HIRATA SEPARABLE EXTENSION OVER THE DESCENDING CHAIN OF SUBRINGS

Ryukyu Math. J., 24 (2011), 117-121

HIRATA SEPARABALE EXTENSION OVER THE DESCENDING CHAIN OF SUBRINGS

Y. YAMASHIRO

Abstract

In [5], we considered the equivalent class of desending chains of subrings.

In this note, we consider Hirata separable extension (H-separable extension) over the desending chain of subrings. Let Λ be a ring, Γ a subring of Λ, C the center of Λ and Δ the commutater of Γ in Λ. Λ is a Hirata separabale extension over Γ if and only if Δ is $C-$finitely generated projective and $\Lambda \otimes_\Gamma \Lambda \cong \mathrm{Hom}_C(\Delta\ ,\ \Lambda)$ by the map $a \otimes b \mapsto [\ d \mapsto adb\ ]$, where $a \otimes b \in \Lambda \otimes_\Gamma \Lambda$ and $d \in \Delta$ (Theorem 1.1 in [3]).

1. Tensor Product over the descending chain of subrings

Throughout this note, Λ is a ring, Ω is a class of subrings of Λ with Λ. We can regard Ω as a poset under the inclusion relation. Now we consider the set of all monotone functions of $\mathbb{N}$ to Ω^{op}, denoted by $\mathcal{C}(\Omega)$, where $\mathbb{N}$ is the poset of all nutural numbers. Throughout this noter, B and B' are members of $\mathcal{C}(\Omega)$. $\mathcal{C}(\Omega)$ is also a poset by taking the relation $B \leq B'$, if $Bi \leq B'i$ in Ω, for every $i \in \mathbb{N}$. We will write $B \preceq B'$ if there exists a monotone function ρ of $\mathbb{N}$ to $\mathbb{N}$ such that $B\rho \leq B'$. Then $\mathcal{C}(\Omega)$ is a preordered set and if we define the relation $B \sim B'$ when $B \preceq B'$ and $B' \preceq B$, the reration "$\sim$" is an equivalence relation and the set of eqivalent classes of the reration "$\sim$" is a poset. Use $\tilde{\mathcal{C}}(\Omega)$ denote this poset. We can observe that Ω is contained in $\tilde{\mathcal{C}}(\Omega)$ (cf. [5]).

For any $i \in \mathbb{N}$, we will call left (right) $Bi-$module simply left (right) $B-$module. If $B \preceq B'$ then $B'-$module is $B-$module. So we can define $[B]-$module, for any

Received November 30, 2011.

$[B] \in \mathcal{C}(\Omega)$. Let N and M be left $B-$module and right $B-$module, respectively, then there exist numbers i and j such that N is left $Bi-$module and M is right $Bj-$module. Put $max\{i,j\} = k$. Then for any $l \geq k$, M and N are $Bl-$module, as left and as right, respectively So we have the sequence:

$$M \otimes_{Bk} N \leftarrow M \otimes_{B(k+1)} N \leftarrow \cdots$$

where homomorphisms are canonical epimorphism: $m \otimes n \longmapsto m \otimes n$. Now we consider the inverse limit $\varprojlim_{l} M \otimes_{Bl} N$, denoted by $M \otimes_B N$. We will denote $\bigcap_l Bl = \bar{B}$. Then there exist the canonical epimorphisms $M \otimes_{\bar{B}} N \to M \otimes_{Bl} N : m \otimes n \longmapsto m \otimes n$, and the epimorphism $M \otimes_{\bar{B}} N \to M \otimes_B N$. So we may write the element of $M \otimes_B N$, $\sum_s m_s \otimes n_s$, for some elements m_s of M and some elements n_s of N. Let $B \preceq B'$ and ρ the monotone function of $\mathbb{N}$ to $\mathbb{N}$ for $B \preceq B'$. Then for any $l \in \mathbb{N}$, there exist the composition homomorphism $M \otimes_B N \to M \otimes_{B\rho l} N \to M \otimes_{B'l} N : m \otimes n \mapsto m \otimes n$. So there exists the homomorphism $M \otimes_B N \to M \otimes_{B'} N : m \otimes n \mapsto m \otimes n$. Therefore we obtain the following statement.

