Some Differentials in the mod 3 Adams Sepectral Sequence

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by

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Let A_p be the mod p Steenrod algebra. J. F. Adams [1] introduced a spectral sequence which has as its E_2 term $Ext_{A_p}(H*(X), Z_p)$ and which converges to a graded algebra associated to $\pi_*^s(X;p)$, i. e., the p-primary stable homotopy groups of X. In this paper we will study this sequence for $X=S^n$, $p=3$. The first problem in any use of the Adams spectral sequence is to obtain $E_2 = Ext_{A_3}^{s,t}(Z_3, Z_3)$. We do this by the technique of J. P. May [5]. J. P. May constructed another spectral sequence which has as its E_∞ term an algebra E^0Ext, i. e. a tri-graded algebra associated to $E_2 = Ext$. In [9], we extended (and corrected) May's computations to obtain complete information on E_0Ext through dimension 158. The next problem is to obtain the differentials in the Adams spectral sequence. J. P. May [5] and S.Oka have previously determined all differentials at least in the range $t-s \leqq 77$ by using the results of the 3-components of stable homotopy groups of sphere which have been calculated by H.Toda [12,13,14,15], J.P.May [5] and S.Oka [10]. The purpose of this paper is to evaluate the differentials in the range $78 \leqq t-s \leqq 104$. Our main result is Theorem 3.19.

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§ 1 Algebra Structure of $Ext_{A_3}^{**}(Z_3, Z_3)$

From now on we will write $H^{**}(A_3)$ instead of $Ext_{A_3}^{**}(Z_3, Z_3)$ for the E_2 term of the Adams spectral sequence. The table of $H^{**}(A_3)$ which will be needed for this paper is given in Appendix. Relations involving a_0 and h_0 are indicated by vertical and slanting lines respectively. Since we have computed $H^{**}(A_3)$ by May's techniques, the products which we naturally obtained are actually the products according to the algebra structure of $E^0H^{**}(A_3)$. The product in $H^{**}(A_3)$ of

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two elements always contains as a summand their product in $E^0H^{**}(A_3)$ but may possibly contain also other terms of the same bi-grading (s,t) but of lower weight in the sense of J. P. May [5]. The relations holding in $E^0H^{**}(A_3)$ which cannot be listed for reasons of space are easily obtained from [9, Theorem 3.3 and 4.3].

The following relations in $H^{**}(A_3)$ differ from ones in $E^0H^{**}(A_3)$. This list is by no means complete.

PROPOSITION 1.1. *We have the following relations in* $H^{**}(A_3)$.

a). $u\cdot h_2 = -a_0^2c$ (58 *stem*)

b). $h_0\cdot h_1 b_{02} = -b_{11}k$ (60 *stem*)

c). $h_1\cdot h_1 b_{02} = b_{11}^2$ (68 *stem*)

d). $h_1\cdot h_2 b_{02} = b_{01}d - cb_{11}$ (92 *stem*)

$h_2\cdot h_1 b_{02} = b_{01}d - cb_{11}$

e). $b_{02}h_2\cdot g_2 = b_{01}v_0$ (99 *stem*)

f). $h_2\cdot h_0 u b_{02} = -h_0 b_{01} G$ (107 *stem*)

PROOF. We first consider the relation b). Let $\overline{b_{02}}$ be a cochain

$$[\xi_2 | \xi_2^2] + [\xi_2^2 | \xi_2] + [\xi_1^3 | \xi_1\xi_2^2] + [\xi_1^3\xi_2^2 | \xi_1]$$
$$- [\xi_1^3\xi_2 | \xi_1\xi_2] + [\xi_1^6 | \xi_1^2\xi_2] + [\xi_1^6\xi_2 | \xi_1^2]$$

in the cobar construction $F^*(A_3^*)$. Let $\overline{b_{01}}$ and $\overline{b_{11}}$ be cocycles

$$[\xi_1 | \xi_1^2] + [\xi_1^2 | \xi_1] \text{ and } [\xi_1^3 | \xi_1^6] + [\xi_1^6 | \xi_1^3]$$

