Note on Homotopy Type

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Ryuji MAEHARA*

Abstract. It will be shown that if spaces A and B have the same homotopy type then there exists a fibration $p:E \longrightarrow I$ such that $p^{-1}(0), p^{-1}(1)$ are homeomorphic to A, B, respectively, where I denotes the closed interval $0 \leqq t \leqq 1$.

1. Let A and B be spaces. We will write $A \simeq B$ if A and B can be imbedded in a third space X in such a way that they are both *strong* deformation retracts of X. We will write $A \approx B$ if A and B can be imbedded in a space X and if there exists a homotopy $h_t:X \longrightarrow X$ $(0 \leqq t \leqq 1)$ satisfying $h_0(X) = A$, $h_1(X) = B$, $h_t \cdot h_t = h_t$ for each t (the last condition means that each h_t is a retraction of X onto its image). The purpose of this paper is to prove

Theorem. *The following statements are equivalent:*

(1) *A and B have the same homotopy type,*

(2) $A \simeq B$,

(3) $A \approx B$, *and*

(4) *there is a fibration $p:E \longrightarrow I$ such that $p^{-1}(0), p^{-1}(1)$ are homeomorphic to A, B, respectively.*

2. Let $f:A \longrightarrow B$ be a map. The *mapping cylinder* of f is the quotient space obtained from the topological sum $A \cup (A \times I) \cup B$ by identifying $a \in A$ with $(a, 0) \in A \times I$ and $(a, 1) \in A \times I$ with $f(a) \in B$.

Lemma. *Let $j:A \longrightarrow M$ be an inclusion map, and let X be the mapping cylinder of j. If $j(A)$ is a deformation retract of M, then $A (= A \times 0)$ is a strong deformation retract of X.*

Proof. By the assumption, there is a retraction $r:M \longrightarrow j(A) \subset M$ and a homotopy $h_t:M \longrightarrow M$ such that $h_0 = 1_M$ (= the identity map of M), $h_1 = r$. Define retractions R_1 of X onto $A \times I$ and R_0 of X onto $A \times 0$ respectively by

$$R_1(x) = \begin{cases} r(x) & \text{for } x \in M \\ x & \text{for } x \in A \times I, \end{cases}$$

$$R_0(x) = PR_1,$$

where P is the projection of $A \times I$ onto $A \times 0$. Then to complete the proof, we need only construct homotopies H_t between 1_M and R_1, and K_t between R_1 and R_0 in such a way that under these homotopies each point of $A \times 0$ remains fixed.

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* Dept. of Math.

For each $a \in A$, let a^* denote the path in X given by

$$a^*(t)=\begin{cases} h_{1-2t}(ja) & 0 \leqq t \leqq 1/2 \\ (a, 2-2t) & 1/2 \leqq t \leqq 1. \end{cases}$$

First, we define homotopies $f_t: M \longrightarrow M$ and $g_t: A \times I \longrightarrow X$ respectively by

$$f_t(m)=\begin{cases} h_{2t}(m) & 0 \leqq t \leqq 1/2 \\ h_{2-2t}(r(m)) & 1/2 \leqq t \leqq 1. \end{cases}$$

$$g_t(a, s)=a^*(1-s+|\frac{1}{2}-t|s).$$

It is easy to verify that these homotopies satisfy the following properties:

a) $f_0=1_M \quad f_1=r,$

$g_0=g_1=$ the inclusion map of $A\times I$ into X,

b) $g_t(a, 0)=(a, 0)$

c) $g_t(a, 1)=f_t(j(a))$

Now define $H_t: X \to X$ by

$$H_t(x)=\begin{cases} f_t(x) & \text{if } x\in M \\ g_t(a, s) & \text{if } x=(a, s)\in A\times I. \end{cases}$$

By property *c*), H_t is well defined on X. By *a*), H_t is a homotopy between 1_X and R_1. By *b*), each point of $A \times 0$ remains fixed under H_t. Next, let $P_t: A \times I \longrightarrow A \times I$ be the homotopy given by

$$P_t(a, s)=(a, (1-t)s).$$

and define $K_t: X \longrightarrow X$ by $K_t = P_t \cdot R_1$. Then K_t is a homotopy between R_1 and R_0 under which the points of $A \times 0$ are fixed. This completes the proof.

Corollary. *If $f: A \longrightarrow B$ is a homotopy equivalence, then A and B are both strong deformation retracts of the mapping cylinder M_f of the map f.*

Proof. The assertion on B is trivial. It is also well known that A is a deformation retract of M_f (see [1], p.45).

Consider the inclusion map $j: A \longrightarrow M_f$ and its mapping cylinder X. By the lemma, A is a strong deformation retract of X. But (X, A) is homeomorphic to $(M_f, j(A))$. Thus the assertion follows.

3. Proof of the theorem.

(1) ⇨ (2). This follows from the preceeding corollary.

(2) ⇨ (3). Suppose that A and B are both strong deformation retracts of a space X. Then there are homotopies $h_t, k_t: X \longrightarrow X$ satisfying the conditions: $h_0 = k_0 = 1_X$, $h_1(X) = A$, $k_1(X) = B$, $h_t(a) = a$ *for* $a \in A$, $k_t(b) = b$ for $b \in B$. Let X^I be the space of all the paths in X with the compact-open topology, and for $x \in X$, let $x^*, x^\#$ denote the paths given by

$$x^*(t)=k_t(x), \quad x^\#(t)=h_{1-t}(x).$$

We define three subspaces of X^I as follows:

$$W=[w\in X^I:\ w(0)\in A,\ \ w(1)\in B],$$
$$A^* =[a^* \in X^I:\ a\in A],$$
$$B^{\#} =[b^{\#} \in X^I:\ b\in B]$$

Clearly, A^* and $B^{\#}$ are homeomorphic to A and B, respectively, and are contained in W. Thus, in order to show $A \simeq B$, we need only construct a homotopy $H_t: W \longrightarrow W$ satisfying the conditions:

(*) $H_0(W)=A^*,\ H_1(W)=B^{\#},\ H_t\cdot H_t=H_t\ (0\leq t\leq 1)$.

