On the Cohomology of the mod p Steenrod Algebra

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by

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Introduction

Let p be an odd prime and A denote the mod p Steenrod algebra. A. Li levi i s made some computations of the cohomology $H^*(A)$ of A in his paper [1] and proved the vanishing theorem for $H^*(A)$ in [2]. Later J. P. May made extensive computations of this algebra for the range $t-s \leqq 2(2p^2+p+2)(p-1)-4$ in his thesis [3]. His procedure is to construct a spectral sequence which passes from the cohomology $H^*(E^0A)$ of E^0A, the associated graded Hopf algebra of A, to an algebre $E^0H^*(A)$ associated to the cohomology $H^*(A)$ and to compute the cohomology $H^*(E^0A)$ by the Koszul resolution.

The purpose of this paper is to compute the algebra $E^0H^*(A)$ for the range $t-s \leqq 2(3p^2+3p+4)(p-1)-2$, using the above spectral sequence together with a spectral sequence which passes from the cohomology of E^0E^0A, the associated graded algebra of E^0A, to an algebra associated to the cohomology $H^*(E^0A)$. Our main results is Theorem 4.4.

§ 1. Spectral Sequences

Let $I(A)$ be the augmentation ideal of the mod p Steenrod algebra A and $J(A^*)$ be its dual.

Let $\Phi_1 : I(A) \longrightarrow I(A)$ be the identity,

$\Phi_2 : I(A) \otimes I(A) \longrightarrow I(A)$ themultiplication, and

$$\Phi_n = \Phi_{n-1}(1 \otimes \cdots \otimes 1 \otimes \Phi_2) : \underbrace{I(A) \otimes \cdots \otimes I(A)}_{n \text{ factors}} \longrightarrow I(A).$$

Define $F_iA = A,\ i \geq 0;\ F_{-i}A = \mathrm{Im}\,\Phi_i,\ i > 0;$

$F^iJ(A^*) = 0,\ i \geq 0;\ F^{-i}J(A^*) = [I(A)/\mathrm{Im}\,\Phi_{i+1}]^*,\ i > 0.$

Then the associated graded Hopf algebra E^0A of A is defined by

$$E^0_{i,j}A = (F_iA/F_{i-1}A)_{i+j}.$$

THEOREM 1.1. ([3]. Theorem II.2.9.) *E^0A is primitively generated Hopf algebra and a basis for the primitive elements of E^0A is $\{Q_k \mid k \geq 0\} \cup \{P^i_j \mid i \geq 0, j \geq 1\}$. The relations of E^0A is given by*

i). $Q_iQ_j + Q_jQ_i = 0.$

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ii). $P^i_j Q_k - Q_k P^i_j = \delta_{i,k} Q_{j+k}$.

iii). $P^i_j P^k_l - P^k_l P^i_j = \delta_{i,k+l} P^k_{j+l}$, $\quad i \geq k$.

iv). $(P^i_j)^p = 0$.

Next we define a filtration of the cobar construction $C(A^*)$ by

$$F^p T^n J(A^*) = \sum_{i_1+\cdots+i_n+n=p} F^{i_1} J(A^*) \otimes \cdots \otimes F^{i_n} J(A^*).$$

Then the resulting spectral sequence is the May spectral sequence.

THEOREM 1.2. ([3]. Theorem I. 5. 1.) *There exists a spectral sequence* $\{E_r, d_r\}$, *converging to* $E^0H^*(A)$ *and having as its* E_2 *term* $H^*(E^0A)$. *Each* E_r *is a tri-graded algebra and each* d_r *is a homomorphism*

$$d_r : E_r^{v,u,t} \longrightarrow E_r^{u+r,v-r+1,t}$$

which is a derivation with respect to the algebra structure.

For the computations of $H^*(E_0A)$, we make use of a spectral sequence (See [8]). Let T be a free associative algebra generated by $\{x^i_j \mid i \geq 0, j \geq 1\} \cup \{y_k \mid k \geq 0\}$ and $\alpha : T \longrightarrow E^0A$ be an epimorphism defined by $\alpha(x^i_j) = P^i_j$ and $\alpha(y_k) = Q_k$. Now T is filtered with $F_p T$ spanned by all monomials of length $\leq p$, with 1 assigned length 0 and then E^0A is filtered by $F_p E^0A = \alpha(F_p T)$. Using this filtration, we define a filtration of the cobar construction $C(E^0A^*)$ by

$$F^p T^n J(E^0A^*) = \sum_{i_1+\cdots+i_n=p} F^{i_1} J(E^0A^*) \otimes \cdots \otimes F^{i_u} J(E^0A^*),$$

where $F^i J(E^0A^*) = [I(E^0A)/F_{i+1} I(E^0A)]^*$.

The resulting spectral sequence $\{\tilde{E}_r, \tilde{d}_r\}$ converges to $E^0H^*(E^0A)$ and each $\tilde{E}_r$ is a tetra-graded algebra and each $\tilde{d}_r$ is a homomorphism

$$\tilde{d}_r : \tilde{E}_r^{p,q,u,t} \longrightarrow \tilde{E}_r^{p+r,q-r+1,u,t}$$

which is a derivation with respect to the algebra structure, where p is the filtration degree, $p+q$ is the homological degree, u is May's filtration degree and t is the degree associated with the grading of A. Since the associated graded Hopf algebra E^0E^0A is a tensor product of an exterior algebra $E\{Q_k \mid k \geq 0\}$ and a twisted polynomial algebra $P\{P^i_j \mid i \geq 0, j \geq 1\} /((P^i_j)^p)$, the $\tilde{E}_1$ term $H^*(E^0 E^0 A)$ is equal to

$E\{R^i_j \mid i \geq 0, j \geq 1\} \otimes P\{S_k \mid k \geq 0\} \otimes P\{\tilde{R}^i_j \mid i \geq 0, j \geq 1\}$ as an algebra, where R^i_j, S_k and $\tilde{R}^i_j$ are represented by $[\xi_j^{p^i}]$, $[\tau_k]$ and $\sum_{k=1}^{p-1} (p-k, k)/p$ $[\xi_j^{(p-k)p^i} \mid \xi_j^{kp^i}]$, respectively. Similarly, we have the dual spectral sequence $\{\tilde{E}^r, \tilde{d}^r\}$, converging to $E^0H_*(E^0A)$ and having as its $\tilde{E}^1$ term $H_*(E^0E^0A)$ $= E\{(R^i_j)^* \mid i \geq 0, j \geq 1\} \otimes \Gamma\{(S_k)^* \mid k \geq 0\} \otimes \Gamma\{(\tilde{R}^i_j)^* \mid i \geq 0, j \geq 1\}$, where Γ denotes a divided polynomial algebra. Dualizing the differential $\bar{d}^r$ in the homology spectral sequence, we calculate the differential $\bar{d}_r$ in this spectral sequence.

We consider first the differential $\bar{d}_1$. Since $\bar{d}_1$ is a derivation, it is sufficient to calculate $\bar{d}_1(R^i_j)$, $\bar{d}_1(S_k)$ and $\bar{d}_1(\bar{R}^i_j)$. Ignoring the grading u and t, we have R^i_j, $S_k \in \bar{E}_1^{1,0}$ and $\bar{R}^i_j \in \bar{E}_1^{p,-p+2}$. Then we compute the differential $\bar{d}^1$ on $\bar{E}^1_{2,0}$ and $\bar{E}^1_{p+1,-p+2}$: that is, $\bar{d}^1(R^i_j\ R^k_l)^*$, $\bar{d}^1(R^i_j\ S_k)^*$, $\bar{d}^1(S_k\ S_l)^*$, $\bar{d}^1(R^i_j\ \bar{R}^k_l)^*$ and $\bar{d}^1(S_k\ \bar{R}^i_j)^*$. $(R^i_j\ R^k_l)^*$ is represented by $\{P^i_j\} * \{P^k_l\}$ in the bar construction $\bar{B}(E^0 A)$, where $*$ denotes the shuffle product. By the relation iii) in Theorem 1. 1, $d\{P^i_j\} * \{P^k_l\} = -\{[P^i_j, P^k_l]\} = -\delta_{i,k+l}\{P^k_{j+l}\}$ if $i \geqq k$, and $\delta_{k,i+j}\{P^i_{j+l}\}$ if $k \geqq i$. Then we have $\bar{d}^1(R^i_j\ R^k_l)^* = -\delta_{i,k+l}(R^k_{j+l})$ if $i \geqq k$ and $\delta_{k,i+j}(R^i_{j+l})^*$ if $k \geqq i$. $(R^i_j\ S_k)^*$ is represented by $\{P^i_j\} * \{Q_k\}$ and $d\{P^i_j\} * \{Q_k\} = -\{[P^i_j, Q_k]\} = -\delta_{i,k}\{Q_{j+k}\}$ by the relation ii) in Theorem 1. 1. Then we have $\bar{d}^1(R^i_j\ S_k)^* = -\delta_{i,k}(S_{k+j})^*$. $(S_k\ S_l)^*$ is represented by $\{Q_k \mid Q_k\}$ if $k = l$ and $\{Q_k\} * \{Q_l\}$, if $k \neq l$. $d\{Q_k \mid Q_k\} = \{(Q_k)^2\} = 0$ and $d\{Q_k\} * \{Q_l\} = \{[Q_k, Q_l]\} = 0$ by the relation i) in Theorem 1. 1. Then we have $\bar{d}^1(S_k\ S_l)^* = 0$. $(R^i_j\ \bar{R}^k_l)^*$ is represented by $-\{P^i_j\} * \{P^k_l \mid (P^k_l)^{p-1}\}$ and $d(-\{P^i_j\} * \{P^k_l \mid (P^k_l)^{p-1}\}) = \{[P^i_j, P^k_l] \mid (P^k_l)^{p-1}\} + \{P^k_l \mid [P^i_j, (P^k_l)^{p-1}]\}$ by the relation iv) in Theorem 1. 1. If $i \geqq k$, then $d(-\{P^i_j\} * \{P^k_l \mid (P^k_l)^{p-1}\}) = \delta_{i,k+l}\{P^k_{j+l} \mid (P^k_l)^{p-1}\} - \delta_{i,k+l}\{P^k_l \mid P^k_{j+l}(P^k_l)^{p-2}\}$. Since $d(\{P^k_{j+l} \mid P^k_l \mid (P^k_l)^{p-2}\} - \{P^k_l \mid P^k_{j+l} \mid (P^k_l)^{p-2}\}) = \{P^k_{j+l} \mid (P^k_l)^{p-1}\} - \{P^k_l \mid P^k_{j+l}(P^k_l)^{p-2}\}$, we have $\bar{d}^1(R^i_j\ \bar{R}^k_l)^* = 0$ if $i \geqq k$. Similarly, we have $\bar{d}^1(R^i_j\ \bar{R}^k_l)^* = 0$ if $k \geqq i$. $(S_k\ \bar{R}^i_j)^*$ is represented by $-\{Q_k\} * \{P^i_j \mid (P^i_j)^{p-1}\}$ and $d(-\{Q_k\} * \{P^i_j \mid (P^i_j)^{p-1}\}) = \{[P^i_j, Q_k] \mid (P^i_j)^{p-1}\} - \{P^i_j \mid [(P^i_j)^{p-1}, Q_k]\} = \delta_{i,k}\{Q_{j+k} \mid (P^i_j)^{p-1}\} + \delta_{i,k}\{P^i_j \mid Q_{j+k}(P^i_j)^{p-2}\}$. Since $d(\{Q_{j+k} \mid P^i_j \mid (P^i_j)^{p-2}\} + \{P^i_j \mid Q_{j+k} \mid (P^i_j)^{p-2}\}) = -\{Q_{j+k} \mid (P^i_j)^{p-1}\} - \{P^i_j \mid Q_{j+k}(P^i_j)^{p-2}\}$, we have $\bar{d}^1(S_k\ \bar{R}^i_j)^* = 0$. Dualizing above results, we have following proposition.

