COMPUTER-AIDED VERIFICATION OF THE GAUSS-BONNET FORMULA FOR CLOSED SURFACES

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COMPUTER-AIDED VERIFICATION OF THE GAUSS-BONNET FORMULA FOR CLOSED SURFACES

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Abstract

If X, a compact connected closed C^∞-surface with Euler-Poincaré characteristic $\chi(X)$, has a Riemannian metric, and if $K : X \to \mathbb{R}$ is the Gauss-curvature and dV is the absolute value of the exterior 2-form which represents the volume, then according to the theorem of Gauss-Bonnet, which holds for orientable as well as non-orientable surfaces, $\frac{1}{2\pi}\int_X KdV = \chi(X)$.

When X is the standard sphere or torus in $\mathbb{R}^3$, the Gaussian curvature is well-known and we can compute the left-hand side explicitly.

Let X be a compact connected closed C^∞-surface of any genus. In this paper, we construct an embedding of X into $\mathbb{R}^3$ or $\mathbb{R}^4$ according as X is orientable or non-orientable. We equip X with the Riemannian metric as a Riemannian submanifold of $\mathbb{R}^3$ or $\mathbb{R}^4$. Then, with the aid of a computer, we compute the left-hand side numerically for the cases that the genus of X is small. The computer data are sufficiently nice and coincide with the right-hand side without errors. Such nice data are obtained by converting double integrals to infinite integrals.

1 Introduction

If X, a compact connected closed C^∞-surface with Euler-Poincaré characteristic $\chi(X)$, has a Riemannian metric, and if $K : X \to \mathbb{R}$ is the Gauss-curvature and dV is the absolute value of the exterior 2-form which represents the volume, then according to the theorem of Gauss-Bonnet, which holds for orientable as well as non-orientable surfaces,

$$\frac{1}{2\pi}\int_X KdV = \chi(X). \tag{1}$$

There are two proofs for (1). The first proof is as follows (see, for example, [1] and [8]): We first prove a similar formula for a triangle in X. Next, if X is orientable, then we give a triangulation and apply the formula for each triangle. Summing up the results, we obtain (1). On the other hand, if X is non-orientable,

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then we take a double covering $p : \tilde{X} \to X$ such that $\tilde{X}$ is orientable. We pull back the Riemannian metric on X by p and define a metric on $\tilde{X}$. Then

$$\frac{1}{2\pi}\int_X K dV = \frac{1}{4\pi}\int_{\tilde{X}} K dV = \frac{1}{2}\chi(\tilde{X}) = \chi(X).$$

(Alternatively, if triangulate X by geodesic triangles, then we can give a proof regardless of the orientability of X.)

The other proof is valid for the case when X is a hypersurface in $\mathbb{R}^3$. (See, for example, [5].) Let $\nu : X \to S^2$ be the Gauss map. Then (1) is a consequence of the following two results: One is $2\deg\nu = \chi(X)$ and the other is $\nu^*\omega_{S^2} = K\omega_X$, where ω_M denotes the volume form on M.

In this paper, we consider the following question: Let X be a compact connected closed C^∞-surface of any genus. Is it possible to construct an embedding of X into Euclidean space explicitly such that the left-hand side of (1) is numerically computable?

For special cases, we know answers to the question. Firstly, for $a > 0$, we define a sphere by

$$S^2(a) = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = a^2\}.$$

Since $K = \dfrac{1}{a^2}$, (1) is nothing but an assertion that the surface area of $S^2(a)$ is $4\pi a^2$.

Secondly, for $0 < r < R$, we consider a torus whose parametrization is given by

$$p(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \tag{2}$$

where $0 \leq u, v \leq 2\pi$. Since $K = \dfrac{\cos u}{r(R + r\cos u)}$, (1) is nothing but an assertion that

$$\frac{1}{2\pi}\iint_{[0,2\pi]\times[0,2\pi]} \cos u \, dudv = 0.$$

Thirdly, we can embed a torus into $\mathbb{R}^4$ and a Klein's bottle into $\mathbb{R}^5$ such that $K = 0$. (See §4.) In this case, (1) clearly holds.

But to the best of the author's knowledge, an example for higher genus case is not known. One reason of this is that computation of K is complicated. The other reason is that we do not have an effective method for the numerical computation of double integrals.

The purpose of this paper is to construct an embedding of X into $\mathbb{R}^3$ or $\mathbb{R}^4$ according as X is orientable or non-orientable. We equip X with the Riemannian metric as a Riemannian submanifold of $\mathbb{R}^3$ or $\mathbb{R}^4$. Then, with the aid of a computer, we compute the left-hand side of (1) numerically for the cases that the genus of X is small.

The key to our method is as follows: We transform a double integral to an infinite integral. Thanks to this, we get a sufficiently nice data about the numerical computation of the left-hand side of (1).

This paper is organized as follows. In §2, we state our main results. In §3, we prove the main theorems, In §4, for our reference, we recall flat embeddings of a torus into $\mathbb{R}^4$ and a Klein's bottle into $\mathbb{R}^5$.