Proposition 1. *If $B \sim B'$ then $M \otimes_B N$ is isomorphic to $M \otimes_{B'} N$.*

2. HIRATA SEPARABALE EXTENSION

For a two-sided $\Lambda-$module M and a subset S of Λ, we denote the commutater of S in M by $V_M(S)$:

$$V_M(S) = \{m \in M \mid sm = ms \ (\forall s \in S)\}.$$

If Γ is a subring of Λ, then $\mathrm{Hom}({}_\Lambda\Lambda \otimes_\Gamma \Lambda_\Lambda \ , \ {}_\Lambda\Lambda_\Lambda)$ is isomorphic to $V_\Lambda(\Gamma)$ by the map $f \mapsto f(1 \otimes 1)$. So we have the isomorphism

$$\begin{aligned}\mathrm{Hom}({}_\Lambda\Lambda \otimes_B \Lambda_\Lambda, \ {}_\Lambda\Lambda_\Lambda) &= \mathrm{Hom}({}_\Lambda\varprojlim_{l} \Lambda \otimes_{Bl} \Lambda_\Lambda, \ {}_\Lambda\Lambda_\Lambda) \\ \cong \varinjlim_{l} \mathrm{Hom}({}_\Lambda\Lambda \otimes_{Bl} \Lambda_\Lambda, \ {}_\Lambda\Lambda_\Lambda) &\cong \varinjlim_{l} V_\Lambda(Bl) \cong V_\Lambda(\bar{B}).\end{aligned}$$

We will put that $\Delta = V_\Lambda(\bar{B})$ and C is the center of Λ, namely $C = V_\Lambda(\Lambda)$

Definition 1. Λ is Hirata separable extension over B if Δ is C−finitely generated projective and $\Lambda \otimes_B \Lambda \cong \mathrm{Hom}_C(\Delta\ ,\ \Lambda)$ by the map $a \otimes b \mapsto [\ d \mapsto adb\ ]$, where $a \otimes b \in \Lambda \otimes_B \Lambda$ and $d \in \Delta$.

If $B \sim B'$ then $\bar{B} = \bar{B}'$. And so Λ is Hirata separable extension over B if and only if Λ is Hirata separable extension over B'.

Now we can state the similar proposition of Proposion 1 in [4].

Proposition 2. *Λ is Hirata separable extension over B if and only if there exist $\delta_i \in \Delta$ $(i = 1, 2, \cdots, n)$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda\otimes_B\Lambda}(\Lambda)$ such that*

$$1 \otimes 1 = \sum_{ij} \delta_i a_{ij} \otimes b_{ij}$$

in $\Lambda \otimes_B \Lambda$.

Proof. $\mathrm{Hom}({}_\Lambda\Lambda \otimes_B \Lambda_\Lambda\ ,\ {}_\Lambda\Lambda_\Lambda) \cong \Delta$ and $\mathrm{Hom}({}_\Lambda\Lambda_\Lambda\ ,\ {}_\Lambda\Lambda \otimes_B \Lambda_\Lambda) \cong V_{\Lambda\otimes_B\Lambda}(\Lambda)$ by the map $f \mapsto f(1 \otimes 1)$ and $g \mapsto g(1)$, respectively.

Suppose Λ is Hirata separable extension over B. Since

$$\mathrm{Hom}_C(\Delta\ ,\ \Lambda) <\oplus\ \mathrm{Hom}_C(\bigoplus_n C,\ \Lambda) \cong \bigoplus_n \Lambda$$

as two-sided Λ−module, the assertion holds.

Conversely, suppose that there exist $\delta_i \in \Delta$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda\otimes_B\Lambda}(\Lambda)$ such that $1 \otimes 1 = \sum_{ij} \delta_i a_{ij} \otimes b_{ij}$ in $\Lambda \otimes_B \Lambda$. Then $\Lambda \otimes_B \Lambda <\oplus (\bigoplus_n \Lambda)$ as two-sided Λ−module and

$$\begin{aligned}\Delta \cong \mathrm{Hom}({}_\Lambda\Lambda \otimes_B \Lambda_\Lambda\ ,\ {}_\Lambda\Lambda_\Lambda) &<\oplus \mathrm{Hom}({}_\Lambda(\bigoplus_n \Lambda)_\Lambda\ ,\ {}_\Lambda\Lambda_\Lambda)\\ &\cong \bigoplus_n \mathrm{Hom}({}_\Lambda\Lambda_\Lambda\ ,\ {}_\Lambda\Lambda_\Lambda) \cong \bigoplus_n C.\end{aligned}$$

Therefore Δ is C−finitely generated projective.

Moreover there exists homomorphism $\mathrm{Hom}_C(\Delta,\ \Lambda) \to \Lambda \otimes_B \Lambda$ by the map $f \mapsto \sum_{ij} f(\delta_i) a_{il} \otimes b_{ij}$. Then the composition map $\Lambda \otimes_B \Lambda \to \mathrm{Hom}_C(\Delta,\ \Lambda) \to \Lambda \otimes_B \Lambda$ and $\mathrm{Hom}_C(\Delta,\ \Lambda) \to \Lambda \otimes_B \Lambda \to \mathrm{Hom}_C(\Delta,\ \Lambda)$ are identity maps. □

We denote the set of δ_i in above proposition by δ.