Proposition 1.3. *The differential $\bar{d}_1$ is given by*

i). $\bar{d}_1(R^i_j) = -\sum_{k=1}^{j-1} R^{i+k}_{j-k}\ R^i_k.$

ii). $\bar{d}_1(S_k) = -\sum_{i=0}^{k-1} R^i_{k-i}\ S_i.$

iii). $\bar{d}_1(\bar{R}^i_j) = 0.$

For further computation of this spectral sequence, we make use of a sequence of spectral sequences. (See [3]. II.4.4.~II.4.6.)

Let $X_n = E\{R_j^i \mid j \leqq n\} \otimes P\{S_k \mid k \leqq n-1\} \otimes P\{\tilde{R}_h^l \mid h \leqq n\}$ be the subcomplex and subalgebra of $(\tilde{E}_1, \tilde{d}_1)$ and $Z_n = E\{R_n^i\} \otimes P\{S_{n-1}\} \otimes P\{\tilde{R}_n^l\}$. Then filter X_n by $x \otimes z \in F^s X_n$ if and only if x has homological degree greater than or equal to s, where $x \in X_{n-1}$ and $z \in Z_n$, and $\sum x_i \otimes z_i \in F^s X_n$ if and only if some $x_i \otimes z_i \in F^s X_n$, $x_i \in X_{n-1}, z_i \in Z_n$. Let $\{{}_nE_r, d_r\}$ denote the resulting spectral sequence. It is easy to see that ${}_nE_1 = {}_nE_0 = X_{n-1} \otimes Z_n$ as an algebra and d_1 is given by $d_1(R_j^i) = -\sum_{k=1}^{j-1} R_{j-k}^{i+k} R_k^i$ if $j < n$, $d_1(S_i) = -\sum_{k=0}^{i-1} R_{i-k}^k S_k$ if $i < n-1$ and $d_1(R_n^i) = d_1(S_{n-1}) = d_1(\tilde{R}_j^i) = 0$. Therefore, ${}_nE_2^{s,t} = H^s(X_{n-1}) \otimes Z_n^t$ and ${}_nE_2$ is an associative differential algebra. d_2 is given by $d_2(R_n^i) = -\sum_{k=1}^{n-1} R_{n-k}^{i+k} R_k^i$, $d_2(S_{n-1}) = -\sum_{k=0}^{n-2} R_{n-k-1}^k S_k$ and $d_2(\tilde{R}_j^i) = 0$. Since $H^{s,t}(X_n) = E_2^{s,t}$ for $t < 2p^{n-1} - 1$, in order to compute $\tilde{E}_2$ term it suffices to calculate the $H^*(X_n)$ successively.

§ 2. The algebra $E^0H^*(E^0A)$

We now begin the calculation of the third spectral sequences. (Compare with II-4 in [3]). $X_1 = H^*(X_1) = Z_1$ as an algebra, hence ${}_2E_2 = Z_1 \otimes Z_2$ as an algebra. The differential d_2 in ${}_2E_2$ is given by $d_2(R_2^i) = -R_1^{i+1} R_1^i$ and $d_2(S_1) = -R_1^0 S_0$.

PROPOSITION 2. 1. ([3]. Proposition II. 4. 3.) *A basis for the indecomposable elements of ${}_2E_3$ consists of*

1). $f_2 : -R_2^0 S_0 - R_1^1 S_1$

$e_{i,3} : -R_1^{i+2} R_2^i - R_2^{i+1} R_1^i,\ i \geqq 0.$

2). $b_{ij} : \tilde{R}_j^i,\ j = 1$ or $2,\ i \geqq 0.$

3). $a_0 : S_0, \quad a_1 : S_1^p$

4). $h_{i+1}(j,k) : (R_2^{i+1})^{j-1} R_1^{i+1} (R_2^i)^{k-1},\ j = 1$ or $2,\ k = 1$ or $2,\ i \geqq 0.$

5). $h_0^l(j,1) : (R_2^0)^{j-1} R_1^0 S_1^l,\ j = 1$ or $2,\ 0 \leqq l \leqq$ p$-1.$

The abbreviated notations $h_i = h_i(1,1)$, $g_{1,l} = h_0^l(1,1)$ and $g_{2,l} = h_l^0(2,1)$ will also be used, as will be $h_{i+1}^0(j,k) = h_{i+1}(j,k)$. Commutativity, associativity and the following relations determine ${}_2E_3$ as an algebra (unless explicitly restricted, i, j, k, l, etc. take all values consistent with the list of generators) :

1. a). $f_2^2 = 0$

b). $h_2 f_2 = e_{0,3} a_0$

c). $e_{i+1,3} h_i = h_{i+3}\, e_{i,3}$

2. a). $e_{i,3}\, h(2,k) = 0$

b). $e_{i,3}\, h_{i+1}(2,2) = 0$

c). $e_{i,3}\ h_{i+2}(j, 2) = 0$

3. a). $f_2\ h_1(j, 2) = 0$

b). $f_2\ g_{1,p-2} = 0$

4. $h_i^{\ell}(j, k)\ h_i^{\ell'}(j', k') = 0$

5. a). $h_{i+1}\ h_i = 0$

b). $h_{i+1}(j, 2)\ h_i^{\ell}(2, k) = 0$

c). $h_{i+1}(j, 1)\ h_i^{\ell}(2, k) = h_{i+1}(j, 2)\ h_i^{\ell}(1, k)$

d). $h_{i+2}\ h_{i+1}(1, 2) = h_{i+1}(2, 1)\ h_i$

6. a). $e_{i,3}\ h_{i+1} = h_{i+1}(2, 1)\ h_i + h_{i+2}\ h_{i+1}(1, 2)$

b). $e_{i,3}\ h_{i+1}(2, 1) = -\ h_{i+2}(1, 2)\ h_{i+1}(1, 2)$

c). $e_{i,3}\ h_{i+1}(1, 2) = -\ h_{i+1}(2, 1)\ h_i(2, 1)$

d). $e_{i,3}\ h_i^{\ell}(1, k) = -\ h_{i+2}\ h_i^{\ell}(2, k)$

e). $e_{i,3}\ h_{i+2}(j, 1) = -\ h_{i+2}(j, 2)\ h_i$

f). $e_{i,3}\ e_{i+1,3} = -\ 2\ h_{i+2}(2, 1)\ h_{i+1}(1, 2)$

g). $(e_{i,3})^2 = -2\ h_{i+2}(1, 2)\ h_i(2, 1)$

h). $e_{0,3}\ f_2 = -2\ h_1(2, 1)\ g_{1,1}$

7. a). $g_{1,\ell}\ a_0 = 0,\ 0 \leqq l \leqq p-2$

b). $(l+1)\ g_{2,\ell}\ a_0 = h_1\ g_{1,\ell+1},\ 0 \leqq l \leqq p-2$

8. a). $f_2\ g_{1,\ell} = -\ g_{2,\ell}\ a_0 - h_1\ g_{1,\ell+1},\ 0 \leqq l \leqq p-3$

b). $f_2\ g_{1,p-1} = -\ g_{2,p-1}\ a_0$

c). $f_2\ g_{2,\ell} = -\ h_1\ g_{2,\ell+1},\ 0 \leqq l \leqq p-2$

b). $f_2\ g_{2,p-1} = -\ h_1\ h_0(2,1)\ a_1$

e). $h_1(j, 1)\ f_2 = -\ h_1(j, 2)\ a_0$

It is easy to see that $d_r = 0,\ r > 2$ and ${}_2E_\infty = H^*(X_2)$ as an algebra. We can now begin the calculation of the spectral sequence $\{{}_3E_r\}$. ${}_3E_2$ is the differential algebra $H^*(X_2) \otimes Z_3$ with differential determined by $d_2(R_3^i) = e_{i,3}$, $d_2(S_2) = f_2$ and $d_2(\tilde{R}_3^i) = 0$. The image of $H^*(X_2)$ in ${}_3E_3$ is therefore $H^*(X_2)/I$, where I is the ideal in $H^*(X_2)$ generated by $\{e_{i,3}\}$ and f_2