2 Main results

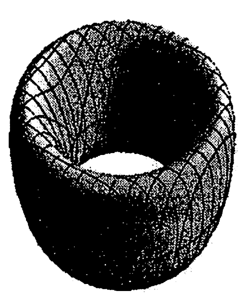
Definition 1. For $g \in \mathbb{N} \cup \{0\}$, we define a subspace X_g of $\mathbb{R}^3$ as follows:
(i) The case of even g. We set

$$X_g = \{(x, y, z) \in \mathbb{R}^3; z^2 + (x^2 + y^2 - 4) \times \prod_{i=1}^{g} \left((x - \cos \frac{2\pi i}{g})^2 + (y - \sin \frac{2\pi i}{g})^2 - \frac{1}{g^2} \right) = 0\}.$$

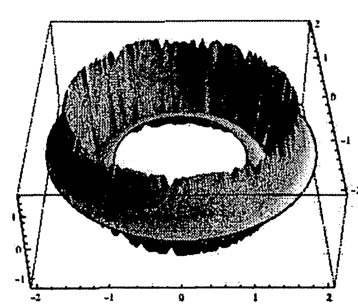
(ii) The case of odd n. We set

$$X_g = \{(x, y, z) \in \mathbb{R}^3; z^2 + (x^2 + y^2 - 4)(x^2 + y^2 - \frac{1}{g^2}) \times \prod_{i=1}^{g-1} \left((x - \cos \frac{2\pi i}{g-1})^2 + (y - \sin \frac{2\pi i}{g-1})^2 - \frac{1}{g^2} \right) = 0\}.$$

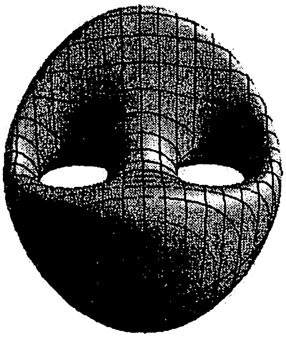
Note that X_g is a closed surface of genus g. (See Figure 1.) Note also that X_g admits an involution $(x, y, z) \mapsto (-x, -y, -z)$.



