delta}$$

$$= c_1 \left(1 + \frac{1}{r}\right) k^{-\varepsilon_1},$$

where $f = 1/4 + \varepsilon$, $c_2 = c_1(1+1/r) = c_2(r)$ and $\varepsilon_1 = \{1/4 + \varepsilon - (r+1+\delta r)/(4r)\}/r > 0$. ε_1 and ε are positive constants depending on ε and independent of t. If k is not cubefree, we take $r = 2$. The proof of Lemma is similar to above one.

3. Proof of Theorem.

For any fixed positive number ε, we choose positive numbers $c_2 = c_2(\varepsilon)$ and $\varepsilon_1 = \varepsilon_1(\varepsilon)$ given by Lemma. Let ε_2 be a fixed positive number $< \varepsilon_1$. By Siegel's theorem[3] we can find a positive number $c_3 = c_3(\varepsilon_2)$ such that

$$L(1, \chi) = \sum_{1}^{\infty} \chi(n) n^{-1} > c_3 k^{-\varepsilon_2}.$$

Hence we have

$$(c_3 k^{-\varepsilon_2}) / (c_2 k^{-\varepsilon_1}) > 1 \quad (k \geqslant k_0(\varepsilon)).$$

On the other hand, by Lemma

$$L(1, \chi) - c_2 k^{-\varepsilon_1} < \sum_{1}^{t} \chi(n) n^{-1}.$$

Therefore

$$L(1, \chi) - c_2 k^{-\varepsilon_1} > c_3 k^{-\varepsilon_2} - c_2 k^{-\varepsilon_1} > 0$$

for $k \geqslant k_0(\varepsilon)$. This completes the proof.

References.

1. Burgess, D. A., *On character sums and* L-*series* II, Proc. London Math. Soc. (3) **13** (1963) 524–36.
2. Chowla, S., *The Riemann zeta and allied functions,* Bull. of Amer. Math. Soc. **58** (1952) 287–305.
3. Siegel, C. L., *Über die Classenzahl quadratischer Körper. Acta Arith.* **1** (1936) 83–86.