in $F^*(A_3^*)$ which represent the elements $b_{01} \in H^{2,12}(A_3)$ and $b_{11} \in H^{2,36}(A_3)$, respectively. By routine calculations, we see that $b_{02}h_1 \in H^{3,60}(A_3)$ is represented by a cocycle $\overline{b_{02}h_1}$ in $F^*(A_3^*)$:

$$\overline{b_{02}h_1} = \overline{b_{02}}\cdot[\xi_1^3] - \overline{b_{11}}\cdot[\xi_1^6] - [\xi_1^9 | \xi_1 | \xi_1^5]$$
$$-[\xi_1^9 | \xi_1^2 | \xi_1^4] - [\xi_1^9 | \xi_1^4 | \xi_1^2] - [\xi_1^9 | \xi_1^5 | \xi_1] - [\xi_2^3]\cdot\overline{b_{01}}.$$

By routine calculations, we have that

$$\delta\ \{ - \overline{b_{02}}\cdot[\xi_2] + [\xi_1^9 | \xi_1^2 | \xi_1\xi_2] + [\xi_1^9 | \xi_1 | \xi_1^2\xi_2] + [\xi_1^9 | \xi_1\xi_2 | \xi_1^2]$$
$$+ [\xi_1^9 | \xi_1^2\xi_2 | \xi_1] + [\xi_1^9 | \xi_1^5 | \xi_1^2] - [\xi_2^3 | \xi_1^2 | \xi_1^2] + [\xi_3 | \xi_1^2 | \xi_1]$$
$$+ [\xi_3 | \xi_1 | \xi_1^2] + [\xi_1^9 | \xi_1^4 | \xi_1^3] + [\xi_1^9 | \xi_1^3 | \xi_1^4]\}$$
$$= \overline{b_{02}h_1}\cdot[\xi_1] - \overline{b_{11}}\cdot([\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1]).$$

Since it is easy to see that the cocycle $[\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1]$ represents the element $k \in H^{2,28}(A_3)$, we have the relation b).

Next we consider the relation c). By tedious but routine calculations, we have

$$\begin{aligned}
\delta \{ & \overline{b_{02}} \cdot [\xi_1^6] - [\xi_1^9 | \xi_1^2 | \xi_1^7] - [\xi_1^9 | \xi_1 | \xi_1^8] + [\xi_1^9 | \xi_1^5 | \xi_1^4] \\
& + [\xi_1^9 | \xi_1^4 | \xi_1^5] - [\xi_1^9 | \xi_1^8 | \xi_1] - [\xi_1^9 | \xi_1^7 | \xi_1^2] + [\xi_2^3 | \xi_1^2 | \xi_1^4] \\
& + [\xi_2^3 | \xi_1 | \xi_1^5] + [\xi_2^3 | \xi_1^5 | \xi_1] + [\xi_2^3 | \xi_1^4 | \xi_1^2] - [\xi_3 | \xi_2 | \xi_1] \\
& + [\xi_3 | \xi_1^3 | \xi_1^2] - [\xi_2^3 | \xi_1 \xi_2 | \xi_1] + [\xi_2^3 | \xi_2 | \xi_1^2] + [\xi_2^3 | \xi_1^4 | \xi_1^2] \\
& + [\xi_1^9 | \xi_2^2 | \xi_1] + [\xi_1^9 | \xi_1^3 \xi_2 | \xi_1^2] \} \\
= {} & \overline{b_{02} h_1} \cdot [\xi_1^3] + \overline{b_{11}} \cdot \overline{b_{11}} \cdot
\end{aligned}$$

Then we have the relation c).

