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## FINITE PART OF $L(1, \chi)$

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FINITE PART OF  $L(1, \chi)$ 

by

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**Abstract.** Let  $k$  be a positive integer and  $\chi(n)$  a real non-principal character (mod  $k$ ). Let  $\varepsilon$  be any positive number, then we have

$$\sum_{n=1}^t \chi(n) n^{-1} > 0,$$

provided  $k \geq k_0(\varepsilon)$  and  $t \geq [k^{\frac{1}{4} + \varepsilon}] + 1$ , if  $k$  is cubefree and  $t \geq [k^{\frac{3}{8} + \varepsilon}] + 1$ , otherwise.

1. Let  $k$  be a positive integer and  $\chi(n)$  be a real non-principal character (mod  $k$ ). S. Chowla [2] proposed the following question: Is it true that

$$\sum_{n=1}^t \chi(n) n^{-1} > 0,$$

where  $t$  is any number?

**Theorem.** Let  $k$  be a positive integer and  $\chi(n)$  a real non-principal character (mod  $k$ ). Let  $\varepsilon$  be any positive number, then we have

$$\sum_{n=1}^t \chi(n) n^{-1} > 0$$

provided  $k \geq k_0(\varepsilon)$  and  $t \geq [k^{\frac{1}{4} + \varepsilon}] + 1$ , if  $k$  is cubefree and  $t \geq [k^{\frac{3}{8} + \varepsilon}] + 1$ , otherwise. Here  $k_0(\varepsilon)$  is a positive constant depending on  $\varepsilon$ .

In § 2 we prove a lemma and in § 3 we prove Theorem.

## 2. Lemma.

Let  $k$  be a positive integer and  $\chi(n)$  a real non-principal character (mod  $k$ ). D. A. Burgess [1] proved following: Let  $\delta$  be a fixed positive number and  $r$  be any fixed integer  $> 0$ . Then if either  $k$  is cubefree or  $r=2$ , for any integer  $H > 0$ , we have

$$\left| \sum_{n=1}^H \chi(n) \right| < c_1 H^{1-1/r} k^{(r+1)/(4r^2)+\delta}$$

where  $c_1 = c_1(\delta, r)$  is a positive number depending on  $\delta$  and  $r$ .

**Lemma.** Let  $\varepsilon$  be any fixed positive number. Then we have

$$\left| \sum_{n=t+1}^{\infty} \chi(n) n^{-1} \right| < c_2 k^{-\varepsilon_1}$$

for  $t \geq [k^{\frac{1}{4} + \varepsilon}] + 1$ , if  $k$  is cubefree and  $t \geq [k^{\frac{3}{8} + \varepsilon}] + 1$ , otherwise. Here  $c_2 = c_2(\varepsilon)$  and  $\varepsilon_1 = \varepsilon_1(\varepsilon)$  are positive numbers depending on  $\varepsilon$ .

*Proof.* Let  $k$  be cubefree. Let  $\varepsilon$  be any fixed positive number. We choose at first an integer  $r$  such that  $(r+1)/(4r) < 1/4 + \varepsilon/2$  and we find a positive number  $\delta$  such that  $\delta r < \varepsilon/2$ . Put

$$S(H) = \sum_{n=1}^H \chi(n).$$

Then

$$\begin{aligned} \sum_{t+1}^{\infty} \frac{\chi(n)}{n} &= \sum_t \{S(n+1) - S(n)\} \frac{1}{n+1} \\ &= -\frac{S(t)}{t+1} + \sum_{t+1}^{\infty} S(n) \left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

Hence

$$\begin{aligned} \left| \sum_{t+1}^{\infty} \right| &\leq \frac{|S(t)|}{t+1} + \sum_{t+1}^{\infty} |S(n)| \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &< C_1 t^{-1/r} k^{(r+1)/(4r^2)+\delta} + C_1 k^{(r+1)/(4r^2)+\delta} \sum_{t+1}^{\infty} n^{-1-1/r} \\ &< C_1 \left(1 + \frac{1}{r}\right) t^{-1/r} k^{(r+1)/(4r^2)+\delta} \\ &< C_1 \left(1 + \frac{1}{r}\right) k^{-f/r+(r+1)/(4r^2)+\delta} \\ &= C_1 \left(1 + \frac{1}{r}\right) k^{-\varepsilon_1}, \end{aligned}$$

where  $f=1/4 + \varepsilon$ ,  $c_2 = c_1 (1+1/r) = c_2(r)$  and  $\varepsilon_1 = 1/4 + \varepsilon - (r+1 + \delta r)/(4r) | / r > 0$ .  $\varepsilon_1$  and  $\varepsilon$  are positive constants depending on  $\varepsilon$  and independent of  $t$ . If  $k$  is not cubefree, we take  $r=2$ . The proof of Lemma is similar to above one.

### 3. Proof of Theorem.

For any fixed positive number  $\varepsilon$ , we choose positive numbers  $c_2 = c_2(\varepsilon)$  and  $\varepsilon_1 = \varepsilon_1(\varepsilon)$  given by Lemma. Let  $\varepsilon_2$  be a fixed positive number  $< \varepsilon_1$ . By Siegel's theorem[3] we can find a positive number  $c_3 = c_3(\varepsilon_2)$  such that

$$L(1, \chi) = \sum_1^{\infty} \chi(n) n^{-1} > c_3 k^{-\varepsilon_2}.$$

Hence we have

$$(c_3 k^{-\varepsilon_2}) / (c_2 k^{-\varepsilon_1}) > 1 \quad (k \geq k_0(\varepsilon)).$$

On the other hand, by Lemma

$$L(1, \chi) - c_2 k^{-\varepsilon_1} < \sum_1^t \chi(n) n^{-1}.$$

Therefore

$$L(1, \chi) - c_2 k^{-\varepsilon_1} > c_3 k^{-\varepsilon_2} - c_2 k^{-\varepsilon_1} > 0$$

for  $k \geq k_0(\varepsilon)$ . This completes the proof.

#### References.

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2. Chowla, S., *The Riemann zeta and allied functions*, Bull. of Amer. Math. Soc. 58 (1952) 287-305.
3. Siegel, C. L., *Über die Classenzahl quadratischer Körper*, Acta Arith. 1 (1936) 83-86.