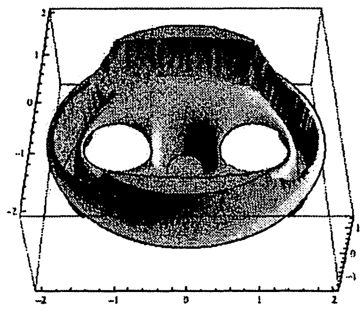
X_1



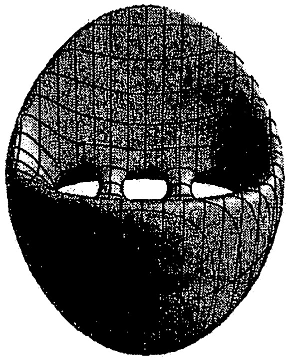
K_{X_1}



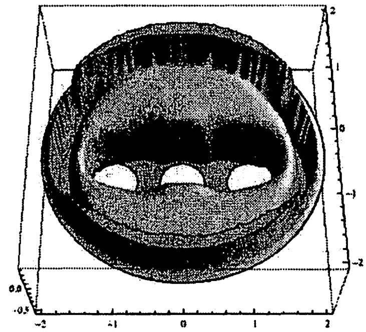
X_2



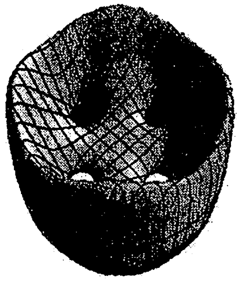
K_{X_2}



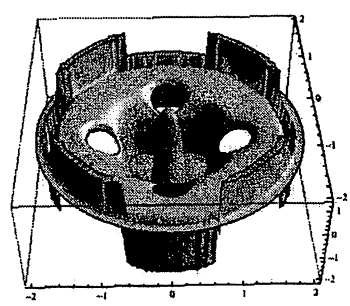
X_3



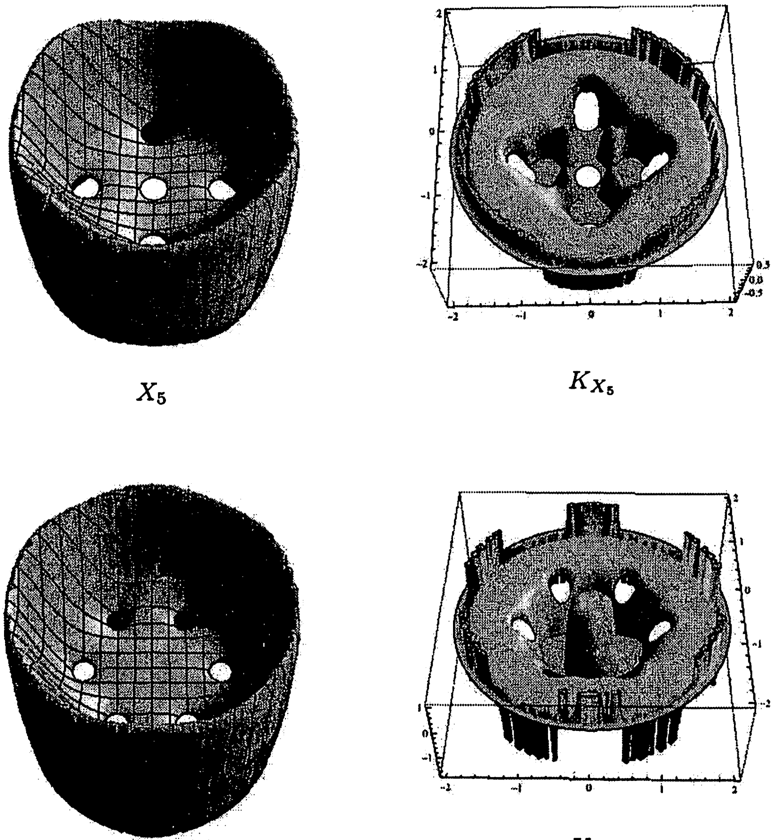
K_{X_3}



X_4



K_{X_4}



X_5 K_{X_5}

X_6 K_{X_6}

Figure 1: X_g and K_{X_g} for $1 \leq g \leq 6$.

Definition 2. We define a map $f : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Then we set

$$Y_{g+1} := f(X_g).$$

Lemma 3. *The space Y_{g+1} is a non-orientable surface of genus $g+1$.*

Proof. We shall show that $f|X_g : X_g \to Y_{g+1}$ is a double covering. First, it is clear that $f(x, y, x) = f(-x, -y, -z)$. Next, we need to check the fact that for any $q \in Y_{g+1}$, $f^{-1}(q)$ consists of two points. The case for $g = 0$ is proved in the appendix 2 of [6]. The proof remains valid for $g > 0$ without changing anything. □

Remark 4. It is known that a Klein's bottle admits an embedding into $\mathbb{R}^4$ with parametrization

$$p(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u \cos\frac{v}{2}, r\sin u \sin\frac{v}{2}),$$

where $0 \leq u, v \leq 2\pi$. (See, for example, [2, p.32].) Note that this is a natural parametrization of $f(T)$, where T is the torus given by (2).

Definition 5. (i) We equip X_g and Y_g with the Riemannian metric as Riemannian submanifolds of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

(ii) We denote by K_{X_g} and K_{Y_g} the Gaussian curvature with respect to the Riemannian metric on X_g and Y_g, respectively. (Note that $K_{X_g} \neq K_{Y_{g+1}} \circ (f|X_g)$, since $f|X_g : X_g \to Y_{g+1}$ is *not* a local isometry.)

Theorem A . *With the aid of a computer, we can compute the Gaussian curvature K_{X_g} and K_{Y_g}.*

Example 6. (i) The graphs of the function K_{X_g} for $1 \leq g \leq 6$ are given in Figure 1. We can understand the well-known fact that the point p with $K(p) > 0$ and $K(p) < 0$ correspond to an elliptic and hyperbolic point, respectively.

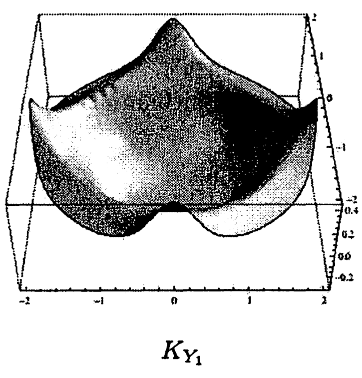
Moreover, for even g, we define $p_0 \in X_g$ to be one of two points of the form $(0, 0, z)$. (Note that such a point is not defined for odd g.) Then the values of $K_{X_g}(p_0)$ are given by the following table.

Table 1: The values of $K_{X_g}(p_0)$.

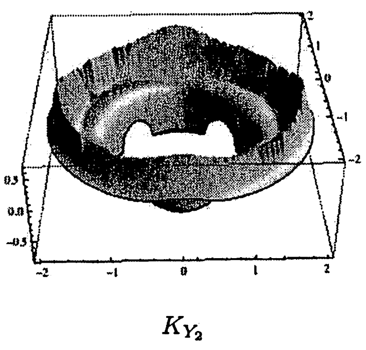
g	0	2	4	6	8	10
$K_{X_g}(p_0)$	0.25	−25.526	0.882572	0.613957	0.50655	0.448304

Note that when g moves in even numbers, $K_{X_g}(p_0)$ attain maximum for $g = 4$

