On the Cohomology of the mod p Steenrod Algebra

16. $L : R_1^2 R_1^1 R_2^0 S_2^{p-2} S_3 - R_3^0 R_1^2 R_1^1 S_2^{p-1} - R_1^2 R_2^1 R_2^0 S_2^{p-1}$
$\in (p+2, (3p^3+2p) q-3)$

The algebra structure of $\tilde{E}_2$ *differs from that of* ${}_4E_\infty$ *as follows :*

1. *all relations involving* $g_{3,p-2}$, $g_{3,p-1}$, j_{p-2}, $k_{2,p-2}$, $v_2(1)$, $v_1(2)$, $\nu_{p-1,0}(2, 1)$, $h_1^{p-1}(1, 2, 1)$, C_1, E_1 *and, if* $p = 3$, j_2 *for* ${}_4E_\infty$ *are omitted for* $\tilde{E}_2$,
2. A_1, X_1, X_2, G_1 *and* L_1 *are replaced by* w, x, x_2, G *and* L *respectively,*
3. *by following relations the coresponding those (listed in Proposition 2.5) are replaced for* $\tilde{E}_2$:

17. b′). $h_1 w = h_0 x$, *if* $p=3$
$= 0$, *if* $p \geqq 5$

c′). $g_{1,l}\, w = a_1 g_{3,l-2}$, $2 \leqq l \leqq p-1$
$g_{2,l}\, w = 0$, $0 \leqq l \leqq p-2$

h′). $k_{1,0}\, w = g_{2,0} X$, *if* $p=3$
$k_{1,2}\, w = h_1 g_{3,0}\, a_2$, *if* $p=3$
$k_{1,l} w = 0$, $0 \leqq l \leqq p-3$ *or* $l = p-1$ *if* $p \geqq 5$
$k_{1,p-2} w = g_{3,p-4}\, \gamma$, *if* $p \geqq 5$
$k_{2,l} w = g_{1,p-3}\, h_1^{l+1}(1, 2, 1)$, $0 \leqq l \leqq p-2$
$= g_{1,p-3} h_1^0\, (1, 2, 1)\, a_2$, $l = p-1$

j′). $w^2 = 0$
$wx = g_{2,0}\, x_2$, *if* $p = 3$
$= 1/2\, h_2\, g_{3,p-4}\, a_2$, *if* $p \geqq 5$

18. a′). $a_0\, x = 0$
$a_0\, x_2 = h_2(1, 2)\, a_2$
$a_1\, x = 1/2\, h_2\, g_{1,p-2}\, a_2$
$a_1\, x_2 = G a_2$

b′). $h_2\, x = 0$

c′). $g_{1,l} x = 0$, $1 \leqq l \leqq p-1$
$g_{1,l}\, x_2 = \nu_{l-1,0}(2, 1) a_2$, $1 \leqq l \leqq p-1$
$g_{2,l}\, x = 0$, $1 \leqq l \leqq p-2$
$g_{2,l}\, x_2 = -\, h_2\, g_{3,l-1}\, a_2$, $1 \leqq l \leqq p-2$

k′). $x^2 = h_1(1, 2) x_2$, *if* $p=3$

e′). $\gamma x = 1/2\, h_2\, k_{1,p-2}\, a_2$

f′). $ux = h_2\, g_{2,p-3}\, a_2$
$ux_2 = h_2\, w a_2$

21. a′). $a_0\, G = h_2(1, 2)\, a_1$

b′). $h_1\, G = h_2(1, 2)\, u$
$h_2\, G = 0$

c′). $g_{1,l}\, G = \nu_{l-1,0}(2, 1)\, a_1$, $1 \leqq l \leqq p-1$
$g_{2,l}\, G = -\, u\, \nu_{0,0}(2, 1)$, $l = 0$
$= -\, h_2\, g_{3,l-1}\, a_1$, $1 \leqq l \leqq p-2$

f′). $uG = h_2 w a_1$

g′). $k_{1,l} G = 0, \ 0 \leqq l \leqq p-3$

$= \nu_{p-3,0}(2, 1)\gamma, \ l = p-2$

$= h_2(1, 2) g_{2,p-2} a_2 - a_1 h_1^{p-2}(1, 2, 1), \ l = p-1$

22. a′). $a_0 L = h_2(1, 2) \gamma$

b′). $h_1 L = - h_2(1, 2) k_{1,p-1}$

c′). $g_{1,l} L = \nu_{l-1,0}(2, 1) \gamma, \ 1 \leqq l \leqq p-1$

$g_{2,l} L = g_{1,l+1} h_1^{p-2}(1, 2, 1), \ 0 \leqq l \leqq p-2$

d′). $uL = a_1 h_1^{p-2}(1, 2, 1)$

4. *the following additional relations for* $\tilde{E}_2$:

24. a). $a_0^2 w = h_1(2, 1) a_1$

$a_0 h_2 w = h_2(1, 2) u$

$a_0 w \gamma = - a_1 k_{2,p-1}$

b). $a_1 j_{p-1} = g_{1,1} \gamma w$, *if* $p \geqq 5$

$a_1 h_1^{p-2}(1, 2, 1) = h_2 \gamma w$

c). $h_0 k_{1,1} w = 0$, *if* $p=3$

$g_{1,p-2} k_{2,p-1} = 0$

$g_{1,1} h_1^1(1, 2, 1) = 1/2 \ h_2 k_{1,1} w$, *if* $p=3$

d). $uk_{2,p-3} = 0$

$uh_1^{p-2}(1, 2, 1) = 0$

$u \nu_{0,0}(2, 1) = h_2 g_{1,1} w$

e). $h_2(1, 2) k_{1,p-2} = 0$

$h_1(2, 2) g_{1,1} a_2 = 0$, *if* $p=3$

5. *the relation* 17, *a*) (*listed in Proposition* 2. 5) *is omited for* $\tilde{E}_2$.

Proof. By the similar proof to Proposition 2. 4, we have the structure of $H^*(X_4)$ in the range $t - s \leqq (3p^2+3p+4)q - 1$. Since $f_4 = 0$ in $H^*(X_n)$ for $n \geqq 5$ which is correspondent to Φ_4 in ${}_4E_\infty$, the structure of $\tilde{E}_2$ is derived from $H^*(X_4)$ by replacing f_4 by zero.

We can now begin the computation of the spectral sequence $\{\tilde{E}_r\}$. From dimensional and filtrational considerations, we find that $\tilde{d}_r = 0$ for $2 \leqq r \leqq p-2$. The next proposition gives the form of $\tilde{d}_{p-1}$ and descrives the structure of $\tilde{E}_p$ by stating how it differs from that of $\tilde{E}_2$.

Proposition 2. 8. *In dimensions* $t - s \leqq (3p^2+3p+4)q - 1$, *the non-zero differentials* $\tilde{d}_{p-1}$ *on* $\tilde{E}_{p-1}$ *are given by the following* :

1. $\tilde{d}_{p-1}(g_{1,p-1}) = a_0^{p-1} b_{01}$

$\tilde{d}_{p-1}(g_{2,p-2}) = h_1 a_0^{p-2} b_{01}$

$\tilde{d}_{p-1}(g_{3,p-3}) = h_1(2,1) a_0^{p-3} b_{01}$

2. $\tilde{d}_{p-1}(\gamma) = - a_0^p b_{02} - a_1 b_{11}$

$\tilde{d}_{p-1}(\Gamma) = -a_0^p b_{03} - a_1 b_{12} - a_2 b_{21}$

3. $\tilde{d}_{p-1}(k_{1,p-2}) = g_{1,p-2} b_{11}$

$\tilde{d}_{p-1}(k_{1,p-1}) = -h_1 a_0^{p-1} b_{02} - u b_{11}$

$\tilde{d}_{p-1}(k_{2,p-3}) = h_2 g_{1,p-3} b_{11}$

$\tilde{d}_{p-1}(k_{2,p-1}) = h_1(2,1) a_0^{p-1} b_{02} + a_0 b_{11} w$

4. $\tilde{d}_{p-1}(h_1^{p-3}(1,2,1)) = -\nu_{p-3,0}(2,1) b_{11}$

$\tilde{d}_{p-1}(h_1^{p-2}(1,2,1)) = h_2 w b_{11} + h_2 h_1(2,1) a_0^{p-2} b_{02}$

