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by

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Abstract. Let k be a positive integer and χ a real non-principal character (mod k). Then under the extended Riemann hypothesis we have

$$\#\{1 \leqslant n \leqslant k^{\frac{1}{4}+\varepsilon} \mid \chi(n) = -1\} > k^{(\frac{1}{4}+\varepsilon)/2-\delta}, [k \geqslant k_0(\varepsilon, \delta)]$$

if k is cubefree and

$$\#\{1 \leqslant n \leqslant k^{\frac{3}{8}+\varepsilon} \mid \chi(n) = -1\} > k^{(\frac{3}{8}+\varepsilon)/2-\delta}, [k \geqslant k_0(\varepsilon, \delta)]$$

otherwise, where ε and δ are any positive real numbers.

1. Let k be a positive integer and χ a real non-principal character modulo k. Littlewood proved under the extended Riemann hypothesis that

$$L(1, \chi) = \sum_{1}^{\infty} \chi(n) \frac{1}{n} = O(\log \log k). \quad (1)$$

In this note, using this result we prove *under the extended Riemann hypothesis that*

$$\#\{1 \leqslant n \leqslant k^{\frac{1}{4}+\varepsilon} \mid \chi(n) = -1\} > k^{(\frac{1}{4}+\varepsilon)/2-\delta}, \ [k \geqslant k_0(\varepsilon, \delta)]$$

if k is cubefree and

$$\#\{1 \leqslant n \leqslant k^{\frac{3}{8}+\varepsilon} \mid \chi(n) = -1\} > k^{(\frac{3}{8}+\varepsilon)/2-\delta}, \ [k \geqslant k_0(\varepsilon, \delta)]$$

otherwise, where ε *and* δ *are any positive real numbers.* Here $\# A$ is the number of elements of a set A.

2. Proof.

Notations are the same as in § 1. Let $[u]$ be the Gaussian symbol of a real u. We need following lemma [1, Lemma].

Lemma. *Let* ε *be any fixed positive number. Then*

$$\left| \sum_{n=t+1}^{\infty} \chi(n) n^{-1} \right| < c\, k^{-\beta}$$

for $t \geqslant [k^{\frac{1}{4}+\varepsilon}] + 1$, *if k is cubefree and* $t \geqslant [k^{\frac{3}{8}+\varepsilon}] + 1$, *otherwise. Here* $c = c(\varepsilon)$ *and* $\beta = \beta(\varepsilon)$ *are positive numbers depending on* ε *but not on k.*

We prove our assertion. Let k be cubefree. Let ε, δ be any positive numbers. Put $g = (1/4 + \varepsilon)/2 - \delta$. Suppose that there exist infinitely many integers k such that

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$$\# \{ 1 \leqslant n \leqslant k^{1/4+\varepsilon} \mid \chi(n) = -1 \} < k^{g}.$$

Put $f = 1/4 + \varepsilon$ and $H = [k^{f}] + 1$. For such integer k we have

$$\sum_{1}^{H} \chi(n)\, n^{-1} \geqslant - \sum_{1}^{k^{g}} n^{-1} + \sum_{k^{g}}^{k^{f}} n^{-1} + O(1)$$

$$> - \int_{1}^{k^{g}} x^{-1}\, dx + \int_{k^{g}}^{k^{f}} x^{-1}\, dx + O(1)$$

$$= 2\,\delta \log k + O(1).$$

On the other hand, by Lemma there exist positive constants c, β depending only on ε such that

$$\left| \sum_{H+1}^{\infty} \chi(n)\, n^{-1} \right| < c k^{-\beta}.$$

Hence

$$L(1, \chi) > \sum_{1}^{H} \chi(n)\, n^{-1} - c k^{-\beta}$$

$$> 2\,\delta \log k + O(1).$$

This contradicts the result (1) of Littlewood. In the case that k is not cubefree, the proof is similar to above one.

Reference.

1. Takaku, A, *Finite part of* L $(1, \chi)$, this Bulletine, this issue 1-3.