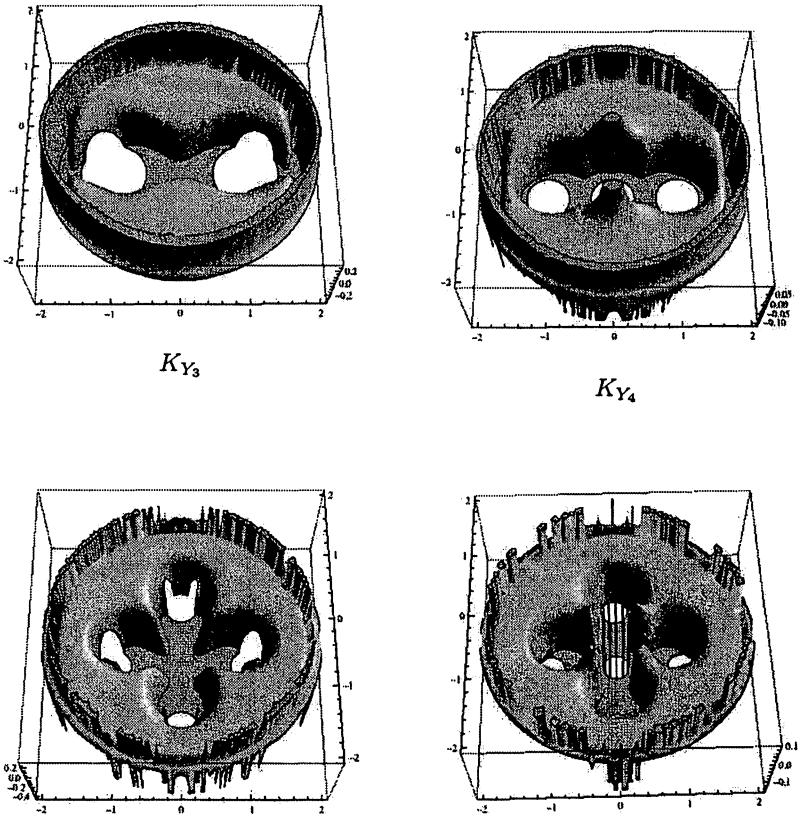
(ii) The graphs of the function K_{Y_g} for $1 \leq g \leq 6$ are given in the following Figure 2.



K_{Y_1}



K_{Y_2}



K_{Y_3} K_{Y_4}

K_{Y_5} K_{Y_6}

Figure 2: K_{Y_g} for $1 \leq g \leq 6$.

For $p_0 \in X_{g-1}$ (where g is an odd number), we set $q_0 := f(p_0) \in Y_g$. Then the values of $K_{Y_g}(q_0)$ are given by the following table.

Table 2: The values of $K_{Y_g}(q_0)$.

g	1	3	5	7	9	11
$K_{Y_g}(q_0)$	-0.3125	-0.9876	-0.5236	-0.4381	-0.4020	-0.3820

Note that when g moves in odd numbers, $K_{Y_g}(q_0)$ attain minimum for $g = 3$.

(ii) See Example 7 for an explicit formula of K_{X_g} for $0 \leq g \leq 2$ and K_{Y_g} for $g = 1$ and 2.

Theorem B . (i) *For X_g with $0 \leq g \leq 8$, there is an effective method for computing the left-hand side of (1) numerically. See the computations below (4) for the cases of $g = 0$ and 1. Moreover, see σ in Table 3 in §3 for the cases of $2 \leq g \leq 8$.*

Note that the results of computations agree with the right-hand side of (1) completely.

(ii) *For Y_g with $1 \leq g \leq 5$, there is an effective method for computing the left-hand side of (1) numerically. See the computations below (10) for the cases of $g = 1$ and 2. Moreover, see σ° in Table 4 in §3 for the cases of $3 \leq g \leq 5$.*

Note that the results of computations agree with the right-hand side of (1) completely.

3 Proof of main theorems

We define $X_g^+ := \{(x,y,z) \in X_g; z \geq 0\}$ and $X_g^- := \{(x,y,z) \in X_g; z \leq 0\}$. We write the equation for X_g in Definition 1 by $z^2 + F(x,y) = 0$. Then X_g^+ admits a parametrization

$$p(u,v) = (u, v, \sqrt{-F(u,v)}). \tag{3}$$

We use $f \circ p$ as a parametrization of Y_{g+1}.

Proof of Theorem A. (i) We show how to compute K_{X_g} with respect to the parametrization (3). Since X_g is a hypersurface of $\mathbb{R}^3$, we can compute from the definition $K := \dfrac{LN - M^2}{EG - F^2}$. We can find a Mathematica program in [4].

Alternatively, we can use the Brioschi formula ([4]), which holds for any Riemannian 2-dimensional manifold: We set

$$E = p_u \cdot p_u, \; F = p_u \cdot p_v \quad \text{and} \quad G = p_v \cdot p_v.$$

Then

$$K = \frac{E(E_v G_v - 2F_u G_v + G_u^2)}{4(EG - F^2)^2} + \frac{F(E_u G_v - E_v G_u - 2E_v F_v - 2F_u G_u + 4F_u F_v)}{4(EG - F^2)^2} + \frac{G(E_u G_u - 2E_u F_v + E_v^2)}{4(EG - F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG - F^2)}.$$

(ii) We use the Brioschi formula for the parametrization $f \circ p$. □

Example 7. (i) Since $X_0 = S^2(2)$, we have $K_{X_0} = \frac{1}{4}$.