Proposition 2. 2. *In dimensions* $t\text{-}s \leqq (3p^2+3p+4)q-1,\ q=2(p-1)$, *a basis for the indecomposable elements of* ${}_3E_3$ *not in* $H^*(X_2)$ *is given by*

1. $\Phi_3 :\ -\ R_3^0\ a_0 - h_2\ S_2$

2. $\varepsilon_{0,4} :\ -\ R_3^1\ h_0 - h_3\ R_3^0$

3. $a_2 : S_2^p$

4. $b_{03} : \tilde{R}_3^0$

5. $\gamma : f_2\ S_2^{p-1}$

6. $\mu_\ell : g_{1,p-2}\ S_2^{\ell},\ 1 \leqq l \leqq p-1$

7. $g_{3,\ell} : R_3^0\ g_{2,\ell},\ 0 \leqq l \leqq p-1$

8. $k_{1,l}$: $h_1(1,2)S_2^l$, $1 \leqq l \leqq p-1$

$k_{2,l}$: $h_1(2,2)\ S_2^l$, $1 \leqq l \leqq p-1$

9. $\nu_{l,m}(1,1)$: $h_2\, g_{1,l}\ S_2^m$, $0 \leqq l \leqq p-3$, $2 \leqq m \leqq p-1$

$\nu_{l,m}(1,2)$: $h_2\, g_{2,l}\ S_2^m$, $0 \leqq l \leqq p-3$ or $l = p-1$, $2 \leqq m \leqq p-1$

$\nu_{l,m}(2,1)$: $R_3^0\, h_2\, g_{1,l}\ S_2^m$, $0 \leqq m \leqq p-1$ if $0 \leqq l \leqq p-2$, $m=0$ if $l=p-1$

$\nu_{l,m}(2,2)$: $R_3^0\, h_2\, g_{2,l}\ S_2^m$, $0 \leqq l \leqq p-1$, $2 \leqq m \leqq p-1$

10. $\sigma_{l,m}$: $g_{1,l}\ h_1(2,1)\ S_2^m$, $2 \leqq l \leqq p-3$, $1 \leqq m \leqq p-1$

11. $h_1^m(1,2,1)$: $R_3^0\, h_1(2,2)\ S_2^m$, $0 \leqq m \leqq p-1$

12. y : $R_3^0\, h_2(1,2)$

13. j_l : $h_1(1,2)\ h_0\ R_3^0\, S_2^l$, $1 \leqq l \leqq p-1$

14. λ_l : $R_3^0\, a_0\, g_{2,l}\ S_2^{p-1}$, $0 \leqq l \leqq p\text{-}3$ or $l = p-1$

15. $\xi_{l,m}$: $h_2(1,2)\ g_{2,l+1}\ S_2^{m-1} - (m-1)\ R_3^0\, h_1(2,1)\ g_{1,l+2}\ S_2^{m-2}$,
$0 \leqq l \leqq p-4$ or $l = p-2$, $2 \leqq m \leqq p$, *where* $g_{1,p} = h_0\, a_1$

16. τ : $h_2(1,2)\ g_{2,p-2}\ S_2 - R_3^0\, h_1(2,1)\ g_{1,p-1}$

In the cited range, a minimal set of relations of ${}_3E_3$ *is given by those relations holding in the image of* $H^*(X_2)$ *and by the following relations*

1. a). $g_{2,l}\ \Phi_3 = -(l+3)\ a_0\ g_{3,l}$, $0 \leqq l \leqq p-1$

b). $k_{1,l-1}\ \Phi_3 = -(l+2)/l\ h_2\, k_{1,l}$, $1 \leqq l \leqq p-1$

c). $\mu_l\ \Phi_3 = h_2\ \mu_{l+1}$, $0 \leqq l \leqq p-2$, *here* $\mu_0 = g_{1,p-2}$

$\mu_{p-1}\ \Phi_3 = h_2\ g_{1,p-2}\ a_2$

d). $h_3\ \Phi_3 = a_0\ \varepsilon_{0,4}$

e). $h_2(1,2)\ \Phi_3 = -a_0\ y$

f). $\nu_{p-1,p-1}(1,2)\ \Phi_3 = a_0\ \nu_{p-1,p-1}(2,2) = \nu_{p-1,0}(2,1)\ \gamma$

g). $h_2\, g_{1,p-1}\ \Phi_3 = -a_0\ \nu_{p-1,0}(2,1)$

h). $g_{1,p-1}\ \gamma\ \Phi_3 = -a_0\ \lambda_{p-1}$

i). $\lambda_{l,p-2}\ \Phi_3 = h_2\ \sigma_{l,p-1}$, $2 \leqq l \leqq p-3$

j). $k_{2,m}\ \Phi_3 = (m+2)\ a_0\ h_1^m(1,2,1)$, $0 \leqq m \leqq p-1$

k). $k_{1,p-1}\, g_{1,p-1}\ \Phi_3 = -h_1\ \lambda_{p-1}$

l). $k_{2,p-1}\, g_{1,p-1}\ \Phi_3 = -h_1(2,1)\ \lambda_{p-1}$

m). $h_2(1,2)\ g_{2,p-2}\ \Phi_3 = h_1(2,1)\ \nu_{p-1,0}(2,1)$

n). $h_1(1,2)\ g_{1,p-1}\ \Phi_3 = -a_0\ \tau$

2. a). $\nu_{l,m}(1,1)\ \Phi_3 = 0$, $0 \leqq l \leqq p-3, 2 \leqq m \leqq p-1$

$\nu_{l,m}(1,2)\ \Phi_3 = 0$, $0 \leqq l \leqq p-3$, $2 \leqq m \leqq p-1$

$\nu_{p-1,m}(1,2)\ \Phi_3 = 0$, $2 \leqq m \leqq p-2$

$\nu_{l,m}(2,1)\ \Phi_3 = 0$, $0 \leqq l \leqq p-2$, $0 \leqq m \leqq p-1$

$\nu_{p-1,0}(2,1)\ \Phi_3 = 0$

b). $\sigma_{l,m}\ \Phi_3 = 0$, $2 \leqq l \leqq p-3, 1 \leqq m \leqq p-3$ *or* $m = p-1$

c). $j_l \Phi_3 = 0,\ 1 \leqq l \leqq p-1$

d). $\lambda_l \Phi_3 = 0,\ 0 \leqq l \leqq p-3$

e). $\Phi_3^2 = 0$

f). $h_0 k_{1,m} \Phi_3 = 0,\ 1 \leqq m \leqq p-1$

g). $a_0 g_{3,p-3} \Phi_3 = 0$

h). $h_1(2,1)\ g_{1,l} \Phi_3 = 0,\ 2 \leqq l \leqq p-3$

i). $g_{1,l}\, \gamma\, \Phi_3 = 0,\ 0 \leqq l \leqq p-4$

j). $h_2 g_{1,l} \Phi_3 = 0,\ 0 \leqq l \leqq p-3$

k). $h_1 g_{3,0} \Phi_3 = 0,$

l). $h_2 g_{3,l} \Phi_3 = 0,\ 0 \leqq l \leqq p-1$

3. a). $a_0 \mu_l = 0,\ 1 \leqq l \leqq p-1$

b). $a_0\ \nu_{l,m}(1,j) = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-3,\ 2 \leqq m \leqq p-1$

$a_0\ \nu_{p-1,m}(1,2) = 0,\ 2 \leqq m \leqq p-2$

$a_0\ \nu_{p-1,p-1}(1,2) = h_2 g_{1,p-1}\ \gamma$

$a_0\ \nu_{l,m}(2,1) = 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-1$

$a_0\ \nu_{l,m}(2,2) = 0,\ 0 \leqq l \leqq p-2,\ 2 \leqq m \leqq p-1,$ *or* $l = p-1,$
$2 \leqq m \leqq p-2$

c). $a_0 k_{j,l} = 0,\ j = 1$ *or* $2,\ 1 \leqq l \leqq p-2$

$a_0 k_{1,p-1} = -h_1\ \gamma$

$a_0 k_{2,p-1} = -h_1(2,1)\ \gamma$

d). $a_0 \sigma_{l,m} = 0,\ 2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-1$

e). $a_0 j_l = 0,\ 1 \leqq l \leqq p-1$

f). $a_0 \lambda_l = 0,\ 0 \leqq l \leqq p-3$

g). $a_0 h_1^{l-1}(1,2,1) = 1/l\ h_2 k_{2,l},\ 1 \leqq l \leqq p-1$

h). $a_0 \xi_{l,m} = 0,\ 0 \leqq l \leqq p-2,\ l \neq p-3,\ 2 \leqq k \leqq p-1$

$a_0 \xi_{l,p} = 1/(l+2)\ h_1(2,1)\ \nu_{l+2,p-1}(1,1),\ 0 \leqq l \leqq p-5$

$a_0 \xi_{p-4,p} = -1/2\ h_2 h_1(2,1)\ \mu_{p-1}$

$a_0 \xi_{p-2,p} = h_2(1,2)\ g_{1,p-1}\ \gamma$

i). $a_0 g_{3,p-3}\ \gamma = 0$

j). $a_0 h_2 g_{3,p-3} = 0$

k). $a_0^2 g_{3,p-3} = 0$

l). $a_0 g_{3,l}\ \gamma = 0,\ 0 \leqq l \leqq p-4$

m). $a_0 h_1 g_{3,0} = 0$

n). $a_0 g_{1,p-1} h_1^{p-1}(1,2,1) = -h_1(2,1)\ \nu_{p-1,p-1}(1,2)$

o). $a_1 j_l = 0,\ 1 \leqq l \leqq p-2$

$a_1 j_{p-1} = g_{3,p-1}\ \gamma$

p). $a_1 h_1 g_{3,0} = 0$

q). $a_1 h_0 k_{1,l} = 0,\ 1 \leqq l \leqq p-2$

4. a). $g_{1,p-3}\ \gamma = 1/2\ h_1\ \mu_{p-1}$

$g_{1,p-2}\ \gamma = 0$

$g_{2,l}\ \gamma = -\ g_{1,l+1}\ k_{1,p-1},\ 0 \leqq l \leqq p-2$

$g_{2,p-1}\ \gamma = -\ h_0\ k_{1,p-1}\ a_1$

b). $g_{j,l}\ \mu_m = 0,\ j = 1$ or $2,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

c). $g_{j,l}\ \nu_{k,m}(l',k') = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,$ *and* $k,\ m,\ l',k'$ *take all values consistent with the list of generators*

d). $g_{j,l}\ g_{3,m} = 0,\ j = 1$ *or* $2,\ 0 \leqq l,\ m \leqq p-1$

e). $g_{1,l}\ k_{1,m} = 0,\ 1 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-2$