First, we define $F_t: W \longrightarrow W$ by

$$F_t(w)(s)=\begin{cases} w(s) & 0\leq s\leq t \\ k(w(t),\ \dfrac{s-t}{1-t}) & t\leq s\leq 1,\ \ t\neq 1, \end{cases}$$

where $k: X \times I \longrightarrow X$ is the map $(x,\ t)\longrightarrow k_t(x)$. F_t is well defined and has the properties:

(**) $F_0(w)=A^*,\ \ F_1=1_W,\ \ F_t\cdot F_t=F_t\cdot$

We show the continuity of $F: W \times I \longrightarrow W$, $F(w, t) = F_t(w)$. Clearly, F is continuous at $(w, t) \in W \times I$ when $t \neq 1$. So, given a neighborhood V of $F(w, 1) = w$, we try to find a neighborhood U of w and a real number $0 \leq u < 1$ such that $w' \in U$ and $u < t \leq 1$ imply $F(w', t) \in V$. We may assume V to be a basic open set (containing w) of the compact open topology; namely, $V = W(C, O)$ for a compact set C in I and an open set O in X, where $W(C, O)=[w\in W:\ w(s)\in O$ for all s in $C]$. If $1 \notin C$, let $U=W(C, O)$ and $u = \sup[t : t \text{ in } C]$. Then, $w' \in U$ and $u < t \leq 1$ imply $F(w', t) \in V = W(C, O)$. In fact, if $s \in C$ then $s \leq u < t$, and hence, $F(w', t)(s) = F_t(w')(s) = w'(s) \in O$; *thus* $F(w', t) \in W(C, O)$. Now assume $1 \in C$. For each t in I, we have $k(w(1), t) = k_t(w(1)) = w(1) \in O$. *Hence,* $w(1) \times I$ is contained in the open subset $k^{-1}(O)$ of $X \times I$. By the compactness of I, there is an open neighborhood G of $w(1)$ such that $G \times I \subset k^{-1}(O)$. By the continuity of $w: I \longrightarrow X$, we can find $u < 1$ so that $w([u, 1]) \subset G$. Let $U = W(C, O) \cap W([u, 1], G)$. Then $w' \in U$, $u < t \leq 1$, $s \in C$ imply $F(w', t)(s) \in O$. In fact, if $s \leq t$ then $F(w', t)(s)=w'(s)\in O$, and if $t\leq s$ then

$$F(w',\ t)(s)=k(w'(t),\ \frac{s-t}{1-t})\in k(w'[u,\ 1]\times I)\subset k(G\times I)\subset O.$$

Thus the continuity of F has been proved.

Next, we define a homotopy $G_t: W \longrightarrow W$ by

$$G_t(w)(s)=\begin{cases} h(w(t),\ \dfrac{t-s}{t}) & 0\leq s\leq t,\ \ t\neq 0 \\ w(s) & t\leq s\leq 1, \end{cases}$$

where $h(x,\ t)=h_t(x)$. By similar arguments as above, we can prove that $G_t(w)$ is continuous with respect to (w, t), and satisfies

(***) $G_0 = 1_W$, $G_1(W) = B^{\#}$, $G_t \cdot G_t = G_t$.
Finally, define $H_t: W \longrightarrow W$ by

$$H_t(w) = \begin{cases} F_{2t}(w) & 0 \leqq t \leqq 1/2 \\ G_{2t-1}(w) & 1/2 \leqq t \leqq 1. \end{cases}$$

H_t is a well-defined homotopy, and satisfies the required conditions (*).

(3) ⇨ (4). Suppose that A and B are subsets of X and there exists a homotopy $h_t: X \longrightarrow X$ satisfying $h_0(X) = A$, $h_1(X) = B$, and $h_t \cdot h_t = h_t$ for all t in I. We define a subset E of $X \times I$ by $E = [(h_t(x), t): x \in X,\ t \in I]$, and a map $p: E \longrightarrow I$ by $p(h_t(x), t) = t$. Then, $p^{-1}(0)$ and $p^{-1}(1)$ are homeomorphic to A, B, respectively. Therefore, it remains only to show that p is a fibration. Let Y be an arbitrary space. Given a homotopy $f_t: Y \longrightarrow I$ and a map $g_0: Y \longrightarrow E$ such that $p \cdot g_0 = f_0$, we have to extend g_0 to a homotopy $g_t: Y \longrightarrow E$ with $p \cdot g_t = f_t$. Because the projection $p: X \times I \longrightarrow I$ is a fibration, there is a covering homotopy $g'_t: Y \longrightarrow X \times I$ of f_t, i.e., $p \cdot g'_t = f_t$. Noting that the map $R: X \times I \longrightarrow E$ given by $R(x, t) = (h_t(x), t)$ is a retraction, we let $g_t = R \cdot g'_t$. Then g_t is what we want.

(4) ⇨ (1). This is well known (see [2], p. 101).

REFERENCES

1. R.H. Fox, *On homotopy type and deformation retracts*, Annals of Mathematics Vol. 44 (1943), 40 – 50.

2. E.D. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York, 1966.