(ii) We have

$$K_{X_1} = -\frac{\alpha_1}{\alpha_2},$$

where

$$\alpha_1 = 4u^8 + 8u^6(2v^2 - 5) + 3u^4(8v^4 - 40v^2 + 41) + 2u^2(8v^6 - 60v^4 + 123v^2 - 80) + 4v^8 - 40v^6 + 123v^4 - 160v^2 + 100$$

and

$$\alpha_2 = \Big(4u^6 + 3u^4(4v^2 - 7) + 6u^2(2v^4 - 7v^2 + 5) + 4v^6 - 21v^4 + 30v^2 - 4\Big)^2.$$

(iii) We have

$$K_{X_2} = -64\frac{\beta_1}{\beta_2},$$

where

$$\begin{aligned}\beta_1 =& 18432u^{14} + 4608u^{12}(28v^2 - 51) + 2304u^{10}(168v^4 - 564v^2 + 473)\\ &+ 576u^8(1120v^6 - 5160v^4 + 7476v^2 - 3755)\\ &+ 24u^6(26880v^8 - 149760v^6 + 271808v^4 - 205152v^2 + 59175)\\ &+ 6u^4(64512v^{10} - 403200v^8 + 771840v^6 - 565184v^4 + 78420v^2 + 75507)\\ &+ 12u^2(10752v^{12} - 71424v^{10} + 122304v^8 - 38848v^6 - 26262v^4\\ &\qquad + 48951v^2 - 22464)\\ &+ (4v^2 + 3)^3(288v^8 - 2592v^6 + 7590v^4 - 10260v^2 + 4901)\end{aligned}$$

and

$$\begin{aligned}\beta_2 =& \Big(2304u^{10} + 768u^8(15v^2 - 26) + 96u^6(240v^4 - 704v^2 + 617)\\ &+ 96u^4(240v^6 - 864v^4 + 1083v^2 - 715)\\ &+ 3u^2(3840v^8 - 14336v^6 + 14176v^4 - 7104v^2 + 8619)\\ &+ (4v^2 + 3)^2(144v^6 - 696v^4 + 825v^2 + 64)\Big)^2.\end{aligned}$$

(iv) We have

$$K_{Y_1} = \frac{-12u^2(v^2 - 4) + 48v^2 - 80}{(9u^4v^2 + 3u^2v^2(3v^2 - 8) + 16)^2}.$$

(v) We have

$$K_{Y_2} = \frac{\gamma_1}{\gamma_2},$$

where

$$\begin{aligned}\gamma_1 =& 51u^{16} + 12u^{14}(10v^2 - 73) - 12u^{12}(25v^4 + 231v^2 - 448)\\ &- 4u^{10}(366v^6 + 399v^4 - 4257v^2 + 3716)\\ &+ u^8(-2190v^8 + 2940v^6 + 19728v^4 - 40960v^2 + 18528)\\ &- 4u^6(366v^{10} - 735v^8 - 4038v^6 + 12140v^4 - 9192v^2 + 1552)\\ &- 4u^4(75v^{12} + 399v^{10} - 4392v^8 + 12140v^6 - 9120v^4 - 1104v^2 + 1984)\\ &+ 4u^2(30v^{14} - 693v^{12} + 4257v^{10} - 10240v^8 + 9192v^6 + 1104v^4\\ &\qquad - 5936v^2 + 1728)\\ &+ 51v^{16} - 876v^{14} + 5376v^{12} - 14864v^{10} + 18528v^8 - 6208v^6\\ &\qquad - 7936v^4 + 6912v^2 - 1280\end{aligned}$$

and

$$\begin{aligned}\gamma_2 =\Big(&9u^{12} + 2u^{10}(27v^2 - 59) + u^8(135v^4 - 686v^2 + 564)\\ &+ 4u^6(45v^6 - 367v^4 + 714v^2 - 304)\\ &+ u^4(135v^8 - 1468v^6 + 4584v^4 - 4740v^2 + 1168)\\ &+ u^2(54v^{10} - 686v^8 + 2856v^6 - 4740v^4 + 2816v^2 - 480)\\ &+ 9v^{12} - 118v^{10} + 564v^8 - 1216v^6 + 1168v^4 - 480v^2 + 64\Big)^2.\end{aligned}$$

Next we try to compute the left-hand side of (1) for $X = X_g$.
The case $g = 0$ is clear. For $g \geq 1$. note that

$$\frac{1}{2\pi}\int_{X_g} K_{X_g} dV = \frac{2}{2\pi}\int_{X_g^+} K_{X_g} dV. \tag{4}$$

We consider the case $g = 1$. Applying the polar conversion $(u, v) = (r\cos\theta, r\sin\theta)$, (4) equals to

$$\frac{2}{2\pi}\int_0^{2\pi}\int_1^2 \frac{-r\sqrt{-4r^6 + 21r^4 - 30r^2 + 4}(4r^8 - 40r^6 + 123r^4 - 160r^2 + 100)}{\sqrt{r^4 - 5r^2 + 4}(4r^6 - 21r^4 + 30r^2 - 4)^2} drd\theta$$
$= 0.$

Hereafter we assume that $g \geq 2$. Let $\pi : X_g \to \mathbb{R}^2$ be the projection defined by $\pi(x, y, z) = (x, y)$. We set $D := \pi(X_g)$ and

$$A(u, v) := K_{X_g}(u, v)\sqrt{E(u, v)G(u, v) - F(u, v)^2}, \tag{5}$$

where E, F and G are defined with respect to the parametrization p in (3). Then by the definition of dV, we have

$$(4) = \frac{2}{2\pi}\iint_D A(u, v)\, dudv. \tag{6}$$

Probably, the most simple method to compute (6) using Mathematica is to translate it as an integral over a rectangle: We set

$$\mu(r, \theta) := \begin{cases} r \cdot A(r\cos\theta, r\sin\theta) & \text{if } (r\cos\theta, r\sin\theta) \in D \\ 0 & \text{if } (r\cos\theta, r\sin\theta) \notin D. \end{cases}$$

Moreover, for a fixed g, we set

$$\rho := \frac{2}{2\pi}\iint_{[0,2]\times[0,2\pi]} \mu(r, \theta)\, drd\theta. \tag{7}$$

Then, we have (6) $= \rho$.