5. $\tilde{d}_{p-1}(y) = k_{1,0} b_{21}$, *if* $p=3$

$\tilde{d}_{p-1}(\nu_{p-2,0}(2,1)) = h_2(1,2) a_0^{p-2} b_{01}$

$\tilde{d}_{p-1}(j_{p-1}) = g_{1,1} w b_{11}$, *if* $p \geqq 5$

$\tilde{d}_{p-1}(L) = -G b_{11} - h_2(1,2) a_0^{p-1} b_{02}$

A basis for the indecomposable elements of $\tilde{E}_p$ *is obtained as follows :*

1. $g_{1,p-1}$, $g_{2,p-2}$, $g_{3,p-3}$, γ, Γ, $k_{1,p-2}$, $k_{1,p-1}$, $k_{2,p-3}$, $k_{2,p-1}$, $h_1^{p-3}(1,2,1)$, $h_1^{p-2}(1,2,1)$, $\nu_{p-2,0}(2,1)$, L, y *if* $p=3$ *and* j_{p-1} *if* $p \geqq 5$ *are delected from that of* $\tilde{E}_2$, *and*

2. ア, イ *and ,if* $p=3$, ウ, エ, オ, カ *which are represented by* $h_0 k_{1,p-2}$, $h_0 h_1^{p-3}(1,2,1)$, $h_1 g_{3,0}$, $h_2(1,2) g_{3,0}$, $h_1 y$ *and* $h_2 h_1^0(1,2,1)$ *in* $\tilde{E}_2$, *respectively, are added to those.*

The algebra structure of $\tilde{E}_p$ *differs from that of* $\tilde{E}_2$ *as follows :*

1. *all relations involving the deleted basis elements for* $\tilde{E}_2$ *are omitted for* $\tilde{E}_p$, *and*
2. *the following additional relations for* $\tilde{E}_p$:

25. a). $a_0^{p-1} b_{01} = 0$

$a_0^{p-2} h_1 b_{01} = 0$

$a_0^{p-3} h_1(2,1) b_{01} = 0$

b). $a_0^p b_{02} + a_1 b_{11} = 0$

$a_0^p b_{03} + a_1 b_{12} + a_2 b_{21} = 0$

c). $g_{1,p-2} b_{11} = 0$

$u b_{11} + a_0^{p-1} h_1 b_{02} = 0$

$h_2 g_{1,p-3} b_{11} = 0$

$a_0^{p-1} h_1(2,1) b_{02} + a_0 b_{11} w = 0$

d). $\nu_{p-3,0}(2,1) b_{11} = 0$

$h_2 w b_{11} + h_2 h_1(2,1) a_0^{p-2} b_{02} = 0$

e). $k_{1,0} b_{21} = 0$, *if* $p=3$

$a_0^{p-2} h_2(1,2) b_{01} = 0$

$g_{1,1} w b_{11} = 0$, *if* $p \geqq 5$

$G b_{11} + a_0^{p-1} h_2(1,2) b_{02} = 0$

26. a). a_iア $= 0$, $i=0$ *or* 1

b). $g_{1,l}$ア $= 0,\ 0 \leqq l \leqq p-2$
$g_{2,l}$ア $= 0,\ 0 \leqq l \leqq p-3$
$g_{3,l}$ア $= 0,\ 0 \leqq l \leqq p-4$

c). h_1ア $= 0$
h_2ア $= -a_0$ウ, *if* $p=3$
$= 2\,a_0 j_{p-3}$, *if* $p \geqq 5$

b). $h_1(2,1)$ア $= 0$
$h_2(1,2)$ア $= 0$

e). $k_{1,l}$ア $= 0,\ 0 \leqq l \leqq p-3$
$k_{2,l}$ア $= 0,\ 0 \leqq l \leqq p-4$

f). uア $= 0$

g). $h_1^m(1,2,1)$ア $= 0,\ 0 \leqq m \leqq p-4$

h). j_mア $= 0,\ 1 \leqq m \leqq p-3$

i). $\nu_{l,0}(2,1)$ア $= 0,\ 0 \leqq l \leqq p-3$

j). wア $= 0$
xア $= 0$

k). Gア $= 0$
ア$^2 = 0$
アウ $= 0$, *if* $p=3$

27. a). a_iイ $= 0$, $i=0$ *or* 1

b). $g_{1,l}$イ $= 0,\ 0 \leqq l \leqq p-2$
$g_{2,l}$イ $= 0,\ 0 \leqq l \leqq p-3$

c). h_1イ $= 0$

b). $k_{1,l}$イ $= 0,\ 0 \leqq l \leqq p-3$

e). uイ $= 0$

28. a). a_1ウ $= 0$

b). $g_{1,l}$ウ $= 0,\ 0 \leqq l \leqq p-2$
$g_{2,l}$ウ $= 0,\ 0 \leqq l \leqq p-3$

c). h_iウ $= 0$, $i=1$ *or* 2

d). $h_1(2,1)$ウ $= 0$

e). $k_{1,l}$ウ $= 0,\ 0 \leqq l \leqq p-3$

f). uウ $= 0$

g). wウ $= 0$
xウ $= 0$

29. a). a_0エ $= 0$

b). $g_{1,l}$エ $= 0,\ 0 \leqq l \leqq p-2$

30. a). a_iオ $= 0$, $i=0$ *or* 1

b). $g_{1,l}$オ $= 0,\ 0 \leqq l \leqq p-2$

c). h_1オ $= 0$

d). a_0カ $= 0$

PROOF. These are proved by the imbedding method; that is, we first compute the differential $\bar{d}^{p-1}$ in homology spectral sequence and then by dualizing it we obtain the differential in this spectral sequence. We give the proof of $\tilde{d}_{p-1}(g_{1,l}) = 0$ if $0 \leqq l \leqq p-2$ and $a_0^{p-1}b_{01}$ if $l=p-1$, the proofs of others being similar. Since $g_{1,l} \in \tilde{E}_{p-1}^{l+1,0,l,2lp+2p-l-2}$, $\tilde{d}_{p-1}(g_{1,l}) \in \tilde{E}_{p-1}^{p+1,2-p,l,2lp+2p-l-2}$. There are no indecomposable elements having non-zero second degree in $\tilde{E}_{p-1}$ except for b_{ij}. For the dimensional and filtrational reasons, $\tilde{d}_{p-1}(g_{1,l}) = 0$ for $0 \leqq l \leqq p-2$ and $\tilde{d}_{p-1}(g_{1,p-1}) = \alpha a_0^{p-1}b_{01}$, $\alpha \in Z_p$. Then we determine $\bar{d}^{p-1}((a_0^{p-1}b_{01})^*)$. $(a_0^{p-1}b_{01})^*$ is represented by $-\{Q_0\}^{p-1} * \{P_1^0 | (P_1^0)^{p-1}\}$ in $B(E^0A)$.