$h_0\ k_{2,l} = 0,\ 1 \leqq l \leqq p-1$

$g_{2,l}\ k_{j,m} = 0,\ 0 \leqq l \leqq p-1,\ j = 1$ *or* $2,\ 1 \leqq m \leqq p-1$

f). $g_{j,l}\ \sigma_{l',m} = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,\ 2 \leqq l' \leqq p-3,\ 1 \leqq m \leqq p-1$

g). $g_{j,l}\ j_m = 0,\ 0 \leqq l \leqq p-1,\ j = 1$ *or* $2,\ 1 \leqq m \leqq p-1$

h). $g_{1,l}\ h_1^0\ (1,\ 2,) = h_1(2,\ 1)\ g_{3,l},\ 0 \leqq l \leqq p-1$

$g_{2,l} h_1^m(1,\ 2,\ 1) = 0,\ 0 \leqq l,\ m \leqq p-1$

i). $g_{2,l}\ y = h_2(1,\ 2)\ g_{3,l},\ 0 \leqq l \leqq p-1$

j). $g_{j,l}\ \lambda_m = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-3$ *or* $m = p-1$

k). $g_{j,l}\ \xi_{l',m} = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,\ 0 \leqq l' \leqq p-4$ *or* $l' = p-2,\ 2 \leqq m \leqq p$

l). $g_{j,l}\ \tau = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1$

5. a). $h_1(2,\ 1)\ \mu_1 = 0,\ 0 \leqq l \leqq p-1$ *if* $p=3$

b). $h_1(2,\ 1)\ k_{1,l} = 0,\ 1 \leqq l \leqq p-1$

c). $h_1(2,\ 1)\ \nu_{l,m}(1,\ 1) = 0,\ 0 \leqq l \leqq p-3$ *if* $2 \leqq m \leqq p-2,$ $0 \leqq l \leqq 1$ *if* $m = p-1$

$h_1(2,\ 1)\ \nu_{l,p-1}(1,1) = h_2\ \sigma_{l,p-1} = 1/(l+1) h_2(1,\ 2) g_{1,l-1}\ \gamma,$ $2 \leqq l \leqq p-3$

$h_1(2,\ 1)\ \nu_{l,m}\ (1,\ 2) = 0,\ 2 \leqq m \leqq p-1$ *if* $0 \leqq l \leqq p-3,$ $2 \leqq m \leqq p-2$ *if* $l = p-1$

$h_1(2,\ 1)\ \nu_{l,m}(2,\ 1) = 0,\ 0 \leqq m \leqq p-1$ *if* $l = 0,\ 0 \leqq m \leqq p-2$ *if* $1 \leqq l \leqq p-2$

$h_1(2,\ 1)\ \nu_{l,m}(2,\ 2) = 0,\ 1 \leqq l \leqq p-1,\ 2 \leqq m \leqq p-2$

d). $h_1(2,\ 1)\ k_{2,l} = 0,\ 1 \leqq l \leqq p-1$

e). $h_1(2,\ 1)\ \sigma_{l,m} = 0,\ 2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-1$

f). $h_1(2,\ 1)\ h_1^m(1,\ 2,\ 1) = 0,\ 0 \leqq m \leqq p-1$

g). $h_1(2,\ 1)\ y = 0$

h). $h_1(2,\ 1)\ j_l = 0,\ 1 \leqq l \leqq p-1$

i). $h_1(2,\ 1)\ \lambda_l = 0,\ 0 \leqq l \leqq p-3$

j). $h_1(2,\ 1)\ \tau = 0$

6. a). $k_{j,l}\ \gamma = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1$

b). $k_{1,l}\ \mu_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1,\ l+m \neq p-1$

$k_{1,l}\ \mu_{p-l-1} = -\ g_{2,p-3}\ \gamma,\ 0 \leqq l \leqq p-1,$ *here* $\mu_0 = g_{1,p-2}$

$k_{2,l}\ \mu_m = k_{2,l'}\ \mu_{m'}$ *if* $0 \leqq l,\ l',m,\ m' \leqq p-1,\ l+m=l'+m'$

$= k_{2,l''} \mu_{m''} a_2$ *if* $0 \leqq l, l'', m, m'' \leqq p-1,\ l+m = l''+m''+p$

e). $k_{j,l}\, \nu_{k,m}(l', k') = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1$, *and* k, m, l', k' *take all values consistent with the list of generators*

d). $k_{j,l}\, g_{3,m} = 0,\ j = 1$ *or* $2,\ 0 \leqq l, m \leqq p-1$

e). $k_{j,l}\, k_{j',l'} = 0,\ j = 1$ *or* $2,\ j' = 1$ *or* $2,\ 0 \leqq l, l' \leqq p-1$

f). $k_{j,l}\, \sigma_{l',m} = 0, j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,\ 2 \leqq l' \leqq p-3,\ 1 \leqq m \leqq p-1$

g), $k_{2,l}\, y = 0,\ 0 \leqq l \leqq p-1$

h). $k_{j,l}\, j_m = 0,\ j = 1$ *or* $2,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

i). $k_{1,l}\, h_2(1, 2) = h_2\, k_{2,l},\ 1 \leqq l \leqq p-1$

j). $k_{1,l}\, h_1^m(1, 2, 1) = 0,\ 0 \leqq l, m \leqq p-1$

k). $k_{1,l}\, \lambda_m = 0,\ 0 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-3$ *or* $m = p-1$

l). $k_{1,l}\, \xi_{l',m} = 0,\ 0 \leqq l \leqq p-1,\ 0 \leqq l' \leqq p-4$ *or* $l' = p-2,\ 2 \leqq m \leqq p$

m). $k_{1,l}\, \tau = 0,\ 1 \leqq l \leqq p-1$

7. a). $h_1\, \mu_l = 0,\ 1 \leqq l \leqq p-2$

b). $h_i\, \nu_{l,m}(k, j) = 0,\ i = 1$ *or* 2, l, m, k, j *take all values consistent with the list of generators*

c). $h_1\, g_{3,l} = 0,\ 1 \leqq l \leqq p-1$

d). $h_1\, k_{j,l} = 0,\ j = 1$ *or* $2,\ 1 \leqq l \leqq p-1$

e). $h_1\, \sigma_{l,m} = 0,\ 2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-1$

f). $h_i\, j_m = 0,\ i = 1$ *or* $2,\ 1 \leqq m \leqq p-1$

g). $h_i\, \lambda_m = 0,\ i = 1$ *or* $2,\ 0 \leqq m \leqq p-3$

$h_2\, \lambda_{p-1} = -\nu_{p-1,0}(2, 1)\, \gamma$

h). $h_1\, h_1^m(1, 2, 1) = 0,\ 0 \leqq m \leqq p-1$

$h_2\, h_1^0(1, 2, 1) = 0,$

i). $h_1\, \xi_{l,m} = 0,\ 0 \leqq l \leqq p-4$ *or* $l = p-2,\ 2 \leqq m \leqq p-1,$

or $0 \leqq l \leqq p-4,\ k = p$

$h_1\, \xi_{p-2,p} = h_1(2, 1)\, \nu_{p-1,p-1}(1, 2)$

$h_2\, \xi_{l,m} = 0,\ 0 \leqq l \leqq p-4$ *or* $l = p-2,\ 2 \leqq m \leqq p$

j). $h_1\, \tau = 0$

$h_2\, \tau = -a_0\, h_2(1, 2)\, g_{3,p-2}$

k). $h_2\, h_0\, k_{1,p-2} = 0$

l). $h_2\, g_{1,l}\, \gamma = 0,\ 0 \leqq l \leqq p-4$

m). $h_2\, g_{3,l}\, \gamma = 0,\ 0 \leqq l \leqq p-2$

8. a). $h_2(1, 2)\, \nu_{l,m}(j, k) = 0$, l, m, j, k *take all values consistent with the list of generators*

b). $h_2(1, 2)\, k_{2,l} = 0,\ 1 \leqq l \leqq p-1$

c). $h_2(1, 2)\, \sigma_{l,m} = 0,\ 2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-1$

d). $h_2(1, 2)\, h_1^m(1, 2, 1) = 0,\ 0 \leqq m \leqq p-1$

e). $h_2(1, 2)\, y = 0$

f). $h_2(1, 2)\, \lambda_m = 0,\ 0 \leqq m \leqq p-4$

$h_2(1, 2)\ \lambda_m = 1/(m+2)\ yg_{1,m}\gamma,\ 0 \leqq m \leqq p-3$

9. a). $\mu_l\ \gamma = 0,\ 1 \leqq l \leqq p-1$

b). $\mu_l\ \mu_{l'} = 0,\ 1 \leqq l,\ l' \leqq p-1$

c). $\mu_l\ \nu_{l',m}(j,\ k) = 0,\ 1 \leqq l \leqq p-1,\ l'$ *,m, j, k take all values consistent with the list of generators*

d). $\mu_l\ g_{3,m} = 0,\ 1 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-1$

e). $\mu_l\ h_1^m(1,2,1) = \mu_{l'}\ h_1^{m'}(1,2,1)$ *if* $0 \leqq l, l', m, m' \leqq p-1,\ l+m=l'+m'$