Next we consider the relation d). By routine calculations, we see that $h_2 b_{02} \in H^{3,84}(A_3)$ is represented by a cocycle $\overline{h_2 b_{02}}$ in $F^*(A_3^*)$:

$$\overline{h_2 b_{02}} = [\xi_1^9] \cdot \overline{b_{02}} - [\xi_1^{18}] \cdot \overline{b_{01}} - [\xi_2^3] \cdot \overline{b_{11}} + [\xi_1^9 | \xi_1^6 | \xi_1^6].$$

By routine calculations, we have

$$\begin{aligned}
\delta (& - [\xi_1^{12}] \cdot \overline{b_{02}} + [\xi_2^3] \cdot \overline{b_{02}} + [\xi_1^9 \xi_2^3] \cdot \overline{b_{01}} + [\xi_1^{21}] \cdot \overline{b_{01}} \\
& + [\xi_1^3 \xi_2^3] \cdot \overline{b_{11}} - [\xi_1^{12} | \xi_1^6 | \xi_1^6] + [\xi_2^3 | \xi_1^6 | \xi_1^6]) \\
= {} & [\xi_1^3] \cdot \overline{h_2 b_{02}} - ([\xi_1^9 | \xi_2^3] - [\xi_1^{18} | \xi_1^3]) \cdot \overline{b_{01}} \\
& + ([\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]) \cdot \overline{b_{11}} \cdot
\end{aligned}$$

Since it is easy to see that the cocycles $[\xi_1^9 | \xi_2^3] - [\xi_1^{18} | \xi_1^3]$ and $[\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]$ represent the elements $d \in H^{2,84}(A_3)$ and $c \in H^{2,60}(A_3)$, respectively, we have the relation $h_1 \cdot h_2 b_{02} = b_{01} d - c b_{11}$. Similarly, we have

$$\begin{aligned}
\delta (& - \overline{b_{02}} \cdot [\xi_1^{12}] + \overline{b_{02}} \cdot [\xi_2^3] + \overline{b_{11}} \cdot [\xi_1^{15}] + \overline{b_{11}} \cdot [\xi_1^3 \xi_2^3] \\
& - [\xi_1^9 | \xi_1^2 | \xi_1 \xi_2^3] - [\xi_1^9 | \xi_1 | \xi_1^2 \xi_2^3] - [\xi_1^9 | \xi_1^2 \xi_2^3 | \xi_1] \\
& - [\xi_1^9 | \xi_1 \xi_2^3 | \xi_1^2] + [\xi_2^3 | \xi_1^2 | \xi_1^{10}] + [\xi_2^3 | \xi_1 | \xi_1^{11}] \\
& + [\xi_2^3 | \xi_1^{11} | \xi_1] + [\xi_2^3 | \xi_1^{10} | \xi_1^2] + [\xi_1^9 \xi_2^3] \cdot \overline{b_{01}} \\
& + [\xi_1^9 | \xi_1 | \xi_1^{14}] + [\xi_1^9 | \xi_1^2 | \xi_1^{13}] + [\xi_1^9 | \xi_1^{14} | \xi_1] \\
& + [\xi_1^9 | \xi_1^{13} | \xi_1^2] + [\xi_1^9 | \xi_1^4 | \xi_1^{11}] + [\xi_1^9 | \xi_1^5 | \xi_1^{10}]) \\
= {} & \overline{b_{02} h_1} \cdot [\xi_1^9] - \overline{b_{11}} \cdot ([\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]) \\
& + ([\xi_1^9 | \xi_2^3] - [\xi_1^{18} | \xi_1^3]) \cdot \overline{b_{01}} \cdot
\end{aligned}$$

Then we have the relation $b_{02} h_1 \cdot h_2 = b_{11} c - d b_{01}$.

Next we consider the relation e). Let α, β and γ be cochains

$$[\xi_1^2 | \xi_1^{10}] + [\xi_1 | \xi_1^{11}] + [\xi_1^{10} | \xi_1^2] + [\xi_1^{11} | \xi_1],$$

$$[\xi_1^3 \xi_3 | \xi_1] - [\xi_1^3 \xi_2^3 | \xi_1^2] + [\xi_3 | \xi_2] - [\xi_1^9 | \xi_2^2] + [\xi_2^3 | \xi_1 \xi_2]$$

and

$$- [\xi_1^2 | \xi_1^{19}] - [\xi_1 | \xi_1^{20}] + [\xi_1^{11} | \xi_1^{10}] + [\xi_1^{10} | \xi_1^{11}] - [\xi_1^{19} | \xi_1^2]$$