$$d\Big(\sum_{i=0}^{p-2}(-1)^{i+1}\, i!\{Q_0\}^{p-i-1} * \{[P_1^0]*[Q_1]^i | (P_1^0)^{p-i-1}\}\Big)$$
$$= \sum_{i=1}^{p-2}(-1)^i\, i!\{Q_0\}^{p-i-1} * \{[Q_1]^i | (P_1^0)^{p-i}\}$$
$$+ \sum_{i=1}^{p-2}(-1)^i\, i!\{Q_0\}^{p-i-1} * \{[P_1^0]*[Q_1]^{i-1} | Q_1(P_1^0)^{p-i-1}\}$$
$$+ \sum_{i=0}^{p-2}(-1)^i\, i!\{Q_0\}^{p-i-2} * \{[[P_1^0, Q_0]]*[Q_1]^i | (P_1^0)^{p-i-1}\}$$
$$+ \sum_{i=0}^{p-2}(-1)^{i+1}\, i!\{Q_0\}^{p-i-2} * \{[P_1^0]*[Q_1]^i | [(P_1^0)^{p-i-1}, Q_0]\}$$
$$= \sum_{i=0}^{p-3}(-1)^{i+1}(i+1)!\{Q_0\}^{p-i-2} * \{[Q_1]^{i+1} | (P_1^0)^{p-i-1}\}$$
$$+ \sum_{i=0}^{p-3}(-1)^{i+1}(i+1)!\{Q_0\}^{p-i-2} * \{[P_1^0]*[Q_1]^i | Q_1(P_1^0)^{p-i-2}\}$$
$$+ \sum_{i=0}^{p-2}(-1)^i(i+1)!\{Q_0\}^{p-i-2} * \{[Q_1]^{i+1} | (P_1^0)^{p-i-1}\}$$
$$+ \sum_{i=0}^{p-2}(-1)^i(i+1)!\{Q_0\}^{p-i-2} * \{[P_1^0]*[Q_1]^i | Q_1(P_1^0)^{p-i-2}\}$$
$$= \{P_1^0\} * \{Q_1\}^{p-1}.$$

Then we have $\bar{d}^{p-1}((a_0^{p-1}b_{01})^*) = (g_{1,p-1})^*$ and $\tilde{d}_{p-1}(g_{1,p-1}) = a_0^{p-1}b_{01}$ by dualizing it. The algebre structure of $\tilde{E}_p$ is obtained by the tedious but routine calculations. We omit this.

PROPOSITION 2.9. *In dimensions $t-s \leqq (3p^2+3p+4)q-1$, the non-zero differentials $\tilde{d}_{2p-3}$ on $\tilde{E}_{2p-3}$ are given by the following :*

1. $\tilde{d}_{2p-3}(ア) = a_0^{p-2}b_{01}b_{11}$
2. $\tilde{d}_{2p-3}(イ) = a_0^{p-3}h_2(1,2)b_{01}b_{11}$
3. $\tilde{d}_3(ウ) = h_2b_{01}b_{11}$, *if* $p=3$

   $\tilde{d}_{2p-3}(j_{p-3}) = h_2a_0^{p-3}b_{01}b_{11}$, *if* $p \geqq 5$
4. $\tilde{d}_3(オ) = h_0b_{11}b_{21}$, *if* $p=3$
5. $\tilde{d}_3(カ) = -g_{2,0}b_{11}b_{21}$, *if* $p=3$

A basis for the indecomposable elements of $\tilde{E}_{2p-2} = E_\infty = E^0H^*(E^0A)$ is obtained by deleting ア・イ・ウ・オ・カ and j_{p-3} if $p \geqq 5$ from the basis for the indecomposable elements of $\tilde{E}_p$. The algebra structure on $E^0H^*(E^0A)$ differs from that of $\tilde{E}_p$ only in that all relations involving the deleted basis elements for $\tilde{E}_p$ are omited for $E^0H^*(E^0A)$ and the following relations are added to the list for $E^0H^*(E^0A)$:

31. a). $a_0^{p-2}b_{01}b_{11} = 0$

b). $a_0^{p-3}h_2(1,2)b_{01}b_{11} = 0$

c). $a_0^{p-3}h_2b_{01}b_{11} = 0$

d). $h_0b_{11}b_{21} = 0$, *if* $p=3$

e). $g_{2,0}b_{11}b_{21} = 0$, *if* $p=3$

Proof. These are similarly proved to Proposition 2.8. We omit it.

§ 3. The algebra structure of $H^*(E^0A)$

Since we have computed $H^*(E^0A)$ by spectral sequence, the relations in Proposition 2.9 are actually the relations according to the algebra structure of $E^0H^*(E^0A)$. In this section we determine the algebra structure of $H^*(E^0A)$. The product in $H^*(E^0A)$ of two elements always contains as a summand their product in $E^0H^*(E^0A)$ but may possibly contain also other terms of the same tri-grading $(p+q,u,t)$ but of higher weight p in the sense of section 1. Then checking the tetra-grading of the product in Proposition 2.9. we can show that almost all relations hold in $H^*(E^0A)$. For example, we consider the relation $a_0g_{1,l} = 0$, $0 \leqq l \leqq p-2$, for $E^0H^*(E^0A)$. The product $a_0g_{1,l}$ has tetra-grading $(l+2,0,l,(l+1)q+l+1)$ and there exists no element having tetra-grading $(l+2-u,u,l,(l+1)q+l+1)$ for $u < 0$ in $E^0H^*(E^0A)$. Then the relation $a_0g_{1,l} = 0$, $0 \leqq l \leqq p-2$, holds in $H^*(E^0A)$. When we can not determine the product in $H^*(E^0A)$ by dimensional and filtrational considerations, we deteimine it by the computation of its coproduct in $H_*(E^0A)$; sample calculations are given in the following lemmas.

Lemma 3.1. *$g_{1,l}g_{1,p-l-2} = -1/(l+1)\, a_0^{p-2}b_{01}$, $0 \leqq l \leqq p-2$, in $H^*(E^0A)$.*

Proof. The relation $g_{1,l}g_{1,l'} = 0$, $0 \leqq l,l' \leqq p-2$, holds in $E^0H^*(E^0A)$. The product $g_{1,l}g_{1,l'}$ has tetra-grading $(l+l'+2,0,l+l',(l+l'+2)q+l+l')$. There exists only one element $a_0^{p-2}b_{01}$ having the same tri-grading $(l+l'+2,l+l',(l+l'+2)q+l+l')$ in $E^0H^*(E^0A)$. Then the relation $g_{1,l}g_{1,l'} = 0$, $0 \leqq l,l' \leqq p-2$ and $l+l' \neq p-2$, holds in $H^*(E^0A)$ and the product $g_{1,l}g_{1,p-l2}$ $(0 \leqq l \leqq p-2)$ is $\alpha\, a_0^{p-2}b_{01}$ $(\alpha \in z_p$, possibly $\alpha \neq 0)$ in $H^*(E^0A)$. For the determination of α we compute the coproduct of the dual homology class $(a_0^{p-2}b_{01})^*$ in $H_*(E^0A)$ of $a_0^{p-2}b_{01}$. It is easy to see that a cycle $\sum_{i=0}^{p-2}(-1)^{i+1}\, i!\, \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{p-i-2}$ represents $(a^{p-2}b_{01})^*$ in the bar construction $\bar{B}(E^0A)$. Then $D(\sum_{i=0}^{p-2}(-1)^{i+1}\, i!\, \{[P_1^0]*$

$[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{p-i-2}) = \sum_{k=2}^{p} \sum_{i=0}^{k-2} (-1)^{i+1}\, i!\ \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{k-i-2} \otimes \{Q_0\}^{p-k} + \sum_{k=1}^{p-1} \sum_{m=1}^{k} \sum_{i=m-1}^{p-k+m-2} (-1)^{i+1}\, i!\ \{P_1^0\} * \{Q_1\}^{m-1} * \{Q_0\}^{k-m} \otimes \{Q_1 \mid \cdots \mid Q_1 \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{p+m-k-i-2} + \sum_{k=0}^{p-2} \sum_{m=0}^{k} \sum_{i=m}^{p-k+m-2} (-1)^{i+1}\, i!\ \{Q_1\}^m * \{Q_0\}^{k-m} \otimes \{[P_1^0]*[Q_1]^{i-m} \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{p+m-k-i-2}$

is homologous to

$\sum_{k=2}^{p} \sum_{i=0}^{k-2} (-1)^{i+1}\, i!\ \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{k-i-2} \otimes \{Q_0\}^{p-k} + \sum_{k=1}^{p-1} -1/k\ \{P_1^0\} * \{Q_1\}^{k-1} \otimes \{P_1^0\} * \{Q_1\}^{p-k-1} + \sum_{k=0}^{p-2} \sum_{i=0}^{p-k-2} (-1)^i\, i! \{Q_0\}^k \otimes \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{Q_0\}^{p-k-i-2}$ in $\bar{B}(E_0A) \otimes \bar{B}(E_0A)$. Then we have $\psi\ a_0^{p-2}b_{01})^* = \sum_{k=2}^{p} (a_0^{k-2}b_{01})^* \otimes (a_0^{p-k})^* + \sum_{k=2}^{p-1} -1/k\ (g_{1,k-1})^* \otimes (g_{1,p-k-1})^* + \sum_{k=0}^{p-2} (a_0^k)^* \otimes (a_0^{p-k-2}b_{01})^*$. Therefore we have the relation $g_{1,l}g_{1,p-l-2} = -1/(l+1)\ a_0^{p-2}b_{01}$ $(0 \leqq l \leqq p-2)$ for $H^*(E_0A)$ by dualizing above result.