The number ρ is computable using the Mathematica command "NIntegrate[$\frac{2}{2\pi}\mu[r, \theta], \{r, 0, 2\}, \{\theta, 0, 2\pi\}$]". The data are given in Table 3, but unfortunately they are not so good. Hence, we convert (7) to an infinite integral.

Definition 8. We fix $g \geq 2$.

(i) We set

$$B(r,\theta) := r \cdot A(r\cos\theta, r\sin\theta).$$

(ii) We set

$$k_1(\theta) := \frac{g^2 + g^2\cos 2\theta - \sqrt{2}\sqrt{-g^4\cos^2\theta + g^4\cos^2\theta\cos 2\theta + 2g^2\cos^2\theta}}{2g^2\cos\theta}$$

and

$$k_2(\theta) := \frac{g^2 + g^2\cos 2\theta + \sqrt{2}\sqrt{-g^4\cos^2\theta + g^4\cos^2\theta\cos 2\theta + 2g^2\cos^2\theta}}{2g^2\cos\theta}.$$

(iii) We set

$$R := \left\{(s,\theta); 1 \leq s < \infty \quad \text{and} \quad 0 \leq \theta \leq \arcsin\frac{1}{g}\right\}$$
$$Q_3 := \left\{(r,\theta); 0 \leq r \leq 2 \quad \text{and} \quad \arcsin\frac{1}{g} \leq \theta \leq \frac{\pi}{g}\right\}$$

and

$$\tilde{Q}_3 := \left\{(r,\theta); \frac{1}{g} \leq r \leq 2 \quad \text{and} \quad \arcsin\frac{1}{g} \leq \theta \leq \frac{\pi}{g-1}\right\}.$$

(We define Q_1 and Q_2 later, but we do not need them for the moment.)

(iv) We set

$$\lambda_1 := \begin{cases} \displaystyle\iint_R B\left(\frac{k_1(\theta)}{s}, \theta\right)\frac{k_1(\theta)}{s^2}\,ds\,d\theta & \text{if } g \text{ is even} \\ \displaystyle\iint_R B\left(\frac{k_1(\theta)-\frac{1}{g}}{s} + \frac{1}{g}, \theta\right)\frac{k_1(\theta)-\frac{1}{g}}{s^2}\,ds\,d\theta & \text{if } g \text{ is odd} \end{cases}$$

$$\lambda_2 := \iint_R B\left(\frac{k_2(\theta)-2}{s} + 2, \theta\right)\frac{-k_2(\theta)+2}{s^2}\,ds\,d\theta$$

and

$$\lambda_3 := \begin{cases} \displaystyle\iint_{Q_3} B(r,\theta)\,drd\theta & \text{if } g \text{ is even} \\ \displaystyle\iint_{\tilde{Q}_3} B(r,\theta)\,drd\theta & \text{if } g \text{ is odd.} \end{cases}$$

(v) We set

$$\sigma := \begin{cases} \dfrac{4g}{2\pi}\displaystyle\sum_{i=1}^{3}\lambda_i & \text{if } g \text{ is even} \\[2ex] \dfrac{4g-4}{2\pi}\displaystyle\sum_{i=1}^{3}\lambda_i & \text{if } g \text{ is odd.} \end{cases}$$

Lemma 9. *For all $g \geq 2$, we have (7) $= \sigma$.*

Proof. (i) The case of even g.

We define closed subspaces P_i $(1 \leq i \leq 3)$ of D as follows:

- P_1: The domain surrounded by a circle $(u-1)^2 + v^2 = \frac{1}{g^2}$ and lines $v = 0$ and $v = (\arcsin \frac{1}{g})u$.
- P_2: The domain surrounded by circles $(u-1)^2 + v^2 = \frac{1}{g^2}$ and $u^2 + v^2 = 4$ and a line $v = (\arcsin \frac{1}{g})u$.
- P_3: The domain surrounded by a circle $u^2 + v^2 = 4$ and lines $v = (\arcsin \frac{1}{g})u$ and $v = \frac{\pi}{g}u$.

(See Figure 3.)

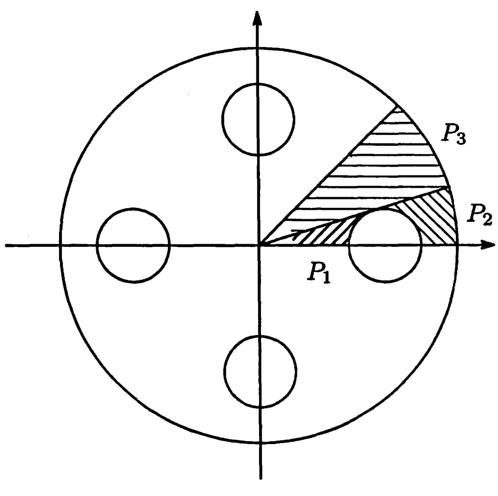


Figure 3: P_i for $g = 4$.