$- [\xi_1^{20} | \xi_1]$,

respectively. Then $b_{02}h_2 \in H^{3,84}(A_3)$ is represented by a cocycle $\overline{b_{02}h_2}$ in $F^*(A_3^*)$:

$$\overline{b_{02}h_2} = \overline{b_{02}} \cdot [\xi_1^9] + \overline{b_{11}} \cdot [\xi_1^{12}] - \overline{b_{11}} \cdot [\xi_2^3] + [\xi_1^{18}] \cdot \overline{b_{01}} - [\xi_1^9] \cdot \alpha .$$

By routine calculations, we have

$$\begin{aligned} &\delta \{ \overline{b_{02}} \cdot (- [\xi_3 | \xi_1] + [\xi_2^3 | \xi_1^2]) - \overline{b_{11}} \cdot \beta \\ &\qquad - \alpha \cdot (- [\xi_3 | \xi_1] + [\xi_2^3 | \xi_1^2]) + \gamma \cdot ([\xi_2 | \xi_1] - [\xi_1^3 | \xi_1^2]) \} \\ &= \overline{b_{02}h_2} \cdot ([\xi_2 | \xi_1] - [\xi_1^3 | \xi_1^2]) \\ &- \overline{b_{01}} \cdot (- [\xi_1^9 | \xi_3 | \xi_1] + [\xi_1^9 | \xi_2^3 | \xi_1^2] + [\xi_1^{18} | \xi_2 | \xi_1] \\ &- [\xi_1^{18} | \xi_1^3 | \xi_1^2]). \end{aligned}$$

It is easy to see that

$$[\xi_2 | \xi_1] - [\xi_1^3 | \xi_1^2]$$

and

$$- [\xi_1^9 | \xi_3 | \xi_1] + [\xi_1^9 | \xi_2^3 | \xi_1^2] + [\xi_1^{18} | \xi_2 | \xi_1] - [\xi_1^{18} | \xi_1^3 | \xi_1^2]$$

are representatives of $g_2 \in H^{2,20}(A_3)$ and $v_0 \in H^{3,92}(A_3)$, respectively. Then we have the relation e).

Next we consider the relation a). It is easy to see that $u \in H^{3,26}(A_3)$ is represented by a cocycle $\bar{u}$ in $F^*(A_3^*)$:

$$\begin{aligned} \bar{u} = {} & [\tau_2 | \xi_1 | \tau_1] - [\xi_2 | \tau_1 | \tau_1] + [\xi_2 | \tau_1 | \xi_1\tau_0] \\ & - [\xi_2 | \tau_1\tau_0 | \xi_1] - [\xi_2 | \tau_0 | \xi_1\tau_1] + [\xi_2 | \xi_1\tau_0 | \tau_1] \\ & - [\tau_2 | \xi_1^2 | \tau_0] - [\xi_2 | \xi_1\tau_0 | \xi_1\tau_0] - [\xi_1^3 | \xi_1\tau_1 | \tau_1] \\ & + [\xi_1^3 | \xi_1 | \xi_1\tau_1\tau_0] - [\xi_1^3 | \xi_1^2 | \tau_1\tau_0] + [\xi_1^3 | \xi_1^2\tau_1 | \tau_0] \\ & - [\xi_1^6 | \tau_0 | \tau_0]. \end{aligned}$$

Let ρ be a cochain

$$\begin{aligned} & - [\tau_2 | \xi_1 | \xi_1^9\tau_1] - [\tau_2 | \xi_1^{10} | \tau_1] - [\xi_1^9\tau_2 | \xi_1 | \tau_1] \\ & + [\tau_3 | \xi_1 | \tau_1] + [\xi_2 | \tau_1 | \xi_1^9\tau_1] + [\xi_2 | \xi_1^9\tau_1 | \tau_1] \\ & + [\xi_1^9\xi_2 | \tau_1 | \tau_1] - [\xi_3 | \tau_1 | \tau_1] - [\tau_3 | \xi_1^2 | \tau_0] \\ & - [\xi_3 | \xi_1\tau_0 | \tau_1] + [\xi_3 | \xi_1 | \tau_1\tau_0] - [\xi_3 | \xi_1\tau_1 | \tau_0] \\ & + [\xi_3 | \xi_1^2\tau_0 | \tau_0] - [\xi_2^3 | \xi_1\tau_1 | \tau_1] + [\xi_2^3 | \xi_1^2\tau_1 | \tau_0] \\ & - [\xi_2^3 | \xi_1^2 | \tau_1\tau_0] + [\xi_1^9\tau_2 | \xi_1^2 | \tau_0] + [\tau_2 | \xi_1^{11} | \tau_0] \\ & + [\tau_2 | \xi_1^2 | \xi_1^9\tau_0] - [\xi_1^9\xi_2 | \xi_1 | \tau_1\tau_0] + [\xi_1^9\xi_2 | \xi_1\tau_0 | \tau_1] \\ & + [\xi_1^9\xi_2 | \xi_1\tau_1 | \tau_0] - [\xi_2 | \xi_1^{10} | \tau_1\tau_0] + [\xi_2 | \xi_1^{10}\tau_0 | \tau_1] \\ & + [\xi_2 | \xi_1^{10}\tau_1 | \tau_0] - [\xi_2 | \xi_1 | \xi_1^9\tau_1\tau_0] + [\xi_3 | \xi_1\tau_0 | \xi_1^9\tau_1] \\ & + [\xi_2 | \xi_1\tau_1 | \xi_1^9\tau_0] - [\xi_2 | \xi_1\tau_1\tau_0 | \xi_1^9] - [\xi_1^9\xi_2 | \xi_1^2\tau_0 | \tau_0] \end{aligned}$$

$$- [\xi_2 | \xi_1^{11}\tau_0 | \tau_0] - [\xi_2 | \xi_1^2\tau_0 | \xi_1^9\tau_0] + [\xi_1^{12} | \xi_1\tau_1 | \tau_1]$$
$$+ [\xi_1^3 | \xi_1^{10}\tau_1 | \tau_1] + [\xi_1^3 | \xi_1\tau_1 | \xi_1^9\tau_1] - [\xi_1^{12} | \xi_1^2\tau_1 | \tau_0]$$
$$+ [\xi_1^{12} | \xi_1^2 | \tau_1\tau_0] - [\xi_1^3 | \xi_1^{11}\tau_1 | \tau_0] + [\xi_1^3 | \xi_1^{11} | \tau_1\tau_0]$$
$$+ [\xi_1^3 | \xi_1^2 | \xi_1^9\tau_1\tau_0] - [\xi_1^3 | \xi_1^2\tau_1 | \xi_1^9\tau_0] + [\xi_1^{15} | \tau_0 | \tau_0]$$
$$+ [\xi_1^6 | \xi_1^9\tau_0 | \tau_0] + [\xi_1^6 | \tau_0 | \xi_1^9\tau_0] .$$

By tedious but routine calculations, we have

$$\delta(\rho) = \bar{u}.[\xi_1^9] + ([\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]).[\tau_0 | \tau_0].$$

Then we have the relation a).

Last we consider the relation e). Let ρ be the cochain defined in proof of a) and μ be $[\tau_0 | \xi_1^2\tau_0 | \xi_1] - [\tau_0 | \xi_1^2 | \xi_1\tau_0] + [\tau_0 | \xi_1\tau_0 | \xi_1^2]$

$$- [\tau_0 | \xi_1 | \xi_1^2\tau_0] + [\tau_0 | \xi_1^2 | \tau_1] - [\xi_1\tau_0 | \xi_1 | \tau_1]$$
$$+ [\xi_1\tau_0 | \xi_1^2 | \tau_0] + [\tau_1 | \xi_1 | \tau_1] - [\tau_1 | \xi_1^2 | \tau_0] .$$