LEMMA 3.2. *$k_{1,l}g_{3,m} = -1/6\ a_0^{p-4}h_2b_{01}b_{11}$ if $l=p-4$ and $m=0$, 0 if otherwise for $p \geqq 5$.*

PROOF. Checking the tetra-grading of $k_{1,l}g_{3,m}$, it is easy to see that $k_{1,l}g_{3,m} = 0$ except in the case $l=p-4$ and m=0, and that $k_{1,p-4}g_{3,0} = \beta\ a_0^{p-4}h_2b_{01}b_{11}$ ($\beta \in Z_p$, possibly $\beta \neq 0$) in $H^*(E^0A)$. By the tedious but routine calculations, we see that a cycle

$\sum_{i=0}^{p-4} (-1)^i\, i!\ \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{P_1^1 \mid (P_1^1)^{p-1}\} * \{P_1^2\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} (-1)^{i+1} (i+1)!\ \{[P_1^0]*[P_2^0]*[Q_1]^i \mid (P_1^0)^{p-i-2} \mid (P_1^1)^{p-1}\} * \{P_1^2\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} \sum_{k=0}^{i} \sum_{j=0}^{min(1,k)} (-1)^{i+k} i!(k-j+1)!\ \{[[P_1^1]*[P_2^0]^{1-j} * [Q_2]^k \mid (P_1^1)^{p-k+j-2}] * [P_1^0]^j * [Q_1]^{i-k} \mid (P_1^0)^{p-i-1}\} * \{P_1^2\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} \sum_{k=0}^{p-i-2} \sum_{j=0}^{i} (-1)^{i+j+1} (k+j)!(k+i+1)!/k!\ \{[P_1^0]*[P_1^1]*[P_2^0]*[Q_1]^{i-j}*[Q_2]^j \mid (P_2^0)^k (P_1^0)^{p-k-i-2} (P_1^1)^{p-k-j-2}\} * \{P_1^2\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} (-1)^i (i+1)!\ \{[[P_1^0]*[P_2^0]*[Q_1]^i \mid (P_1^0)^{p-i-2}]*[P_2^1] \mid (P_1^1)^{p-2}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} (-1)^{i+1}\, i!\ \{[P_1^0]*[Q_1]^i \mid (P_1^0)^{p-i-1}\} * \{[P_1^1]*[P_2^1] \mid$

$(P_1^1)^{p-2}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} (-1)^{i+1}(i+1)! \{[[P_1^0]*[P_3^0]*[Q_1]^i | (P_1^0)^{p-i-2}] * [P_1^1] | (P_1^1)^{p-2}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4} (-1)^{i+1}(i+2)! \{[P_1^0]*[P_2^0]*[P_3^0]*[Q_2]^i | (P_1^0)^{p-i-3} | (P_1^1)^{p-2}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4}\sum_{k=0}^{i} (-1)^{i+k} i!(k+2)! \{[[P_1^1]*[P_2^0]*[P_2^1]*[Q_2]^k | (P_1^1)^{p-k-3}]*[Q_1]^{i-k} | (P_1^0)^{p-i-1}\} * \{Q_0\}^{p-i-4} + \sum_{i=1}^{p-4}\sum_{k=1}^{i} (-1)^{i+k+1} i!(k+1)! \{[[P_1^1]*[P_2^1]*[Q_2]^k | (P_1^1)^{p-k-2}] *[P_1^0]*[Q_1]^{i-k} | (P_1^0)^{p-i-1}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4}\sum_{k=0}^{p-i-2}\sum_{j=0}^{\min(i,p-k-3)} (-1)^{i+j}(k+j)!(k+i+1)!(k+j+2)/k! \{[P_1^0]*[P_1^1]*[P_2^0]*[P_2^1]*[Q_1]^{i-j} * [Q_2]^j | (P_2^0)^k (P_1^0)^{p-k-i-2}(P_1^1)^{p-k-j-3}\} * \{Q_0\}^{p-i-4} + \sum_{i=0}^{p-4}\sum_{k=0}^{p-i-3}\sum_{j=0}^{i} (-1)^{i+j}(k+j)!(k+i+2)!/k! \{[P_1^0]*[P_1^1]*[P_2^0]*[P_3^0]*[Q_1]^{i-j}*[Q_2]^j | (P_2^0)^k (P_1^0)^{p-k-i-3}(P_1^1)^{p-k-j-3}\} * \{Q_0\}^{p-i-4}$

represents the homology class $(a_0^{p-4}h_2b_{01}b_{11})^*$. For the determination of β it is sufficient to determine the coefficient of $(k_{1,p-4})^* \otimes (g_{3,0})^*$ in ψ $(a_0^{p-4}h_2b_{01}b_{11})^*$. Then it is sufficient to consider the following terms of D(above representative cycle of $(a_0^{p-4}h_2b_{01}b_{11})^*$) in $\bar{B}(E^0A) \otimes \bar{B}(E^0A)$; $(p-4)!(p-3)! \{P_1^1\} * \{P_2^0\} * \{Q_2\}^{p-4} \otimes \{(P_1^1)^2 | (P_1^0)^3\} * \{P_1^2\} - \sum_{k=0}^{2} (k+p-4)!(k+p-3)!/k! \{P_1^1\} * \{P_2^0\} * \{Q_2\}^{p-4} \otimes \{P_1^0 | (P_2^0)^k (P_1^0)^{2-k}(P_1^1)^{2-k}\} * \{P_1^2\} + (p-4)!(p-2)! \{P_1^1\} * \{P_2^0\} * \{Q_2\}^{p-4} \otimes \{P_2^1 | P_1^1 | (P_1^0)^3\} + \sum_{k=0}^{1} (k+p-4)!(k+p-2)!/k! \{P_1^1\} * \{P_2^0\} * \{Q_2\}^{p-4} \otimes \{[P_1^0]*[P_2^1] | (P_2^0)^k (P_1^0)^{2-k}(P_1^1)^{1-k}\} + \sum_{k=0}^{1} (k+p-4)!(k+p-2)!/k! \{P_1^1\} * \{P_2^0\} * \{Q_0\}^{p-4} \otimes \{[P_1^0]*[P_3^0] | (P_2^0)^k (P_1^0)^{1-k}(P_1^1)^{1-k}\}$ is homologous to
$2(p-3)!(p-4)! \{P_1^1\} * \{P_2^0\} * \{Q_2\}^{p-4} \otimes \{P_1^0\} * \{P_2^0\} * \{P_3^0\}$.
Then $\beta = -1/6$ and we have a relation $k_{1,p-4}g_{3,0} = -1/6\, a_0^{p-4}h_2b_{01}b_{11}$.