Corollary 1. *If Λ is Hirata separable extension over B and $\delta \subset V_\Lambda(Bk)$ for some k. Then Λ is Hirata separable extension over Bl, for any $l \geq k$.*

Proof. By the epimorphism $\Lambda \otimes_B \Lambda \to \Lambda \otimes_{Bl} \Lambda$, $V_{\Lambda \otimes_B \Lambda}(\Lambda)$ is mapped into $V_{\Lambda \otimes_{Bl} \Lambda}(\Lambda)$. So $\sum_j a_{ij} \otimes b_{ij}$ in above proposition are belong to $V_{\Lambda \otimes_{Bl} \Lambda}(\Lambda)$. By the application Proposion 1 in [4] to Λ and Bl, Λ is a Hirata separable extension over Bl. □

Corollary 2. *Let $\overset{*}{B} = V_\Lambda(\Delta)$. If Λ is Hirata separable extension over B and $Bk \subset \overset{*}{B}$, for some k. Then Λ is a Hirata separable extension over $\overset{*}{B}$.*

Proof. The composition homohorphism $\Lambda \otimes_B \Lambda \to \Lambda \otimes_{Bk} \Lambda \to \Lambda \otimes_{\overset{*}{B}} \Lambda$ maps $V_{\Lambda \otimes_B \Lambda}(\Lambda)$ into $V_{\Lambda \otimes_{\overset{*}{B}} \Lambda}(\Lambda)$. The lest of proof is same to that of Corollary 1. □

Λ is a strongly separable extension over a siubring Γ if and only if there exsit $\delta_i \in V_\Lambda(\Gamma)$ $(i = 1, 2, \cdots, n)$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_\Gamma \Lambda}(\Lambda)$ such that $d = \sum_{ij} \delta_i a_{ij} d b_{ij}$ for any $d \in V_\Lambda(\Gamma)$ (Theorem3.5 in [2]).

Theorem 1. *If Λ is Hirata separable extension over B and the epimorphism $\Lambda \otimes_{\bar{B}} \Lambda \to \Lambda \otimes_B \Lambda$ maps $V_{\Lambda \otimes_{\bar{B}} \Lambda}(\Lambda)$ onto $V_{\Lambda \otimes_B \Lambda}(\Lambda)$. Then Λ is a strongly separable extension over $\bar{B}$.*

Proof. Let $\delta_i \in \Delta$ and $\sum_j a_{ij} \otimes b_{ij} \in V_{\Lambda \otimes_B \Lambda}(\Lambda)$ in Proposition 2. Then $\delta_i \in V_\Lambda(\bar{B})$ and, by assumption, in $\Lambda \otimes_{\bar{B}} \Lambda$, $\sum_j a_{ij} \otimes b_{ij}$ may be belong to $V_{\Lambda \otimes_{\bar{B}} \Lambda}(\Lambda)$. So there exists the homomorphism $\mathrm{Hom}_C(\Delta\ ,\ \Lambda) \to \Lambda \otimes_{\bar{B}} \Lambda$ by $f \mapsto \sum_{ij} f(\delta_i) a_{ij} \otimes b_{ij}$.

On the other hand, the homomorphism $\Lambda \otimes_{\bar{B}} \Lambda \to \mathrm{Hom}_C(\Delta\ ,\ \Lambda)$ is equal to the composion of $\Lambda \otimes_{\bar{B}} \Lambda \to \Lambda \otimes_B \Lambda \to \mathrm{Hom}_C(\Delta\ ,\ \Lambda)$. We can observe that the composition of $\mathrm{Hom}_C(\Delta\ ,\ \Lambda) \to \Lambda \otimes_{\bar{B}} \Lambda \to \mathrm{Hom}_C(\Delta\ ,\ \Lambda)$ maps the identity map to $[d \mapsto \sum_{ij} \delta_i a_{ij} d b_{ij}]$, and the composition of $\mathrm{Hom}_C(\Delta,\ \Lambda) \to \Lambda \otimes_{\bar{B}} \Lambda \to \Lambda \otimes_B \Lambda \to \mathrm{Hom}_C(\Delta,\ \Lambda)$ maps the identity map to $[d \mapsto d]$. Therefore $d = \sum_{ij} \delta_i a_{ij} d b_{ij}$ for any $d \in \Delta$ and Λ is a strongly separable extension of $\bar{B}$. □

References

[1] K. Hirata, Some type of separable extensions of rings, Nagoya Math. J., 33 (1968), 107-115.

[2] E. Mcmahon and A. C. Mewbern, Separable extensions of noncommutative rings, Hokkaido Math. J., 13 (1984), 74-88.

[3] K. Sugano, Note on semisimple extensions and separable extensions, Osaka J. Math., 4 (1967), 265-270.

[4] K. Sugano, Separable extensions and Frobenius extensions, Osaka J. Math., 7 (1970), 291-299.

[5] Y. Yamashiro, On equivalent class of descending chains of subrings, Ryukyu Math. J., 23 (2010), 117-120.

Faculty of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN