On Relations among Structurally Stable Systems

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by

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In the first section we study several relations among structurally stable systems, and in the second section we study the set of dynamical systems which have hyperbolic and quasi-hyperbolic singular points in the space of C^r tangent vector fields on a smooth compact manifold.

1.

In the first place we recall that two dynamical systems X, Y on a smooth manifold M are topologically equivalent if there exists a homeomorphism of M which maps positively oriented orbits of X onto positively oriented orbits of Y. We remark that only a homeomorphism, not a diffeomorphism, is required. Let $\mathfrak{X}$ be the space of dynamical systems on M, endowed with the C^r-topology. A dynamical system X is structurally stable if there exists a neighbourhood U of X in $\mathfrak{X}$ such that every Y of U is topologically equivalent to X. We consider a dynamical system which has a saddle singular point at the north pole on the two-dimensional sphere. If we add to it a small rotational force with center at the pole, then in the neighbourhood of the pole, we still have a dynamical system which has a saddle singular point at the pole. If we give a stronger rotational force to it, then we can expect that the state of orbits will be changed. Since the neighbourhood of the pole on the two-dimensional sphere is homeomorphic to the plane, we consider it as the plane, and so we treat systems on the plane. We denote by $X=(X_1, X_2)=(\dot{x}, \dot{y})$ the vector at the point (x, y) in the plane. To begin with, we take a dynamical system $L=(L_1, L_2)$ whose singular points form a line, and whose orbits have the same inclination except singular points. Then $L=(L_1, L_2)$ can be denoted by $(p(x, y), kp(x, y))$, where k is a constant real number. We consider the case of that $p(x, y)$ is a linear form for x, y. By an appropriate coordinate transformation, we can put $L=(p(x, y), 0)$. We suppose the coefficient of x in $p(x, y)$ is zero and that of y is positive. Even if the coefficient of y is negative, there is no essential differerence. Let A be the vector field $(-y, x)$, and ε be a small positive real number. If we add εA to L, then the new vector field is a structurally stable system which has only one saddle singular point. If we add $-\varepsilon A$ to L, then the new vector field is an unstable system which has many periodic orbits. Next, let B denote the vector field $(-x, -y)$. If we add εB to L, then this new vector field has one node singular point.

We define $f(\lambda)=\begin{cases} L+\lambda A & (\lambda \geqq 0) \\ L-\lambda B & (\lambda \leqq 0) \end{cases}$, $|\lambda|$ is small.

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This means that the bifurcation value for f is zero. Similarly, the dynamical system which has a saddle singular point can be connected to the system such that all orbits except a singular point are spiral, going exponentially toward the origin, because we need only take as the path

$$f(\lambda) = \begin{cases} L + \lambda A & (\lambda \geqq 0) \\ L + \lambda (A - B) & (\lambda \leqq 0) \end{cases}$$

We summarize the above to the following;

Theorem. *Let $L = (p(x, y), kp(x, y))$ be a dynamical system on a bounded open set in the plane, where $p(x, y)$ is a linear form and the coefficient of x is zero. Then any neighbourhood of L contains a structurally stable system having a saddle singular point and also a structurally stable system having a node singular point.*

2.

Let M be a compact C^{∞} manifold, and $\mathfrak{X}^r = \mathfrak{X}^r(M)$ be the space of C^r tangent vector field on M. Denote by $\Phi^r = \Phi^r(\Lambda)$ the space of C^r-maps ξ from a manifold Λ into $\mathfrak{X}^r(M)$, endowed with C^r-topology. This mean that each map $\hat{\xi}$ defined by $\hat{\xi}(\lambda, x) = \xi(\lambda)x$ is of class C^r from $\Lambda \times M$ to TM, and that two such maps ξ, η are close if $\hat{\xi}$, $\hat{\eta}$ are close. For the definition of a quasi-hyperbolic singular point, refer to [2].

Theorem. *If $r \geqq 5$, then the set of all dynamical systems which have only hyperbolic and quasi-hyperbolic singular point is dense in $\mathfrak{X}^r(M)$.*

Proof. Let S^1 is a one dimensional sphere. For every system X in $\mathfrak{X}^r(M)$, denote by η the C^r-map of S^1 to $\mathfrak{X}^r(M)$ such that $\eta(S^1) = X$. Let Γ be the set of C^r-maps of S^1 to $\mathfrak{X}^r(M)$ under which the image of each element of S^1 has only hyperbolic and quasi-hyperbolic singular points, then by the theorem of J.Sotomayor [2], Γ is dense in Φ^r. Therefore there exists a C^r-map $\xi: S^1 \longrightarrow \mathfrak{X}^r(M)$ which is in Γ and close to η. This means $\xi(\lambda)$ is close to $\eta(\lambda)$. The theorem now follows.

References

1. J. Sotomayor, *Structural Stability and Bifurcation Theory*, Dynamical Systems, M. Peixoto ed., Academic Press, N. Y., 1973, 549-560.
2. J. Sotomayor, *Generic Bifurcations of Dynamical Systems*, Dynamical Systems, M. Peixoto ed., Academic Press, N. Y., 1973, 561-582.
3. E. C. Zeeman, *Lecture Note on Dynamical Systems*, Aarhus Universitet, Math. Ins., Nordic Summer School in Math., 1968.