Remark. In the special case we can use manipulative method to determine the product in $H^*(E^0A)$. For example, we consider the relations $wh_2(1,2) = 0$ and $h_1(2,1)G = 0$ in $E^0H^*(E^0A)$ in the case $p=3$. In May spectral sequence $d_3(wh_2(1,2)) = -a_0h_1b_{11}b_{21} \neq 0$ and $d_3(h_1(2,1)G) = -a_0^2h_1b_{11}b_{21} \neq 0$ in E_3. Then $wh_2(1,2) \neq 0$ and $h_1(2,1)G \neq 0$ in $H^*(E^0A)$. It is only possible that $wh_2(1,2) = -a_0b_{21}b_{02} - a_0b_{01}b_{12}$ and $h_1(2,1)G = -a_0^2b_{21}b_{02}$ in $H^*(E^0A)$.

We extend Lemma 3.1 and 3.2 to obtain the following results.

Theorem 3.3. *In the dimensions* $t-s \leq (3p^2+3p+4)q-1$, *a basis for the indec-*

omposable elements of $H^*(E_0A)$ *is given by the following*:

1. $a_0 : S_0 \in (1,0,0) \cdots\cdots (s,t-s,u)$, u: *May's weight*

   $a_i : S_i^p \in (p,(p^i+p^{i-1}+\cdots+p)q,ip)$, $i \geqq 1$

2. $b_{i1} : \tilde{R}_1^i \in (2,p^{i+1}q-2,p-2)$

   $b_{i2} : \tilde{R}_2^i \in (2,p^i(p^2+p)q-2,2p-2)$

   $b_{i3} : \tilde{R}_3^i \in (2,p^i(p^3+p^2+p)q-2,3p-2)$

3. $h_i : R_1^i \in (1,p^iq-1,0)$

4. $h_1(2,1) : R_2^1R_1^1 \in (2,(p^2+2p)q-2,1)$

   $h_2(1,2) : R_1^2R_2^1 \in (2,(2p^2+p)q-2,1)$

5. $g_{1,l} : R_1^0S_1^l \in (l+1,(l+1)q-1,l)$, $0 \leqq l \leqq p-2$

   $g_{2,l} : R_2^0R_1^0S_1^l \in (l+2,(p+l+2)q-2,l+1)$, $0 \leqq l \leqq p-3$

   $g_{3,l} : R_3^0R_2^0R_1^0S_1^l \in (l+3,(p^2+2p+l+3)q-3,3+l)$, $0 \leqq l \leqq p-4$

6. $k_{1,l} : R_1^1R_2^0S_2^l \in (l+2,(2p+lp+l+1)q-2,2l+1)$, $0 \leqq l \leqq p-3$

   $k_{2,l} : R_2^1R_1^1R_2^0S_2^l \in (l+3,(p^2+3p+lp+l+1)q-3,2l+2)$, $0 \leqq l \leqq p-4$

7. $u : R_1^0S_1^{p-2}S_2 - R_2^0S_1^{p-1} \in (p,2pq-1,p)$

8. $h_1^m(1,2,1) : R_3^0R_2^1R_1^1R_2^0S_2^m \in (m+4,(2p^2+4p+mp+m+2)q-4,2m+4)$, $0 \leqq m \leqq p-4$

9. $j_m : R_1^1R_2^0R_1^0R_3^0S_2^m \in (m+4,(p^2+3p+mp+m+3)q-4,2m+3)$, $1 \leqq m \leqq p-4$

10. $v_m : R_3^0R_1^2R_1^0S_1^m \in (m+3,(2p^2+p+m+2)q-3,m+2)$, $0 \leqq m \leqq p-3$

11. $y : R_3^0R_1^2R_2^1 \in (3,(3p^2+2p+1)q-3,3)$, *if* $p \geqq 5$

12. $w : R_2^0R_1^0S_1^{p-3}S_3 - R_3^0R_1^0S_1^{p-3}S_2 + R_3^0R_2^0S_1^{p-2} \in (p,(p^2+3p)q-2,p+1)$

13. $x : R_1^1R_2^0S_2^{p-3}S_3 + 1/2\ R_3^0R_1^1S_2^{p-2} - 1/2\ R_2^1R_2^0S_2^{p-2} \in (p,(2p^2+p-1)q-2,2p-2)$

    $x_2 : R_1^1R_2^0S_2^{p-3}S_3^2 + R_3^0R_1^1S_2^{p-2}S_3 - R_2^1R_2^0S_2^{p-2}S_3 - R_3^0R_2^1S_2^{p-1} \in (p+1,(3p^2+2p)q-2,2p+1)$

14. $G : R_1^2R_1^0S_1^{p-2}S_3 + R_3^0R_1^2S_1^{p-1} \in (p+1,(2p^2+2p)q-2,p+1)$

15. エ : $R_1^2R_2^1R_3^0R_2^0R_1^0 \in (5,(3p^2+3p+3)q-5)$, *if* $p=3$.

In the cited range, the algebra structure of $H^*(E_0A)$ *is determined by the following relations*:

1. a). $a_0g_{1,l} = 0$, $0 \leqq l \leqq p-2$

   b). $a_0g_{2,l} = 0$, $0 \leqq l \leqq p-3$

   c). $a_0g_{3,l} = 0$, $0 \leqq l \leqq p-4$

   d). $a_0k_{1,l} = 0$, $0 \leqq l \leqq p-3$

   e). $a_0k_{2,l} = 0$, $0 \leqq l \leqq p-4$

   f). $a_0u = h_1a_1$

g). $a_0 h_1^m(1,2,1) = 0,\ 0 \leqq m \leqq p-4$

h). $a_0 j_m = 0,\ 1 \leqq m \leqq p-4$ *if* $p \geqq 5$

i). $a_0 v_m = 0,\ 0 \leqq m \leqq p-3$

j). $a_0 y = 0$

k). $a_0 x = 0$

$a_0 x_2 = h_2(1,2) a_2$

l). $a_0 G = h_2(1,2) a_1$

m). $a_0 \perp = 0$

2. a). $a_1 g_{2,l} = g_{1,l+1} u,\ 0 \leqq l \leqq p-3$

b). $a_1 g_{3,l} = g_{1,l+2} w,\ 0 \leqq l \leqq p-4$

c). $a_1 h_1(2,1) = a_0^2 w$

d). $a_1 k_{1,l} = 0,\ 0 \leqq l \leqq p-3$

e). $a_1 k_{2,l} = 0,\ 0 \leqq l \leqq p-4$

f). $a_1 h_1^m(1,2,1) = 0,\ 0 \leqq m \leqq p-4$

g). $a_1 j_m = 0,\ 1 \leqq m \leqq p-4$

h). $a_1 v_m = g_{1,m+1} G,\ 0 \leqq m \leqq p-3$

i). $a_1 y = 0$

j). $a_1 x = 1/2\ h_2 g_{1,p-2} a_2$

$a_1 x^2 = G a_2$

3. a). $a_2 v_m = g_{1,m+1} x_2,\ 0 \leqq m \leqq p-3$

b). $a_2 h_1^{p-4}(1,2,1) = 2\ x^2$ *if* $p \geqq 5$

4. a). $g_{1,l} g_{1,l'} = -1/(l+1)\ a_0^{p-2} b_{01},\ 0 \leqq l,l' \leqq p-2$ *and* $l+l' = p-2$

$= 0,\ 0 \leqq l,l' \leqq p-2$ *and* $l+l' \neq p-2$

b). $g_{1,l} g_{2,l'} = -1/(l+1) a_0^{p-3} h_1 b_{01},\ 0 \leqq l \leqq p-2,\ 0 \leqq l' \leqq p-3,\ l+l' = p-3$

$= 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq l' \leqq p-3,\ l+l' \neq p-3$

c). $g_{3,l} g_{1,l'} = -1/(1+3)\ a_0^{p-4} h_1(2,1) b_{01},\ 0 \leqq l \leqq p-2,\ 0 \leqq l' \leqq p-4$ *and* $l+l' = p-4$