$\mu_{l^*}\ h_1^{m^*}(1,2,1)a_2$ *if* $0 \leqq l, l^*, m, m^* \leqq p-1,\ l+m=l^*+m^*+p$

f). $\mu_l\ \sigma_{l',m}, = 0,\ 1 \leqq l, m \leqq p-1,\ 2 \leqq l' \leqq p-3$

g). $\mu_l\ j_m = 0,\ 1 \leqq l, m \leqq p-1$

h). $\mu_l\ \lambda_m = 0,\ 0 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-3$ *or* $m = p-1$

i). $\mu_l\ \xi_{l',m} = 0,\ 1 \leqq l \leqq p-1,\ 0 \leqq l' \leqq p-4$ *or* $l' = p-2,\ 2 \leqq m \leqq p$

j). $\mu_l\ \tau = 0,\ 1 \leqq l \leqq p-1$

10. a). $\gamma^2 = 0$

b). $\gamma\ \nu_{l,m}(1,1) = 0,\ 0 \leqq l \leqq p-3,\ 2 \leqq m \leqq p-1$

$\gamma\ \nu_{l,m}(1,2) = 0,\ 0 \leqq l \leqq p-3$ *or* $l = p-1,\ 2 \leqq m \leqq p-1$

$\gamma\ \nu_{l,m}(2,1) = 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-1$

c). $\gamma j_l = 0,\ 1 \leqq l \leqq p-1$

d). $\gamma\ \lambda_l = 0,\ 0 \leqq l \leqq p-3,$ *or* $l = p-1$

e). $\gamma\ \sigma_{l,m} = 0,\ 2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-1$

11. a). $\nu_{l,m}(j,k)\ \nu_{l'm'},(j',k') = 0,\ l, l'\ m, m', j, j', k, k'$ *take all values consistent with the list of generators*

b). $\nu_{l,m}(j,k)\ g_{3,l'} = 0,\ 0 \leqq l' \leqq p-1,\ l,\ m,\ j,\ k$ *take all values consistent with the list of generators*

c). $\nu_{l,m}(j,k)\ \sigma_{l',m'} = 0,\ 2 \leqq l' \leqq p-3,\ 1 \leqq m' \leqq p-1,\ l, m, j, k$ *take all values consistent with the list of generators*

d). $\nu_{l,m}(j,k)\ j_{m'} = 0,\ 1 \leqq m' \leqq p-1,\ l,\ m,\ j,\ k$ *take all values consistent with the list of generators*

e). $\nu_{p-1,l}(1,2)\ h_1^m(1,2,1) = 0,\ 1 \leqq m \leqq p-1,\ 2 \leqq l \leqq p-1$

12. a). $g_{3,l}g_{3,l'} = 0,\ 0 \leqq l,\ l' \leqq p-1$

b). $g_{3,l}\ \sigma_{l',m} = 0,\ 0 \leqq l \leqq p-1,\ 2 \leqq l' \leqq p-3,\ 1 \leqq m \leqq p-1$

c). $g_{3,l}h_1^m(1,2,1) = 0,\ 0 \leqq l,\ m \leqq p-1$

d). $g_{3,l}\ j_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

e). $\sigma_{l,m}\ \sigma_{l',m'} = 0,\ 2 \leqq l, l' \leqq p-3,\ 1 \leqq m,\ m' \leqq p-1$

f). $j_l\ \sigma_{l',m} = 0,\ 1 \leqq l,\ m \leqq p-1,\ 2 \leqq l' \leqq p-3$

g). $j_l\ j_{l'} = 0,\ 1 \leqq l, l' \leqq p-1$

PROOF. The proof for the indecomposable elements consists of a rather tedious but routine inspection of the structure of $H^*(X_2)$, in particular, of the annihilators of generators of the ideal I. It is easily seen by considering all possible cochains that no other indecomposable classes occur. Many of the relations listed are implied by the algebra structure of ${}_3E_2$. The other are easy to derive. For example, $d_2(R_3^0\, g_{1,l}\, S_2) = e_{0,3}g_{1,l}\, S_2 + R_3^0\, g_{1,l}\, f_2 = -g_{2,l}\, h_2\, S_2 + (l+2)\, R_3^0\, g_{2,l}\, a_0 = g_{2,l}\, \Phi_3 + (l+3)\, a_0\, g_{3,l}$, for $0 \leqq l \leqq p-1$, which proves 1.a). That the relations listed generate all others is seen by examining all possible products.

PROPOSITION 2. 3. *The following list gives all non-zero higher differentials on those indecomposable elements of* ${}_3E_3$ *satisfying*
$t - s \leqq (3p^2+3p+4)q-1,\ q=2(p-1)$.

1. $d_3(\mu_l) = l(l-1)k_{1,l-2}a_1,\ 2 \leqq l \leqq p-1$
2. $d_3(\nu_{l,2}(1,1)) = 2(l+3)/(l+1)h_1(2,1)g_{1,l+2},\ 0 \leqq l \leqq p-4$
   $d_3(\nu_{l,m}(1,1)) = m(m-1)(l+3)/(l+1)\sigma_{l+2,m-2},\ 3 \leqq m \leqq p-1,\ 0 \leqq l \leqq p-5$
   $d_3(\nu_{p-4,m}(1,1)) = m(m-1)/3\ h_1(2,1)\mu_{m-2},\ 3 \leqq m \leqq p-1$
   $d_3(\nu_{l,m}(1,2)) = m(m-1)k_{2,m-2}\, g_{1,l+2},\ 0 \leqq l \leqq p-3,\ 2 \leqq m \leqq p-1$
   $d_3(\nu_{p-1,m}(1,2)) = m(m-1)a_1\, k_{2,m-2}\, g_{1,1},\ 2 \leqq m \leqq p-1$
   $d_3(\nu_{l,1}(2,1)) = (l+3)/(l+1)\ h_2(1,2)\ g_{2,l+1},\ 0 \leqq l \leqq p-4$ *or* $l = p-2$
   $d_3(\nu_{l,m}(2,1)) = m\ (l+3)/(l+1)\ \xi_{l,m},\ 0 \leqq l \leqq p-4$ *or* $l = p-2$,
   $2 \leqq m \leqq p-1$
   $d_3(\nu_{l,m}(2,2)) = m(m-1)\ g_{1,l+2}\, h_1^{m-2}(1,2,1),\ 0 \leqq l \leqq p-3,\ 2 \leqq m \leqq p-1$
   $= m(m-1)a_1\, g_{1,l+2}\, h_1^{m-2}(1,2,1),\ l=p-2$ *or* $p-1,\ 2 \leqq m \leqq p-1$
3. $d_3(\lambda_l) = 1/(l+2)g_{1,l+2}k_{2,p-2},\ 0 \leqq l \leqq p-3$
   $= a_1\, g_{1,l}k_{2,p-2},\ l = p-1$
4. $d_4(\nu_{p-3,m}(1,1)) = -m(m-1)(m-2)/2\ a_1\, k_{2,m-3},\ 3 \leqq m \leqq p-1$
   $d_4(\nu_{p-3,m}(2,1)) = m(m-1)(m-2)/2\ a_1\, h_1^{m-3}(1,2,1),\ 2 \leqq m \leqq p-1$

A basis for the indecomposable elements of ${}_3E_\infty$ *is obtained by deleting* μ_l $(2 \leqq l \leqq p-1)$, $\nu_{l,m}(j,k)$ *(except for* $\nu_{p-3,2}(1,1)$, $\nu_{l,0}(2,1)$, $0 \leqq l \leqq p-1$, *and* $\nu_{p-3,m}(2,1)$, $m=1$ *or* 2), λ_m $(0 \leqq m \leqq p-3$ *or* $m=p-1)$, $\sigma_{l,m}$ $(2 \leqq l \leqq p-3, 1 \leqq m \leqq p-3)$ *and* $\xi_{l,m}$ $(0 \leqq l \leqq p-4$ *or* $l = p-2$, $2 \leqq m \leqq p-1)$ *from, and by adding* ρ *and, if* $p \geqq 5$, η_m $(p-2 \leqq m \leqq p-1)$ *which are represented by* $h_2(1,2)\ g_{1,p-2}\ S_2^2$ *and* $h_1(2,1)g_{1,p-2}\ S_2^m$ $(p-2 \leqq m \leqq p-1)$ *respectively to the basis for the indecomposable elements of* ${}_3E_3$. *The algebra structure on* ${}_3E_\infty$ *differs from that of* ${}_3E_3$ *as follows:*

1. *all relations involving* μ_l $(2 \leqq l \leqq p-1)$, $\nu_{l,m}(j,k)$ *(except for* $\nu_{p-3,2}(1,1)$, $\nu_{l,0}(2,1)$, $0 \leqq l \leqq p-1$, *and* $\nu_{p-3,m}(2,1)$, $m = 1$ *or* 2) *and* λ_m $(0 \leqq m \leqq p-3)$ *for* ${}_3E_3$ *are omitted for* ${}_3E_\infty$,
2. $\sigma_{l,m}$ $(2 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-3)$ *and* $\xi_{l,m}$ $(0 \leqq l \leqq p-4$ *or* $l = p-2$, $2 \leqq m \leqq p-1)$ *are replaced by zero,*
3. *the following additional relations for* ${}_3E_\infty$:

   13. a). $a_1\, k_{1,l} = 0,\ 0 \leqq l \leqq p-3$
       b). $h_1(2,1)\ g_{1,l} = 0,\ 2 \leqq l \leqq p-2$
       c). $h_1(2,1)\ \mu_l = 0$, *if* $p \geqq 5$
       d). $g_{1,l}\, k_{2,m} = 0,\ 2 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-3$

e). $a_1 g_{1,1} k_{2,m} = 0,\ 0 \leqq m \leqq p-3$

f). $h_2(1, 2)\ g_{2,l} = 0,\ 1 \leqq l \leqq p-3$ *or* $l = p-1$

g). $g_{1,1}\ h_1^m(1, 2, 1) = 0,\ 2 \leqq l \leqq p-1,\ 0 \leqq m \leqq p-3$

h). $a_1 g_{1,l} h_1^m(1, 2, 1) = 0,\ 0 \leqq l \leqq 1,\ 0 \leqq m \leqq p-3$

14. a). $a_0\ \rho = 0$

b). $g_{1,l}\ \rho = 0,\ 0 \leqq l \leqq p-1$

$g_{2,l}\ \rho = 0,\ 0 \leqq l \leqq p-1$

c). $h_1(1, 2)\ \rho = 0$

d). $h_i\ \rho = 0,\ i = 1$ *or* 2

e). $\mu_1\ \rho = 0$

f). $k_{1,l}\ \rho = 0,$ *if* $1 \leqq l \leqq p-1$ *and* $l \neq p-3$

$= -k_{2,p-3}\ \mu_l\ \Phi_3,$ *if* $l = p-3$

15. a). $a_0\ \eta_m = 0,\ p-2 \leqq m \leqq p-1$

b). $g_{1,l}\ \eta_m = 0,\ 0 \leqq l \leqq p-1,\ p-2 \leqq m \leqq p-1$

$g_{2,l}\ \eta_m = 0,\ 0 \leqq l \leqq p-1,\ p-2 \leqq m \leqq p-1$

c). $h_1\ \eta_m = 0,\ p-2 \leqq m \leqq p-1$

$h_2\ \eta_{p-2} = 0$

$h_2\ \eta_{p-1} = -\Phi_3\ \eta_{p-2} = 2h_2(1, 2)\ g_{1,p-3}\ \gamma$

d). $\Phi_3\ \eta_{p-1} = 0$

e). $\gamma\ \eta_m = 0,\ p-2 \leqq m \leqq p-1$

f). $\mu_1\ \eta_m = 0,\ p-2 \leqq m \leqq p-1$

g). $k_{1,l}\ \eta_m = 0,\ 0 \leqq l \leqq p-1,\ p-2 \leqq m \leqq p-1$

16. a). $a_1 k_{2,m} = 0,\ 0 \leqq m \leqq p-4$

b). $a_1 h_1^m(1, 2, 1) = 0,\ 0 \leqq m \leqq p-4$

PROOF. These are proved by making explicit use of the definition of the differentials in a spectral sequence. We give the proof of 3, the proofs of 1, 2 and 4 being similar. λ_l is represented by the image in ${}_3E_0^{l+3,p}$ of the cochain $R_3^0 R_2^0 R_1^0 S_0 S_1^l S_2^{p-1} \in F^{l+3}X_3$. In X_3, $d(R_3^0 R_2^0 R_1^0 S_0 S_1^l S_2^{p-1}) = (p-1)\ R_3^0 R_2^0 R_1^0 R_1^1 S_0 S_1^{l+1} S_2^{p-2} \in F^{l+5} X_3$, which, since $d_2\ (\lambda_l) = 0$, must be congruent in $F^{l+4} X_3$ to an element of $F^{l+6} X_3$. $d(R_3^0 R_2^0 R_1^1 S_1^{l+2} S_2^{p-2}) = -R_2^1 R_1^0 R_2^0 R_1^1 S_1^{l+2} S_2^{p-2} + (l+2) R_3^0 R_2^0 R_1^1 R_1^0 S_0 S_1^{l+1} S_2^{p-2}$. Therefore $d(R_3^0 R_2^0 R_1^0 S_0 S_1^l S_2^{p-1} - 1/(l+2)\ R_3^0 R_2^0 R_1^1 S_1^{l+2} S_2^{p-2}) = 1/(l+2)\ R_2^1 R_1^0 R_2^0 R_1^1 S_1^{l+2} S_2^{p-2}$. By the definition, the image of $1/(1+2)\ R_2^1 R_1^0 R_2^0 R_1^1 S_1^{l+2} S_2^{p-2}$ in ${}_3E_0^{l+6,p-2}$ represents $d_3(\lambda_l)$ in ${}_3E_3$, and therefore $d_3(\lambda_l) = 1/(l+2)\ g_{1,l+2}\ k_{2,p-2}$ if $0 \leqq l \leqq p-3$ and $a_1 g_{1,l} k_{2,p-2}$ if $l = p-1$ as claimed. By the routine inspection of the structure of ${}_3E_3$ and the differentials d_3, we have two new indecomposable elements ρ and η_m $(p-2 \leqq m \leqq p-1,\ p \geqq 5)$ which are permanent cocycles. The algebra structure of ${}_3E_\infty$ is easy to derive. We omit it.

Next we state how the algebra structure of $H^*(X_3)$ differs from that of ${}_3E_\infty$ in the following proposition, the proof of which consists of finding representative cocycles of the indecomposable elements in X_3 and checking the relations for ${}_3E_\infty$.

PROPOSITION 2. 4. *In dimensions* $t-s \leqq (3p^2+3p+4)\, q-1,\ q=2(p-1)$, *a basis for the indecomposable elements of* $H^*(X_3)$ *not in the image of* $H^*(X_2)$ *is given by* :

1. $f_3 : -R_3^0 S_0 - R_1^2 S_2 - R_2^1 S_1$
2. $e_{0,4} : -R_3^1 R_1^0 - R_2^2 R_2^0 - R_1^3 R_3^0$
3. $a_2 : S_2^p$
4. $b_{03} : \tilde{R}_3^0$
5. $\gamma : -R_2^0 S_0 S_2^{p-1} - R_1^1 S_1 S_2^{p-1}$
6. $u : R_0^1 S_1^{p-2} S_2 - R_2^0 S_1^{p-1}$
7. $g_{3,l} : R_3^0 R_2^0 R_1^0 S_1^l,\ 0 \leqq l \leqq p-1$
8. $k_{1,l} : R_1^1 R_2^0 S_2^l,\ 1 \leqq l \leqq p-1$

   $k_{2,l} : R_2^1 R_1^1 R_2^0 S_2^l,\ 1 \leqq l \leqq p-1$
9. $v_2(1) : R_1^2 R_1^0 S_1^{p-3} S_2^2 + R_2^1 R_1^0 S_1^{p-2} S_2 + R_2^0 R_1^2 S_1^{p-2} S_2 - R_2^1 R_2^0 S_1^{p-1}$

   $\nu_{l,0}(2,1) : R_3^0 R_1^2 R_1^0 S_1^l,\ 0 \leqq l \leqq p-2$

   $v_1(2) : R_3^0 R_1^2 R_1^0 S_1^{p-3} S_2 - 1/2\ R_3^0 R_1^2 R_2^0 S_1^{p-2} + 1/2\ R_3^0 R_2^1 R_1^0 S_1^{p-2}$

   $v_2(2) : R_3^0 R_1^2 R_1^0 S_1^{p-3} S_2^2 - R_3^0 R_1^2 R_2^0 S_1^{p-2} S_2 + R_3^0 R_2^1 R_1^{p-2} S_2 - R_3^0$
   $R_2^1 R_2^0 S_l^{p-1}$

   $\nu_{p-1,0}(2,1) : R_3^0 R_1^2 R_1^0 S_1^{p-1}$
10. $h_1^m(1,2,1) : R_3^0 R_2^1 R_1^1 R_2^0 S_2^m,\ 0 \leqq m \leqq p-1$
11. $y : R_3^0 R_1^2 R_2^1$
12. $j_l : R_1^1 R_2^0 R_1^0 R_3^0 S_2^l,\ 1 \leqq l \leqq p-1$

The algebra structure of $H^*(X_3)$ *differs from that of* ${}_3E_\infty$ *as follows* :

1. *all relations involving* $g_{2,p-1}$, $\sigma_{l,m}$ $(2 \leqq l \leqq p-3,\ p-2 \leqq m \leqq p-1)$, $\xi_{l,p}$ $(0 \leqq l \leqq p-4$ *or* $l=p-2)$, τ, ρ *and* η_m $(p-2 \leqq m \leqq p-1)$ *are omited for* $H^*(X_3)$,
2. *the relations* 2. *g*), 2. *h*), 2. *i*) and 7. *l*) (*listed in Proposition* 2. 2) *are omited,*
3. Φ_3, $\varepsilon_{0,4}$, μ_1, $\nu_{p-3,2}(1,1)$, $\nu_{p-3,1}(2,1)$ and $\nu_{p-3,2}(2,1)$ *are replaced by* f_3, $e_{0,4}$, u, $v_2(1)$, $v_1(2)$ *and* $v_2(2)$ *respectively, and*
4. *by following relations the corresponding those* (*listed in Proposition* 2, 2) *are replaced for* $H^*(X_3)$;