For $1 \leq i \leq 3$, we set

$$\zeta_i := \iint_{P_i} A(u,v)\, dudv.$$

Then we have

$$(6) = \frac{4g}{2\pi}\sum_{i=1}^{3}\zeta_i. \tag{8}$$

About ζ_1, we apply the polar conversion as in the case for $g = 1$. For that purpose, we fix θ and determine the lengths from the origin to the intersection points of a circle and a line:

$$(u-1)^2 + v^2 = \frac{1}{g^2} \quad \text{and} \quad v = (\tan t)u.$$

For that purpose, we define $h_1(t)$ and $h_2(t)$ (where $h_1(\theta) \leq h_2(\theta)$) as indicated in the following Figure 4. Then the lengths are given by $\frac{h_1(\theta)}{\cos\theta}$ and $\frac{h_2(\theta)}{\cos\theta}$. It is easy to see that these lengths coincide with $k_1(\theta)$ and $k_2(\theta)$ in Definition 8 (ii).

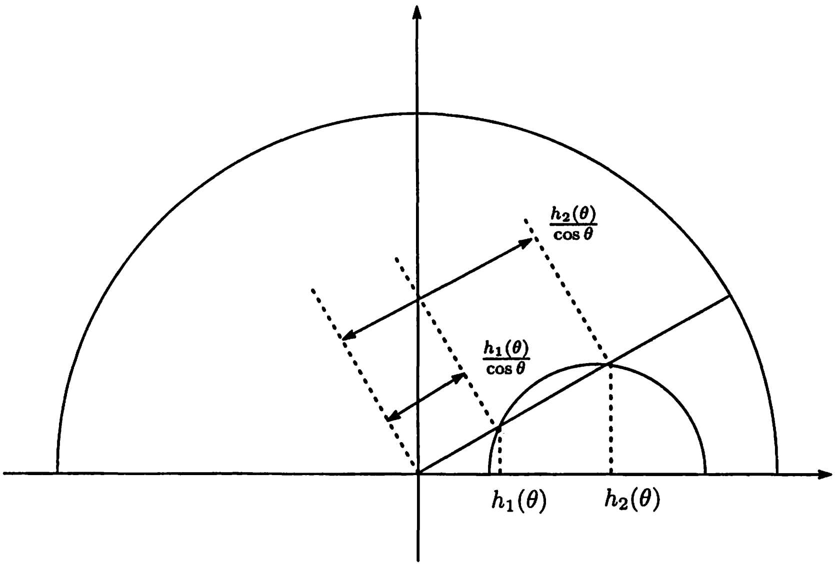


Figure 4: $h_1(t)$ and $h_2(t)$.

If we define

$$Q_1 := \left\{(r,\theta); 0 \leq r \leq k_1(\theta) \quad \text{and} \quad 0 \leq \theta \leq \arcsin\frac{1}{g}\right\},$$

then

$$\zeta_1 = \iint_{Q_1} B(r,\theta)\, dr d\theta. \tag{9}$$

Moreover, we convert r by $r = \frac{k_1(\theta)}{s}$ (where we keep θ), then (9) is converted to λ_1.

Next, we define Q_2 appropriately and convert ζ_2 as in (9). Then, converting r by $r = \dfrac{-2 + k_2(t)}{s} + 2$, we obtain λ_2.

Finally, using Q_3 in Definition 7 (iii), we convert ζ_3 as in (9). Then we obtain λ_3. This completes the proof for even g.

(ii) The case of odd g.

For $1 \leq i \leq 3$, we define $\tilde{P}_i$ by the following Figure 5.

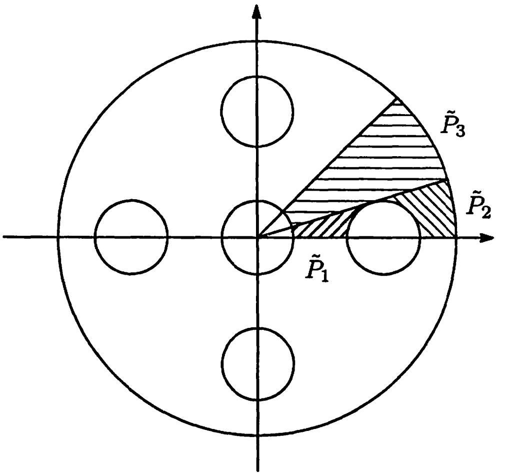


Figure 5: $\tilde{P}_i$ for $g = 5$.

If we define

$$\tilde{\zeta}_i := \iint_{\tilde{P}_i} A(u, v)\, dudv,$$

then, in contrast to (8), we have

$$(6) = \frac{4g - 4}{2\pi} \sum_{i=3}^{3} \tilde{\zeta}_i.$$

First, about $\tilde{\zeta}_i$, we first apply polar conversion, then convert by

$$r = \frac{k_1(t) - \frac{1}{g}}{s} + \frac{1}{g}.$$

Then we obtain λ_1.

Second, since $\tilde{\zeta}_2 = \zeta_2$, the argument for even g remains valid and we obtain λ_2.