$= 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq l' \leqq p-4,\ l+l' \neq p-4$

d). $h_1 g_{1,l} = 0,\ 0 \leqq l \leqq p-2$

e). $h_1(2,1) g_{1,l} = 0,\ 0 \leqq l \leqq p-2$

f). $h_2(1,2) g_{1,l} = 0,\ 0 \leqq l \leqq p-2$

g). $k_{1,0} g_{1,0} = h_1 g_{2,0}$

$k_{1,l} g_{1,m} = 0,\ 0 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-2$

h). $g_{1,l} k_{2,m} = 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-4$

i). $g_{1,0} h_1^0(1,2,1) = h_1(2,1) g_{3,0}$ *if* $p \geqq 5$

$g_{1,l} h_1^m(1,2,1) = 0,\ 1 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-4$

j). $g_{1,l} j_m = 0,\ 0 \leqq l \leqq p-2,\ 1 \leqq m \leqq p-4$

k). $g_{1,l} v_m = -1/(l+1) a_0^{p-3} h_2(1,2) b_{01},\ 0 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-3,\ l+m = p-3$

$= 0,\ 0 \leqq l \leqq p-2,\ 0 \leqq m \leqq p-3,\ l+m \neq p-3$

l). $g_{1,l}y = 0,\ 1 \leqq l \leqq p-2$

m). $g_{1,0}x = h_1w - a_0h_2b_{02}$ *if* $p=3$

$g_{1,l}x = 0,\ 1 \leqq l \leqq p-2$

n). $g_{1,l}$ 工 $= 0,\ 0 \leqq l \leqq p-2$ *if* $p=3$

5. a). $g_{2,l}g_{2,l'} = -k_{1,0}b_{01},\ l=l'=0$ *if* $p=3$

$= 0,\ 0 \leqq l,l' \leqq p-3$ *if* $p \geqq 5$

b). $g_{2,l}g_{3,l'} = 0,\ 0 \leqq l \leqq p-3,\ 0 \leqq l' \leqq p-4$

c). $h_1g_{2,l} = 0,\ 1 \leqq l \leqq p-3$

d). $h_2g_{2,l} = 0,\ 0 \leqq l \leqq p-3$

e). $h_1(2,1)g_{2,l} = 0,\ 0 \leqq l \leqq p-3$

f). $h_2(1,2)g_{2,l} = 0,\ 0 \leqq l \leqq p-3$

g). $k_{1,m}g_{2,l} = -1/2\ a_0^{p-3}b_{01}b_{11},\ m=p-3,\ l=0$

$= 0,$ *otherwise*

h). $g_{2,l}k_{2,m} = 0,\ 0 \leqq l \leqq p-3,\ 0 \leqq m \leqq p-4$

i). $g_{2,l}u = 0,\ 0 \leqq l \leqq p-3$

j). $g2,_l h_1^m(1,2,1) = 1/2\ a_0^{p-4}h_2(1,2)b_{01}b_{11},\ l=0,\ m=p-4$

$= 0,$ *otherwise*

k). $g_{2,l}j_m = 0,\ 0 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-4$

l). $g_{2,l}v_m = 1/(l+2)\ a_0^{p-4}h_1h_2(1,2)b_{01},\ 0 \geqq l,m \geqq p-3,\ 1+m = p-4$

$= 0,$ *otherwise*

m). $g_{2,0}y = h_2(1.2)g_{3,0}$

$g_{2,1}y = 0,\ 1 \leqq l \leqq p-3$

n). $g_{2,l}w = g_{1,l}h_2b_{02} - b_{01}x,\ l=0$ *if* $p=3$

$= 0,\ 0 \leqq l \leqq p-3$ *if* $p \geqq 5$

o). $g_{2,l}x = 0,\ 1 \leqq l \leqq p-3$

p). $g_{2,0}x_2 = wx$ *if* $p=3$

$g_{2,l}x_2 = -h_2g_{3,l-1}a_2,\ 1 \leqq l \leqq p-3$

$= -2\ wx,\ l=p-3$ *if* $p \geqq 5$

q). $g_{2,l}G = -uv_l,\ 0 \leqq l \leqq p-3$

$= -h_2g_{3,l-1}a_1,\ 1 \leqq l \leqq p-3$

6. a). $g_{3,l}g_{3,l'} = 0,\ 0 \leqq l,l' \leqq p-4$

b). $h_1g_{3,l} = 0,\ 1 \leqq l \leqq p-4$

c). $h_1(2,1)g_{3,1} = 0,\ 1 \leqq l \leqq p-4$

d). $h_2(1,2)g_{3,l} = 0,\ 1 \leqq l \leqq p-4$

e). $k_{1,l}g_{3,m} = -1/6\ a_0^{p-4}h_2b_{01}b_{11},\ l=p-4,\ m=0$

$= 0,$ *otherwise*

f). $k_{2,l}g_{3,m} = -1/2\ a_0^{p-4}h_2(1,2)b_{01}b_{11},\ l=p-4,\ m=0$

$= 0,$ *otherwise*

g). $vg_{3,m} = 0,\ 0 \leqq m \leqq p-4$

h). $g_{3,l}j_m = 0,\ 0 \leqq l \leqq p-4,\ 1 \leqq m \leqq p-4$

i). $g_{3,l}w = 0,\ 0 \leqq l \leqq p-4$

j). $g_{3,l}x = 0,\ 0 \leqq l \leqq p-4$

7. a). $h_1^2 = 0$

b). $h_1h_2 = 0$

c). $h_2(1,2)h_1 = h_2h_1(2,1)$

d). $h_1h_1(2,1) = -h_2b_{11}$ *if* $p=3$
$= 0$ *if* $p \geqq 5$

e). $h_1k_{1,l} = -g_{1,p-3}b_{11},\ l=p-3$
$= 0,\ 0 \leqq l \leqq p-4$

f). $h_1k_{2,l} = -h_2g_{1,p-4}b_{11},\ l=p-4$
$= 0,\ 0 \leqq l \leqq p-5$

g). $h_1u = 0$

h). $h_1h_1^m(1,2,1) = v_{p-4}b_{11},\ m=p-4 = 0,\ 0 \leqq m \leqq p-5$

i). $h_1j_m = 0,\ 1 \leqq m \leqq p-4$

j). $h_1v_m = 0,\ 0 \leqq m \leqq p-3$

k). $h_1w = 0$ *if* $p \geqq 5$

l). $h_1G = h_2(1,2)u = a_0h_2w$

8. a). $h_2^2 = 0$

b). $h_2h_3 = 0$

c). $h_2h_2(1,2) = -b_{21}h_1$ *if* $p=3$
$= 0$ *if* $p \geqq 5$

d). $h_2k_{1,l} = 0,\ 0 \leqq l \leqq p-3$

e). $h_2k_{2,l} = 0,\ 0 \leqq l \leqq p-4$

f). $h_2v = 0$

g). $h_2h_1^0(1,2,1) = -h_1(1,2)y$

h). $h_2j_m = 0,\ 1 \leqq m \leqq p-4$

i). $h_2v_m = g_2b_{21}$ *if* $p=3$
$= 0,\ 0 \leqq l \leqq p-3$ *if* $p \geqq 5$

j). $h_2x = 0$

k). $h_2G = -ub_{21}$ *if* $p=3$
$= 0$ *if* $p \geqq 5$

9. a). $h_1(2,1)h_1(2,1) = -h_2(1,2)b_{11}$ *if* $p=3 = 0$ *if* $p \geqq 5$

b). $h_2(1,2)h_1(2,1) = b_{11}b_{21}$ *if* $p=3$
$= 0,$ *if* $p \geqq 5$

c). $h_1(2,1)k_{1,l} = h_2g_{1,p-4}b_{11},\ l=p-4$
$= 0,\ 0 \leqq l \leqq p-5$ *or* $l=p-3$