1. a'). $g_{2,l} f_3 = -(l+3)\, a_0\, g_{3,l},\ 0 \leqq l \leqq p-2$

   $f_3\, h_0\, u = -2\, a_0\, g_{3,p-1}$

2. a'). $f_3\, \nu_{l,0}(2,1) = -y g_{1,l+1},\ 0 \leqq l \leqq p-2$

   $f_3\, v_1(2) = 1/2\ yu$

   $f_3\, v_2(2) = 0$

   $f_3\, \nu_{p-1,0}(2,1) = -y h_0\, a_1$

   c'). $f_3\, j_l = -(l+2)\, h_1^l(1,2,1)\, g_{1,1},\ 1 \leqq l \leqq p-1$

   f'). $h_0\, k_{1,m} f_3 = -(m+3)\, g_{1,1}\, k_{2,m},\ 1 \leqq m \leqq p-1$

j'). $h_2 g_{1,l} f_3 = (l+2)/(l+1)\ h_2(1,2)\ g_{1,l+1},\ 0 \leqq l \leqq p-3$

k'). $h_1 g_{3,0} f_3 = -h_1^0(1,2,1)\ g_{1,1}$

l'). $h_2 g_{3,l} f_3 = y g_{2,l+1},\ 0 \leqq l \leqq p-3$

3. a'). $a_0 u = h_1 a_1$

b'). $a_0 v_2(1) = h_1(2,1)\ a_1$

$a_0 \nu_{l,0}(2,1) = -1/(l+1)\ h_2(1,2)\ g_{1,l+1},\ 0 \leqq l \leqq p-2$

$a_0 v_1(2) = 1/2\ h_2(1,2)\ u$

$a_0 v_2(2) = -v_2(1)\ f_3$

e'). $a_0 j_l = g_{1,1} k_{2,l},\ 1 \leqq l \leqq p-1$

i'). $a_0 g_{3,p-3}\ \gamma = -g_{1,p-1} k_{2,p-1}$

k'). $a_0^2 g_{3,p-3} = h_1(2,1)\ g_{1,p-1}$

l'). $a_0 g_{3,l}\ \gamma = 1/(l+2)\ g_{1,l+2} k_{2,p-1},\ 0 \leqq l \leqq p-4$

m'). $a_0 h_1 g_{3,0} = -h_1(2,1)\ g_{2,1}$

4. a'). $g_{1,p-2}\ \gamma = k_{1,p-2} a_1$

$g_{2,l}\ \gamma = -g_{1,l+1} k_{1,p-1},\ 0 \leqq l \leqq p-2$

b'). $g_{1,l} u = g_{2,l-1} a_1,\ 1 \leqq l \leqq p-1$

$g_{2,l} u = 0,\ 0 \leqq l \leqq p-2$

c'). $g_{1,l} v_2(1) = -l\ a_0 g_{3,p+l-2},\ 0 \leqq l \leqq 1$

$= -l\ a_0 a_1 g_{3,l-2},\ 2 \leqq l \leqq p-1$

$g_{1,l} \nu_{l',0}(2,1) = 0,\ 0 \leqq l \leqq p-1,\ 0 \leqq l' \leqq p-2$

$g_{1,l} v_1(2) = -1/2\ h_2 g_{3,p+l-2},\ 0 \leqq l \leqq 1$

$= -1/2\ h_2 g_{3,l-2} a_1,\ 2 \leqq l \leqq p-1$

$g_{1,l} v_2(2) = -g_{3,p+l-2} f_3,\ 0 \leqq l \leqq 1$

$= -g_{3,l-2} a_1 f_3,\ 2 \leqq l \leqq p-1$

$g_{1,l} \nu_{p-1,0}(2,1) = 0,\ 0 \leqq l \leqq p-1$

$g_{2,l} v_2(1) = 0,\ 0 \leqq l \leqq p-2$

$g_{2,l} \nu_{l',0}(2,1) = 0,\ 0 \leqq l, l' \leqq p-2$

$g_{2,l} v_j(2) = 0,\ 0 \leqq l \leqq p-2,\ j = 1$ *or* 2

$g_{2,l} \nu_{p-1,0}(2,1) = 0,\ 0 \leqq l \leqq p-2$

i'). $g_{2,l}\ y = h_2(1,2)\ g_{3,l},\ 0 \leqq l \leqq p-2$

$h_0 u y = h_2(1,2)\ g_{3,p-1}$

6. c'). $k_{1,l} v_2(1) = -(l+3)\ g_{1,p-2} k_{2,l+1},\ 0 \leqq l \leqq p-2$

$= -2 g_{1,p-2} h_1(2,2)\ a_2,\ l = p-1$

$k_{2,1} v_2(1) = 0,\ 0 \leqq l \leqq p-1$

$k_{1,l} \nu_{l',0}(2,1) = l\ h_1^{l-1}(1,2,1)\ g_{1,l'+1},\ 0 \leqq l \leqq p-1,$
$0 \leqq l' \leqq p-2$

$k_{1,l} v_1(2) = (2l+3)/2\ h_1^l(1,2,1)\ g_{1,p-2},\ 0 \leqq l \leqq p-1$

$k_{1,l} v_2(2) = (l+3)\ h_1^{l+1}(1,2,1)\ g_{1,p-2},\ 0 \leqq l \leqq p-2$

$= 2\ h_1^0(1,2,1)\ g_{1,p-2} a_2,\ l = p-1$

$k_{1,l} \nu_{p-1,0}(2,1) = l\ h_1^{l-1}(1,2,1)\ h_0 a_1,\ 0 \leqq l \leqq p-1$

7. b′). $h_1 v_2(1) = -2 h_1(2, 1) u$

$h_1 \nu_{l,0}(2, 1) = 0, \ 0 \leqq l \leqq p-3$

$= -h_2(1, 2) g_{2,p-2}, \ l = p-2$

$h_1 v_j(2) = 0, \ j = 1 \ or \ 2$

$h_1 \nu_{p-1,0}(2, 1) = 0$

$h_2 v_2(1) = h_2(1, 2) u$

$h_2 \nu_{l,0}(2, 1) = 0, \ 0 \leqq l \leqq p-2$

$h_2 v_1(2) = -1/2 \ y g_{1,p-2}$

$h_2 v_2(2) = -yu$

$h_2 \nu_{p-1,0}(2, 1) = 0$

f′). $h_1 j_m = 0, \ 1 \leqq m \leqq p-1$

$h_2 j_m = m \ h_1^{m-1}(1, 2, 1) \ g_{1,1}, \ 1 \leqq m \leqq p-1$

k′). $h_2 h_0 k_{1,p-2} = 2 \ g_{1,1} k_{2,p-3}$

m′). $h_2 g_{3,l} \gamma = -h_1^{p-2}(1, 2, 1) \ g_{1,l+2}, \ 0 \leqq l \leqq p-3$

$= -h_1^{p-2}(1,2, 1) \ h_0 a_1, \ l = p-2$

9. a′). $u \gamma = -a_1 k_{1,p-1}$

c′). $uv_2(1) = 0$

$u \nu_{l,0}(2, 1) = h_2 g_{3,p-1}, \ l = 0$

$= h_2 g_{3,l-1} a_1, \ 1 \leqq l \leqq p-2$

$uv_1(2) = 1/2 \ g_{3,2p-4} f_3$

$uv_2(2) = 0$

$u \nu_{p-1,0}(2, 1) = h_2 g_{3,p-2} a_1$

10. b′). $v_2(1) \gamma = -a_1 k_{2,p-1}$

$\nu_{p-3,0}(2, 1) \gamma = -h_1^{p-3}(1, 2, 1) \ a_1$

$v_1(2) \gamma = 1/2 \ h_1^{p-2}(1, 2, 1) \ a_1$

$v_2(2) \gamma = -h_1^{p-1}(1, 2, 1) \ a_1$

11. a′). $v_2(1)^2 = 0$

$v_2(1) \nu_{l,0}(2, 1) = -h_2(1, 2) g_{3,p-1}, \ l = 0$

$= -h_2(1, 2) g_{3,l-1} a_1, \ 1 \leqq l \leqq p-2$

$v_2(1) v_j(2) = 0, \ j = 1 \ or \ 2$

Remark. The indecomposable element $g_{2,\ p-1}$ in $H^*(X_2)$ is not indecomposable in $H^*(X_3)$ and is denoted by $h_0 u$.

We can now begin the calculation of the spectral sequence ${}_4E_r$. ${}_4E_2$ is the differential algebra $H^*(X_3) \otimes Z_4$ with differential determined by $d_2(R_4^i) = e_{i,4}$, $d_2(S_3) = f_3$ and $d_2(\tilde{R}_4^i) = 0$. The image of $H^*(X_3)$ in ${}_4E_3$ is therefore $H^*(X_3)/I$, where I is the ideal in $H^*(X_3)$ generated by $\{e_{i,4}\}$ and f_3. The next proposition lists the indecomposable elements and defining relations of ${}_4E_3$. Its proof is similar to that of proposition 2. 2.