Then, by the tedious but routine calculations, we see that $ub_{02}h_0 \in H^{6,78}(A_3)$ is represented by a cocycle $\overline{ub_{02}h_0}$ in $F^*(A_3^*)$:

$$\overline{ub_{02}h_0} = u \cdot \overline{b_{02}} \cdot [\xi_1] + u \cdot \overline{b_{11}} \cdot [\xi_2] - \rho \cdot \overline{b_{01}} \cdot [\xi_1]$$
$$+ ([\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]) \cdot \mu \cdot [\xi_1] .$$

Let ν and $\overline{G}$ be

$$- [\tau_3 | \xi_1 | \tau_1] + [\xi_3 | \tau_1 | \tau_1] + [\tau_3 | \xi_1^2 | \tau_0]$$
$$+ [\xi_3 | \xi_1\tau_0 | \tau_1] - [\xi_3 | \xi_1 | \tau_1\tau_0] + [\xi_3 | \xi_1\tau_1 | \tau_0]$$
$$- [\xi_3 | \xi_1^2\tau_0 | \tau_0] + [\xi_2^3 | \xi_1\tau_1 | \tau_1] - [\xi_2^3 | \xi_1^2\tau_1 | \tau_0]$$
$$+ [\xi_2^3 | \xi_1^2 | \tau_1\tau_0] + [\xi_1^9 | \xi_2 | \xi_1\tau_1\tau_0]$$

and

$- [\xi_1^9] \cdot \rho - \nu \cdot [\xi_1^9]$, respectively. Let $\bar{c}$ be the representative $[\xi_2^3 | \xi_1^3] - [\xi_1^9 | \xi_1^6]$ of $c \in H^{2,60}(A_3)$. Since $\delta([\xi_1^9] \cdot \rho + \nu \cdot [\xi_1^9]) = - [\xi_1^9] \cdot \bar{c} \cdot [\tau_0 | \tau_0] - \bar{c} \cdot [\tau_0 | \tau_0 | \xi_1^9]$ in $F^*(A_3^*)$ and $\delta_3(G) = - a_0^2 h_2 c$ in the May spectral sequence, the cochain $\overline{G}$ is a representative of G in some sense. It is easy to see that

$$\delta(\nu \cdot \overline{b_{02}} \cdot [\xi_1] + \nu \cdot \overline{b_{11}} \cdot [\xi_2] + \tau \cdot [\tau_0 | \tau_0 | \xi_2 | \xi_1\xi_2^2]$$
$$+ \bar{c} \cdot [\tau_0 | \tau_0 | \xi_1\xi_2 | \xi_2^2] + \bar{c} \cdot [\tau_0 | \tau_0 | \xi_2^2 | \xi_1\xi_2]$$
$$+ \bar{c} \cdot [\tau_0 | \tau_0 | \xi_1\xi_2^2 | \xi_2] + \bar{c} \cdot [\tau_0 | \xi_1\tau_0 | \xi_2 | \xi_2^2]$$
$$+ \bar{c} \cdot [\tau_0 | \xi_1\tau_0 | \xi_2^2 | \xi_2] - \bar{c} \cdot [\tau_0 | \tau_1] \cdot \overline{b_{02}})$$

$\equiv [\xi_1^9] \cdot \overline{ub_{02}h_0} - G \cdot \overline{b_{01}} \cdot [\xi_1]$ modulo terms which have the May's weight (the weight associated to May spectral sequence) less than 5 or have the May's weight 5 and the weight, associated to the spectral sequence defined in [9] , greater than 1 . For the dimensional and filtrational reasons, we have the relation

$$h_2 \cdot ub_{02}h_0 = Gb_{01}h_0 + \varepsilon\ a_0^6 h_3 \text{ , where } \varepsilon \in Z_3 \cdot$$

Since $d_2(h_2) = a_0 b_{11}$ (Theorem 2.1.) and $d_2(h_0 w) = h_0 ub_{02}$ (Theorem 2. 4.) in the mod 3 Adams spectral sequence, we have $d_2(h_2 h_0 w) = h_2 \cdot h_0 ub_{02} = - h_0 b_{01} G - \varepsilon \cdot a_0^6 h_3$, up to sign. Then we have a non-zero differential $d_2(h_2 w)$. By the dimensional considerations, we have $d_2(h_2 w) = b_{01} G$, up to sign. Then we have $\varepsilon = 0$ and therefore we have the relation f).