d). $h_1(2,1)k_{2,l} = 0,\ 0 \leqq l \leqq p-4$

e). $h_1(2,1)u = a_0^2h_2b_{02}$ *if* $p=3$
$= 0$ *if* $p \geqq 5$

f). $h_1(2,1)h_1^m(1,2,1) = 0,\ 0 \leqq m \leqq p-4$

g). $h_1(2,1)j_m = 0,\ 1 \leqq m \leqq p-4$

h). $h_1(2,1)v_m = 0,\ 0 \leqq m \leqq p-3$

i). $h_1(2,1)y = 0$

j). $h_1(2,1)w = a_0h_2(1,2)b_{02}$ *if* $p=3$

$= 0$ *if* $p \geqq 5$

k). $h_1(2,1)x = 0$

l). $h_1(2,1)G = a_0^2b_{21}b_{02}$ *if* $p=3$

$= 0$ *if* $p \geqq 5$

10. a). $h_2(1,2)k_{1,l} = 0,\ 0 \leqq l \leqq p-3$

b). $h_2(1,2)k_{2,l} = 0,\ 0 \leqq l \leqq p-4$

c). $h_2(1,2)h_1^m(1,2,1) = 0,\ 0 \leqq m \leqq p-4$

d). $h_2(1,2)v_m = 0,\ 0 \leqq m \leqq p-3$

e). $h_2(1,2)y = 0$

f). $h_2(1,2)w = -a_0b_{21}b_{02} - a_0b_{01}b_{12}$ *if* $p=3$

$= 0$ *if* $p \geqq 5$

11. a). $k_{1,l}k_{1,l'} = 1/(l+1)(l+2)\ g_{2,p-3}b_{11},\ 0 \leqq l,l' \leqq p-3,\ l+l' = p-3$

$= 0$, *otherwise*

b). $k_{1,l}k_{2,l'} = 0,\ 0 \leqq l \leqq p-3,\ 0 \leqq l' \leqq p-4$

c). $k_{1,l}u = 0,\ 0 \leqq l \leqq p-3$

d). $k_{1,l}h_1^m(1,2,1) = 0,\ 0 \leqq l \leqq p-3,\ 0 \leqq m \leqq p-4$

e). $k_{1,l}j_m = 0,\ 0 \leqq l \leqq p-3,\ 1 \leqq m \leqq p-4$

f). $k_{1,l}v_m = 0,\ 0 \leqq l,m \leqq p-3$

g). $k_{1,l}w = 0,\ 0 \leqq l \leqq p-3$

h). $k_{1,l}x = 2\ a_0^{p-2}h_1(2,1)b_{02} + 2\ b_{11}w,\ l=0$

$= 0,\ 1 \leqq l \leqq p-3$

i). $k_{1,0}x_2 = x^2$ *if* $p=3$

j). $k_{1,l}G = 0,\ 0 \leqq l \leqq p-3$

12. a). $k_{2,l}k_{2,l'} = 0,\ 0 \leqq l,l' \leqq p-4$

b). $k_{2,l}u = 0,\ 0 \leqq l \leqq p-4$

c). $k_{2,l}j_m = 0,\ 0 \leqq l \leqq p-4,\ 1 \leqq m \leqq p-4$

d). $k_{2,l}w = 0,\ 0 \leqq l \leqq p-4$

e). $k_{2,l}x = 0,\ 0 \leqq l \leqq p-4$

13. a). $u^2 = 0$

b). $uh_1^m(1,2,1) = 0,\ 0 \leqq m \leqq p-4$

c). $uj_m = 0,\ 1 \leqq m \leqq p-4$

d). $uv_0 = h_2g_{1,1}w$

e). $uy = 0$

f). $uw = a_1h_2b_{02} - b_{01}h_2a_2$ *if* $p=3$; $= 0$ *if* $p \geqq 5$

g). $ux = 0$

h). $ux_2 = h_2wa_2$

i). $uG = h_2 w a_1$

14. a). $j_m j_{m_t} = 0,\ 1 \leqq m, m_t \leqq p-4$

b). $j_m w = 0,\ 1 \leqq m \leqq p-4$

c). $j_m x = 0,\ 1 \leqq m \leqq p-4$

d). $v_m w = g_{1,1} b_{01} b_{12} - g_{11} b_{21} b_{02}$ *if* $p=3$; $= 0,\ 0 \leqq m \leqq p-3$ *if* $p \geqq 5$

e). $w^2 = G b_{02} - b_{01} x_2$ *if* $p=3$; $= 0$ *if* $p \geqq 5$

f). $wx = 1/2\ h_2 g_{3,p-4} a_2$ *if* $p \geqq 5$

15. a). $a_0^{p-1} b_{01} = 0$

b). $a_0^{p-2} h_1 b_{01} = 0$

c). $a_0^{p-3} h_1(2,1) b_{01} = 0$

d). $a_0^p b_{02} + a_1 b_{11} = 0$

e). $a_0^p b_{03} + a_1 b_{12} + a_2 b_{21} = 0$

f). $g_{1\,p-2} b_{11} = 0$

g). $u b_{11} + a_0^{p-1} h_1 b_{02} = 0$

h). $h_2 g_{1,p-3} b_{11} = 0$

i). $a_0^{p-1} h_1(2,1) b_{02} + a_0 b_{11} w = 0$

j). $v_{p-3} b_{11} = 0$

k). $h_2 w b_{11} + h_2 h_1(2,1) a_0^{p-2} b_{02} = 0$

l). $k_{1,0} b_{21} = 0$ *if* $p=3$

m). $a_0^{p-2} h_2(1,2) b_{01} = 0$

n). $G b_{11} + a_0^{p-1} h_2(1,2) b_{02} = 0$

16. a). $a_0^{p-2} b_{01} b_{11} = 0$

b). $a_0^{p-3} h_2(1,2) b_{01} b_{11} = 0$

c). $a_0^{p-3} h_2 b_{01} b_{11} = 0$

d). $h_0 b_{11} b_{21} = 0$ *if* $p=3$

e). $g_{2,0} b_{21}^2 = 0$ *if* $p=3$

REMARK 1. We denote $v_{l,0}(2,1)$, $0 \leqq l \leqq p-3$, *by* v_l in above theorem.

REMARK 2. The following relations are different from those of J.P.May ([3], Theorem II.4.11 and 13).

5. g). $k_{1,p-3} g_{2,0} = -1/2\ a_0^{p-3} b_{01} b_{11}$

n). $g_{2,0} w = h_0 h_2 b_{02} - b_{01} x$ *if* $p=3$

6. e). $k_{1,p-4} g_{3,0} = -/6\ a_0^{p-4} h_2 b_{01} b_{11}$ *if* $p \geqq 5$

9. e). $h_1(2,1) u = a_0^2 h_2 b_{02}$ *if* $p=3$

11. a). $k_{1,l} k_{1,p-l-3} = 1/(l+1)(l+2) g_{2,p-3} b_{11},\ 0 \leqq l \leqq p-3$

§4. The Cohomology of the Steenrod Algebra

In the previous section, we have obtained a good deal of imfomation about the structure of $H^*(E^0A)$. In this section, we study the May spectral sequence.

The following methods of proof are known for the various differentials in May spectral sequence.

1. *The Imbedding Method.* This method is introduced in J.P.May's [3]. In theory this method is effective, but in practice, like any method using the bar construction, it is sometimes prohibitively slow. We therefore supplement it whenever possible with an independent proof by one of the alternative methods below.
2. *Manipulative Methods.*
3. *Application of Liulevicious' Results on $H^*(A)$.*
4. *Application of Known Results in Homotopy Theory.* These three methods are introduced in M.C.Tangora's [9].
5. *Application of Effect of algebraic Steenrod operations in spectral sequence.* This is introduced in J.P.May's [5].
6. *The Matric Massey Products Method.* The matric Massey products in spectral sequence were studied by J.P.May in [4]. The following J.P.May's theorem turn out to be apowerful tool in studying differentials in May spectral sequence.