Proposition 2. 5 *At least in dimensions* $t-s \leqq (3p^2+3p+4)q-1$, *a basis for* *the indecomposable elements of $_4E_3$ not in $H^*(X_3)$ is given by :*

1. $a_3 : S_3^p$
2. $\Gamma : f_3 S_3^{p-1}$
3. $\Phi_4 : R_4^0 a_0 + h_3 S_3$
4. $A_m : g_{2,p-3} S_3^m,\ 1 \leqq m \leqq p-1$
5. $X_m : k_{1,p-3} S_3^m,\ 1 \leqq m \leqq p-1$
6. $C_m : k_{2,p-2} S_3^m,\ 1 \leqq m \leqq p-1$
7. $E_m : j_{p-2} S_3^m,\ 1 \leqq m \leqq p-1$
8. $G_m : h_2 g_{1,p-2} S_3^m,\ 1 \leqq m \leqq p-1$
9. $L_m : h_2 k_{1,p-2} S_3^m,\ 1 \leqq m \leqq p-1$

In the cited range, defining relations of $_4E_3$ are given by those relations holding in the image of $H^*(X_3)$ and by the following relations:

17. a). $a_0 A_m = 0,\ 1 \leqq m \leqq p-1$

b). $h_0 A_m = 0,\ 1 \leqq m \leqq p-1$

$h_1 A_m = h_0 X_m,\ 1 \leqq m \leqq p-1$ *if* $p=3$

$= 0,\ 1 \leqq m \leqq p-1$ *if* $p \geqq 5$

$h_2 A_m = 0,\ 1 \leqq m \leqq p-1$

c). $g_{1,l} A_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

$g_{2,l} A_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$

d). $h_1(2, 1) A_m = 0,\ 1 \leqq m \leqq p-1$

$h_2(1, 2) A_m = 0,\ 1 \leqq m \leqq p-1$

e). $uA_m = 0,\ 1 \leqq m \leqq p-1$

f). $v_2(1) A_m = 0,\ 1 \leqq m \leqq p-1$

$\nu_{l,0}(2, 1) A_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$

g). $g_{3,l} A_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

h). $k_{1,l} A_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$, *where* $k_{1,0} = h_1(1, 2)$

$k_{2,l} A_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$, *where* $k_{2,0} = h_1(2, 2)$

i). $j_l A_m = 0,\ 1 \leqq l, m \leqq p-1$

j). $A_m A_{m'} = 0,\ 1 \leqq m, m' \leqq p-1$

$A_m X_l = 0,\ 1 \leqq m, l \leqq p-1$

18. a). $a_0 X_m = 0,\ 1 \leqq m \leqq p-1$

$a_1 X_m = 0,\ 1 \leqq m \leqq p-1$

b). $h_1 X_m = 0,\ 1 \leqq m \leqq p-1$

$h_2 X_m = 0,\ 1 \leqq m \leqq p-1$ *if* $p = 3$ *or* $1 \leqq m \leqq p-2$ *if* $p \geqq 5$

$= -3\, k_{1,\ p-4}\, \Gamma,\ l = p-1$ *if* $p \geqq 5$

c). $g_{1,l} X_m = 0,\ 1 \leqq l, m \leqq p-1$

$g_{2,l} X_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$

d). $h_1(2, 1)\ X_m = 0,\ 1 \leqq m \leqq p-1$

e). $\gamma X_m = 0,\ 1 \leqq m \leqq p-1$
f). $u X_m = 0,\ 1 \leqq m \leqq p-1$
g). $v_2(1)\ X_m = 0,\ 1 \leqq m \leqq p-1$
h). $g_{3,l}\ X_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$
i). $k_{1,l}\ X_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$
$k_{2,l}\ X_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$
j). $j_l\ X_m = 0,\ 1 \leqq l, m \leqq p-1$
k). $X_m X_{m'} = 0,\ 1 \leqq m, m' \leqq p-1$

19. a). $a_0\ C_m = 0,\ 1 \leqq m \leqq p-1$
b). $h_i\ C_m = 0,\ i = 0$ *or* $1,\ 1 \leqq m \leqq p-1$
c). $g_{1,l}\ C_m = 0,\ l = 1,\ 1 \leqq m \leqq p-2$ *or* $2 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$
$= -h_0\, k_{2,p-2}\, \Gamma,\ l = 1,\ m = p-1$
$g_{2,l} C_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$

20. a). $a_0\ E_m = 0,\ 1 \leqq m \leqq p-2$
$= g_{1,1}\ C_{p-1},\ m = p-1$
$a_1\ E_m = 0,\ 1 \leqq m \leqq p-1$
b). $h_i\ E_m = 0,\ i = 0$ *or* $1,\ 1 \leqq m \leqq p-1$
c). $g_{1,l}\ E_m = 0,\ 1 \leqq l, m \leqq p-1$

21. a). $a_0\ G_m = 0,\ 1 \leqq m \leqq p-1$
b). $h_i\ G_m = 0,\ i = 0,\ 1$ *or* $2,\ 1 \leqq m \leqq p-1$
c). $g_{1,l}\ G_m = 0,\ 1 \leqq l, m \leqq p-1$
$g_{2,l}\ G_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$
d). $h_1(2, 1)\ G_m = 0,\ 1 \leqq m \leqq p-1$
e). $\gamma G_m = a_1\ L_m,\ 1 \leqq m \leqq p-1$
f). $u G_m = 0,\ 1 \leqq m \leqq p-1$
g). $k_{1,l} G_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

22. a). $a_0\ L_m = 0,\ 1 \leqq m \leqq p-1$
b). $h_1\ L_m = 0,\ 1 \leqq m \leqq p-1$
c). $g_{1,l} L_m = 0,\ 1 \leqq l, m \leqq p-1$
$g_{2,l} L_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-1$
d). $u L_m = 0,\ 1 \leqq m \leqq p-1$
e). $k_{1,l} L_m = 0,\ 0 \leqq l \leqq p-1,\ 1 \leqq m \leqq p-1$

PROPOSITION 2.6. *The higher differentials on those indecomposable elements of* ${}_4E_3$ *are zero except for the following* :

1. $d_3(A_2) = -2\ v_2(2)$
2. $d_3(G_2) = -2\ a_1 y$

in the range $t-s \leqq (3p^2+3p+4)q-1$ • *A basis for the indecomposable elements of* ${}_4E_\infty$ *is obtained by deleting* $v_2(2)$ *from the basis for the image of* $H^*(X_3)$ *in* ${}_4E_3$ *and adding* a_3, Γ, Φ_4, A_1, X_m $(1 \leqq m \leqq 2)$, C_1, E_1, G_1 *and* L_1 • *In the cited range, the algebra structure of* ${}_4E_\infty$ *differs from that of* ${}_4E_3$ *only in that all relations involving*

A_2 and G_2 are omited for ${}_4E_\infty$, $v_2(2)$ is replaced by zero and following relation is added :

23. a). $a_1 y = 0$.

Proof. This is similarly proved to Proposition 2. 3. We omit it.

Proposition 2. 7. *At least in dimensions $t-s \leqq (3p^2+3p+4)q-1$, a basis for the indecomposable elements of $\tilde{E}_2$ is given by :*

1. $a_0 : S_0 \in (1, 0)$,

   $a_i : S_i^p \in (p, (p^i+p^{i-1}+\cdots+p)q)$, $i \geqq 1$
2. $b_{i1} : \tilde{R}_1^i \in (2, p^{i+1}q-2)$

   $b_{i2} : \tilde{R}_2^i \in (2, p^i(p^2+p)q-2)$

   $b_{i3} : \tilde{R}_3^i \in (2, p^i(p^3+p^2+p)q-2)$
3. $h_i : R_1^i \in (1, p^i q-1)$
4. $h_1(2, 1) : R_2^1 R_1^1 \in (2, (p^2+2p)q-2)$

   $h_2(1, 2) : R_1^2 R_2^1 \in (2, (2p^2+p)q-2)$
5. $g_{1,l} : R_1^0 S_1^l \in (l+1, (l+1)q-1)$, $1 \leqq l \leqq p-1$

   $g_{2,l} : R_2^0 R_1^0 S_1^l \in (l+2, (p+l+2)q-2)$, $0 \leqq l \leqq p-2$

   $g_{3,l} : R_3^0 R_2^0 R_1^0 S_1^l \in (l+3, (p^2+2p+l+3)q-1)$, $0 \leqq l \leqq p-3$
6. $\gamma : -R_2^0 S_0 S_2^{p-1} - R_1^1 S_1 S_2^{p-1} \in (p+1, (p^2+p)q-1)$

   $\Gamma : -R_3^0 S_0 S_3^{p-1} - R_1^2 S_2 S_3^{p-1} - R_2^1 S_1 S_3^{p-1} \in (p+1, (p^3+p^2+p)q-1)$
7. $u : R_1^0 S_1^{p-2} - R_2^0 S_1^{p-1} \in (p, 2pq-1)$
8. $k_{1,l} : R_1^1 R_2^0 S_2^l \in (1+2, (2p+lp+l+1)q-2)$, $0 \leqq l \leqq p-1$

   $k_{2,l} : R_2^1 R_1^1 R_2^0 S_2^l \in (l+3, (p^2+(l+3)p+l+1)q-3)$, $0 \leqq l \leqq p-3$ or $l = p-1$
9. $h_1^m(1, 2, 1) : R_3^0 R_2^1 R_1^1 R_2^0 S_2^m \in (m+4, (2p^2+(m+4)p+m+2)q-4)$, $0 \leqq m \leqq p-2$
10. $y : R_3^0 R_1^2 R_2^1 \in (3, (3p^2+2p+1)q-3)$
11. $\nu_{l,0}(2, 1) : R_3^0 R_1^2 R_1^0 S_1^l \in (l+3, (2p^2+p+l+2)q-3)$, $0 \leqq l \leqq p-2$
12. $j_l : R_1^1 R_2^0 R_1^0 R_3^0 S_2^l \in (l+4, (p^2+(l+3)p+l+3)q-4)$, $1 \leqq l \leqq p-3$ or $l = p-1$ if $p \geqq 5$
13. $w : R_2^0 R_1^0 S_1^{p-3} S_3 - R_3^0 R_1^0 S_1^{p-3} S_2 + R_3^0 R_2^0 S_1^{p-2} \in (p, (p^2+3p)q-2)$
14. $x : R_1^1 R_2^0 S_2^{p-3} S_3 + 1/2\, R_3^0 R_1^1 S_2^{p-2} - 1/2\, R_2^1 R_2^0 S_2^{p-2}$ $\in (p, (2p^2+p-1)q-2)$

    $x_2 : R_1^1 R_2^0 S_2^{p-3} S_3^2 + R_3^0 R_1^1 S_2^{p-2} S_3 - R_2^1 R_2^0 S_2^{p-2} S_3 - R_3^0 R_2^1 S_2^{p-1}$ $\in (p+1, (3p^2+2p)q-2)$
15. $G : R_1^2 R_1^0 S_1^{p-2} S_3 + R_3^0 R_1^2 S_1^{p-1} \in (p+1, (2p^2+2p)q-2)$