§ 2. Some known results on the mod 3 Adms spectral sequence

From now on we neglect the non-zero coefficient of the differentials and the relations in the mod 3 Adams spectral sequence.

The following five theorems for the differentials and elements surviving to E_∞ were verified by using the results of the stable homotopy groups of sphere and the statement that E_r is a differential algebra by J. P. May [5] (in the range $t-s \leqq 32$ and partially in the range $t-s \geqq 33$) and by S. Oka the rest.

THEOREM 2. 1. (H. H. Gershenson [2] . A. Liulevicius [4]. R.J. Milgram [7] . N. Shimada and T. Yamanoshita [11] . H. Toda [12])

$d_2(h_i) = a_0 b_{i-1\,1}$ for $i \geqq 1$.

THEOREM 2. 2. (H.Toda [12,13,14,15])

a). $d_2(g_2) = g_1 b_{01}$ ([12])

b). $d_2(u) = a_1 b_{01}$ ([12])

c). $d_5(b_{11}) = h_0 b_{01}^3$ ([14])

d). $d_5(h_1 b_{02}) = b_{01}^3 k$ ([15])

e). $d_6(e_1) = b_{01}^6$ ([13]).

THEOREM 2. 3. (J.P.May [5])

a). $d_3(a_0^4 h_2) = b_{01} a_1^2$

b). $d_2(a_1^j u) = a_1^{j+1} b_{01}$ for $0 \leqq j$, $j \not\equiv 1 \mod 3$.

THEOREM 2. 4. (S.Oka [10])

a). $d_2(c) = a_0 h_1 b_{02}$

b). $d_3(a_0^2 c) = h_0 b_{01}^2 b_{11}$

c). $d_2(g_2a_2) = e_2$

d). $d_2(f) = h_0b^3_{01}b_{11}$

e). $d_3(a^8_0a_2u) = b_{01}a^5_1$

f). $d_2(a_2u) = \ell$

g). $d_2(h_0w) = h_0ub_{02}$

h). $d_2(g_1w) = g_1ub_{02}$

REMARK. All other non-zero differentials in the range $t-s \leqq 77$ are easily determined by the differentials listed in the above four theorems and the statement that E_r is a differential algebra.

THEOREM 2. 6. (J. P. May [5] and S. Oka) *E_∞ for $t-s \leqq 76$ (and corresponding generators of $\pi^S_*(S^0:3)$) are given in Table A.*

Table A.

t-s	$\pi^S_*(S^0:3)$	survivor (corresponding generator)
0	Z	a^i_0 (ι)
1	0	
2	0	
3	Z_3	h_0 (α_1)
4	0	
5	0	
6	0	
7	Z_3	g_1 (α_2)
8	0	
9	0	
10	Z_3	b (β_1)
11	Z_9	a_0h_1 , $a^2_0h_1$ (α'_3)
12	0	
13	Z_3	h_0b ($\alpha_1\beta_1$)
14	0	
15	Z_3	h_0a_1 (α_4)
16	0	
17	0	
18	0	
19	Z_3	g_1a_1 (α_5)
20	Z_3	b^2 (β^2_1)
21	0	
22	0	
23	Z_3+Z_9	h_0b^2 ($\alpha_1\beta^2_1$), a^2_0u, a^3_0u (α'_6)
24	0	
25	0	