Third, $\tilde{\zeta}_3$ is converted to λ_3. This completes the proof of Lemma 9. □

Proof of Theorem B (i). The cases for $g = 0$ and 1 are proved already. With the aid of a computer, we compute λ_i and σ for $2 \leq g \leq 8$. The results are as follows.

Table 3: The values ρ, λ_i and σ for X_g with $2 \le g \le 8$.

g	ρ	λ_1	λ_2	λ_3	σ
2	−1.97542	−0.434019	−2.15234	1.01556	−2
3	−3.64478	−2.05477	−2.17813	1.09131	−4
4	−5.97797	−0.491114	−2.22446	0.359384	−6
5	−7.68207	−1.92189	−2.45524	1.23553	−8
6	−9.98756	−0.4837	−2.31766	0.183367	−10
7	−11.9391	−1.86038	−2.59831	1.31709	−12
8	−13.9779	−0.469199	−2.39838	0.118686	−14

Here ρ is defined in (7). Thus σ is much accurate than ρ. □

Proof of Theorem B (ii). The above proof of Theorem B (i) for X_g works for Y_{g+1} after slight modifications. We indicate where to change. Similarly to (5), we define

$$A^\circ(u,v) := K_{Y_{g+1}}(u,v)\sqrt{E(u,v)G(u,v) - F(u,v)^2},$$

where E, F and G are defined with respect to a parametrization $f \circ p$ of Y_{g+1}. Since $f(p(D)) = Y_{g+1}$, instead of (6), we have

$$\frac{1}{2\pi}\iint_{Y_{g+1}} K_{Y_{g+1}}\, dV = \frac{1}{2\pi}\iint_D A^\circ(u,v)\, dudv. \tag{10}$$

We compute this.

First, using Example 7 (iv) and (v), we can compute the integrals of (10) for Y_1 and Y_2. The results are exactly 1 and 0, respectively. For example, similarly to the case X_1, the computation for Y_1 is given as follows:

$$(10) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 \frac{16\sqrt{2}r(3(-1+\cos 4t)r^4 + 96r^2 - 160)}{\sqrt{(-r^2+4)\left(9(1-\cos 4t)r^6 + 24(-1+\cos 4t)r^4 + 128\right)^3}}\, dr d\theta$$
$$= 1.$$

Next, for Y_{g+1} with $g \ge 2$, we change Definition 8 as follows: First, we change (i) to the definition:

$$B^\circ(r,\theta) := r \cdot A^\circ(r\cos\theta, r\sin\theta).$$

We use (ii) and (iii) as it is. About (iv), we substitute B° for B and define λ_i°. About (v), we define

$$\sigma^\circ := \begin{cases} \dfrac{2g}{2\pi}\displaystyle\sum_{i=1}^{3}\lambda_i^\circ & \text{if } g \text{ is even} \\[2ex] \dfrac{2g-2}{2\pi}\displaystyle\sum_{i=1}^{3}\lambda_i^\circ & \text{if } g \text{ is odd.} \end{cases}$$

Lemma 9° . *For all* Y_{g+1} *with* $g \geq 2$, *we have* $(10) = \sigma^\circ$.

Proof. The proof is quite similar to that of Lemma 9. □

Similarly to (7), we define ρ° as follows: Extending the function $B^\circ(r, \theta)$ to $\mu^\circ(r, \theta) : [0, 2] \times [0, 2\pi] \to \mathbb{R}$, we set

$$\rho^\circ := \frac{1}{2\pi} \iint_{[0,2]\times[0,2\pi]} \mu^\circ(r, \theta)\, dr d\theta.$$

Now similarly to Table 3, we have the following table.

Table 4: The values ρ°, λ_i° and σ° for Y_g with $3 \leq g \leq 5$.

g	ρ°	λ_1°	λ_2°	λ_3°	σ°
3	−0.945098	−1.65875	−1.10731	1.19526	−1
4	−1.92215	−2.5315	−1.15509	0.544998	−2
5	−2.94932	−1.6532	−0.785998	0.0830011	−3

Thus σ° is much accurate than ρ°. □

4 Flat embeddings

For our reference, we recall flat embeddings.

Proposition 10. (i) (The Clifford torus.) *We embed* $S^1 \times S^1$ *into* $\mathbb{R}^4$ *by*

$$\{(x, y, z, w) \in \mathbb{R}^4; x^2 + y^2 = 1 \quad \text{and} \quad z^2 + w^2 = 1\}.$$

Then $K = 0$ *everywhere.*

(ii) *We embed a Klein's bottle into* $\mathbb{R}^5$ *by the parametrization*

$$(u, v) \mapsto (\cos u \cos v, \sin u \cos v, 2\cos\frac{u}{2}\sin v, 2\sin\frac{u}{2}\sin v, \cos v),$$

where $(u, v) \in [0, \pi] \times [0, \pi]$. *Then* $K = 0$ *everywhere.*

Proof. (i) Since the covering map $\mathbb{R}^2 \to S^1 \times S^1$ is a local isometry, the result follows. (ii) is taken from [3, p.115]. The paper gives [7] as a reference. We can compute K using the Brioschi formula in §3. □

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