THEOREM 4.1. ([4]. Corollary 4.4) *Let a matric Massey product $< V_1,\cdots,V_n >$ be defined in E_{r+1}. Let $s>r$ be given such that $d_t V_i = 0$ for $t<s$ and $1 \leqq i \leqq n$ and such that the following condition(*) is satisfied.*

(*) *If $(p,q) \in D_{ij}, 1<j-i<n$, then, for each t such that $r<t<s$, $E_t^{p+t,q-t+1}=0$. Assume in addition to above hypotheses that for $1 \leqq k \leqq n$ there is just one matrix $Y_k \in ME_{r+1}$ which survives to $d_s V_k$. By abuse, write $Y_k = d_s V_k$, and suppose that each $< \bar{V}_1,\cdots,\bar{V}_{k-1},d_s V_k,V_{k+1},\cdots,V_n >$ is strictly defined in E_{r+1}. Assume further that all matric Massey products in sight have zero indeterminacy. Then $d_s < V_1,\cdots,V_n >$*

$$= - \sum_{k=1}^{n} < \bar{V}_1,\cdots,\bar{V}_{k-1},d_s V_k,V_{k+1},\cdots,V_n>.$$

The following theorem is proved by J.P.May in his thesis [3].

THEOREM 4.2. ([3]. Theorem II .6.9)

1. $d_{p^{n+1}}(b_{ij})^{p^n} = h_{i+n+1}(b_{i+1\ j-1})^{p^n} - h_{i+j+n}(b_{i\ j-1})^{p^n},\ n \geqq 0,\ j \geqq 2,\ i \geqq 0$
2. $d_{p^{n+1}}(a_1)^{p^n} = -(a_0)^{p^{n+1}} h_{n+1},\ n \geqq 0$
3. $d_{p^{n+1}}(a_i)^{p^n} = -(a_{i-1})^{p^n} h_{n+i},\ n \geqq 0,\ i \geqq 2$

All other non-zero differentials in the range $t-s \leqq 2(p-1)(2p^2+p+2)-4$ are determined by the statement that E_r is a differential algebra and by ;

4. $d_{2p-1}(h_1 a_2) = a_0^p c,\ c = h_1(2,1)$
5. $d_{2p-1}(h_2 a_2) = a_0^p d,\ d = h_2(1,2)$
6. $d_{2p-1}(a_1^l a_2 u) = a_0^{p+1} a_1^l w,\ 0 \leqq l \leqq p-4$ *and* $l=p-2$ *if* $p>3$;
   $d_5(a_1 a_2 u) = a_0^4 m,\ m = a_1 w + a_0 a_2 b_{11}$ *if* $p=3$
7. $d_{p^2-3p+3}(a_1^{p-3} w) = a_0^{p^2-2p-2} h_2 b_{11}$
8. $d_{p^2-2p+2}(a_1^{p-2} u) = a_0^{p^2-p-1} b_{11}$
9. $d_{p^2-2p+2}(a_1^{p-3} u b_{02}) = a_0^{p^2-2p-1} b_{11}^2$

We now begin the calculation.

PROPOSITION 4.3. *All non-zero differentials in the range $t-s \leqq 2(p-1)(3p^2+3p+4)$*

-1 *are determined by the statement that* E_r *is a differential algebra and by the following differentials :*

1. $d_p(a_1) = -a_0^p h_1$
2. $d_{p^2-2p+2}(a_1^{p-2}u) = \alpha a_0^{p^2-p-1}b_{11}$, $\alpha \not\equiv 0$ *(mod p) if* $p \geqq 5$

   $d_5(a_1 u) = a_0^5 b_{11}$ *if* $p=3$
3. $d_{p^2}(a_1^p) = \beta a_0^{h^2} h_2$, $\beta \not\equiv 0$ *(mod p)*
4. $d_p(a_2) = -h_2 a_1$
5. $d_p(b_{02}) = -b_{01}h_2 + h_1 b_{11}$
6. $d_{2p-1}(h_1 a_2) = -2\, a_0^p h_1(2,1)$
7. $d_{2p-1}(a_1^l a_2 u) = -2\, a_0^{p+1} a_1^l w$, $0 \leqq l \leqq p-1$, $l \neq p-3$ *if* $p \geqq 5$

   $d_5(a_1 a_2 u) = -a_0^4 m$, $m = a_1 w - a_0 b_{11} a_2$ *if* $p=3$

   $b_5(a_1^2 a_2 u) = a_0^4 a_1^2 w$ *if* $p=3$
8. $d_{p^2-2p+2}(a_1^{p-3} u b^o{}_2) = -\alpha a_0^{p^2-2p-1} b_{11}$, *where* α *is same in* 2 *if* $p \geqq 5$

   $d_5(ub_{02}) = -a_0^2 b_{11}^2$ *if* $p=3$
9. $d_{p^2-3p+3}(a_1^{p-3}w) = \gamma a_0^{p^2-2p-2} h_2 b_{11}$, $\gamma \not\equiv 0$ *(mod p) if* $p \geqq 5$

   $d_3(w) = -a_0 h_1 h_1(2,1)$ *if* $p=3$
10. $d_{2p-1}(h_2 a_2) = a_0^p h_2(1,2)$
11. $d_p(G) = -a_0^{p-1} h_1 h_2(1,2)$
12. $d_{2p-1}(a_1 h_2 b_{02}) = -2\, a_0^p h_1(2\ 1) b_{11}$
13. $d_{2p-1}(h_1 b_{02}^2) = 2\, b_{01}^2 h_2(1,2) + 2\, b_{01} b_{11} h_1(2,1)$ *if* $p \geqq 5$

    $d_5(h_1 b_{02}^2) = h_0 b_{01} v_0 + b_{11}^3$ *if* $p=3$
14. $d_{2p-1}(h_1 b_{02} a_2) = 2\, a_0 b_{01} G - 2\, a_0^p b_{02} h_1(2,1)$ *if* $p \geqq 5$

    $d_5(h_1 b_{02} a_2) = -a_0 b_{01} G$ *if* $p=3$
15. $d_{2p-1}(h_1 a_2^2) = 2\, a_0 a_1 G - 4\, a_0^p a_2 h_1(2,1)$
16. $d_{17}(a_1^3 u a_2 = \varepsilon a_0^{14} b_{21}$, $\varepsilon = \pm 1$ *if* $p=3$
17. $d_{27}(a_1^9) = \eta a_0^{27} h_3$, $\eta = \pm 1$ *if* $p=3$
18. $d_{2p-1}(g_{1,l} a_2^2) = -2\, g_{1,l+1} a_1 G$, $0 \leqq l \leqq p-3$
19. $d_{2p-1}(g_{2,1} b_{02}^2) = 2\, g_{1,0} k_{1,1} b_{11}^2 - 2\, b_{01}^2 v_1$ *if* $p \geqq 5$

    $d_{2p-1}(g_{2,l} b_{02}^2) = -2\, b_{01}^2 v_1$, $2 \leqq l \leqq p-3$ *if* $p \geqq 5$
20. $d_{2p-1}(g_{2,l} b_{02} a_2) = -2\, g_{1,l+1} b_{01} G$, $1 \leqq l \leqq p-3$ *if* $p \geqq 5$
21. $d_{2p-1}(u b_{02}^2) = 2\, b_{01}^2 G$
22. $d_{2p-1}(u b_{02} a_2) = 2\, b_{01} a_1 G - 2\, a_0^{p+1} w b_{02}$ *if* $p \geqq 5$

    $d_5(u b_{02} a_2) = -b_{01} a_1 G + a_0^2 b_{11}^2 a_2 + a_0^4 w b_{02}$ *if* $p=3$
23. $d_{2p-1}(u a_2^2) = 2\, (a_1^2 G - 3\, a_0^{p+1} w a_2)$ *if* $p \geqq 5$

    $d_5(u a_2^2 (= -a_1^2 G$ *if* $p=3$
24. $d_{2p-1}(k_{1,l} b_{02}^2) = 2/(1+2)\ k_{2,l} b_{01} b_{11}$, $0 \leqq l \leqq p-5$ *if* $p \geqq 5$

    $d_{2p-1}(k_{1,p-4} b_{02}^2) = -k_{2,p-4} b_{01} b_{11}$ *if* $p \geqq 5$
25. $d_{2p-1}(h_0 k_{1,p-3} b_{02}^2) = -g_{2,p-3} b_{11}^3$ *if* $p \geqq 5$