Table A. (continued)

t-s	$\pi_*^S(S^0:3)$	survivor (corresponding generator)
26	Z_3	$k\ (\beta_2)$
27	Z_3	$h_0a_1^2\ (\alpha_7)$
28	0	
29	Z_3	$h_0k\ (\alpha_1\beta_2)$
30	Z_3	$b^3\ (\beta_1^3)$
31	Z_3	$g_1a_1^2\ (\alpha_8)$
32	0	
33	0	
34	0	
35	Z_{27}	$a_0^6h_2$, $a_0^7h_2$, $a_0^8h_2\ (\alpha'_9)$
36	Z_3	$bk\ (\beta_1\beta_2)$
37	Z_3	$h_0b_{11}\ (\varepsilon')$
38	Z_3	$h_0h_2\ (\varepsilon_1)$
39	Z_3+Z_3	$h_0bk\ (\alpha_1\beta_1\beta_2)$, $h_0a_1^3\ (\alpha_{10})$
40	Z_3	$b^4\ (\beta_1^4)$
41	0	
42	Z_3	$g_1h_2\ (\varepsilon_2)$
43	Z_3	$g_1a_1^3\ (\alpha_{11})$
44	0	
45	Z_9	bh_2, $a_0bh_2\ (\varphi)$
46	Z_3	$b^2k\ (\beta_1^2\beta_2)$
47	Z_3+Z_9	$h_0bb_{11}\ (\beta_1\varepsilon')$, $a_0^2a_1^2u$, $a_0^3a_1^2u\ (\alpha'_{12})$
48	0	
49	Z_3	$h_0b^2k\ (\alpha_1\beta_1^2\beta_2)$
50	Z_3	$b^5\ (\beta_1^5)$
51	Z_3	$h_0a_1^4\ (\alpha_{13})$
52	Z_3	$k^2\ (\beta_2^2)$
53	0	
54	0	
55	Z_3+Z_3	$h_0k^2\ (\alpha_1\beta_2^2)$, $g_1a_1^4\ (\alpha_{14})$
56	0	
57	0	
58	0	

Table A' (continued)

t-s	$\pi_*^S(S^0:3)$	survivor (corresponding generator)
59	Z_9	$a_0^2a_1^3u$, $a_0^3a_1^3u$ (α'_{15})
60	0	
61	0	
62	Z_3	bk^2 ($\beta_1\beta_2^2$)
63	Z_3	$h_0a_1^5$ (α_{16})
64	0	
65	Z_3	h_0bk^2 ($\alpha_1\beta_1\beta_2^2$)
66	0	
67	Z_3	$g_1a_1^5$ (α_{17})
68	Z_3	b_{11}^2 (λ)
69	0	
70	0	
71	Z_{27}	$a_0^{10}a_2u$, $a_0^{11}a_2u$, $a_0^{12}a_2u$ (α''_{18})
72	Z_3	b^2k^2 ($\beta_1^2\beta_2^2$)
73	0	
74	Z_3	ka_2 (β_5)
75	$Z_3+Z_3+Z_3$	h_0bk_{02} (μ), $h_0b^2k^2$ ($\alpha_1\beta_1^2\beta_2^2$), $h_0a_1^6$ (α_{19})
76	0	

The matric Massey products in algebraic spectral sequences were studied by J.P.May [6]. We quote some of his results which we use in this paper.

THEOREM 2. 6. (J.P.May). *Let* $<v^1, v^2, v^3>$ *be defined in* E_{r+1} *term of the May spectral sequence. Assume that* $v^i \in E_{r+1}^{p_i, q_i, t_i}$ *and that* v^i *converges to* w^i, *where* $<w^1, w^2, w^3>$ *is defined in* $H^{**}(A_3)$. *Assume further that the following condition (*) is satisfied.*

(*) *If* $(p,q,t) = (p_{i+1}+p_i,\ q_{i+1}+q_i,\ t_{i+1}+t_i)$, $i=1$ *or* 2, *then*

$$E_{r+u+1}^{p-r-u\ \ q+r+u-1,\ t} \subset E_{r+u+1,\ \infty} \quad \text{for } u \geqq 0.$$

Then any element of $<v^1, v^2, v^3>$ *is a permanent cocycle which converges to an element of* $<w^1, w^2, w^3>$.

The matric Massey products in the Adams spectral sequence were studied by R.M.F.Moss [8] and A.F.Lawrence [3]. We quote some of their results which we use in this paper.

THEOREM 2. 7. (R.M.F.Moss).