Some Differentials in the mod 3 Adams Sepectral Sequence

i). *Let* $v_i \in E_r^{s_i, t_i} = E_r^{s_i, t_i}(S^0, S^0)$ $(i=1,2,3)$ *be such that* $v_1 v_2 = 0$ *and* $v_2 v_3 = 0$. *Then*

$$d_r <v_1, v_2, v_3> \subset -<(d_r v_1\ v_1), \begin{pmatrix} v_2 & 0 \\ (-1)^i d_r v_2 & v_3 \end{pmatrix}, \begin{pmatrix} v_3 \\ (-1)^{i+i'} d_r v_3 \end{pmatrix}>.$$

where $i = t_1 - s_1$ *and* $i' = t_2 - s_2$.

ii). *Suppose also that* $v_1 d_r v_2 = 0$ *and* $v_2 d_r v_3 = 0$, *then*

$$d_r < v_1, v_2, v_3 > \subset -<d_r v_1, v_2, v_3> - (-1)^i <v_1, d_r v_2, v_3> - (-1)^{i+i'} <v_1, v_2, d_r v_3>.$$

THEOREM 2. 8. (R. M.F.Moss) *Let* $v_i \in E_r^{s_i, t_i}$ $(i=1,2,3)$ *be permanent cycles such that* $v_1 v_2 = 0$ *and* $v_2 v_3 = 0$. *Let* v_i *be realized in* E_∞ *by homotopy classes* $\omega_i \in \{S^0, S^0\}_{t_i - s_i}$ *such that,* $\omega_1 \omega_2 = 0$ *and* $\omega_2 \omega_3 = 0$. *Assume that the following condition* (**) *is satisfied.*

(**) $E_{r+u+1}^{s_i+s_{i+1}-r-u, t_i+t_{i+1}-r-u+1} \subset E_{r+u+1,\infty}$ *for any* $u \geqq 0$ *and* $i=1,2$.

Then the Massey product $<v_1, v_2, v_3>$ *contains a permanent cycle that is realized in* $E^{s_1+s_2+s_3-r+1, t_1+t_2+t_3-r+2}$ *by an element of the Toda bracket* $<\omega_1, \omega_2, \omega_3>$.

THEOREM 2. 9. (A.F.Lawrence). *Let* $<v^1, v^2, v^3>$ *be a Massey product, where* $v^i \in E_r^{s_i, t_i} = E_r^{s_i, t_i}(S^0, S^0)$ $(i=1,2,3)$ *and* $d_t v^i = 0$, $r-1 \leqq t < n$. *Assume that*

(***) $E_m^{s_i+s_{i+1}-r+m+1, t_i+t_{i+1}-r+m+1} = E_{r+n-m-1}^{s_i+s_{i+1}-r+m+1, t_i+t_{i+1}-r+m+1} = 0$

for $r-1 < m < n$ *and* $1 \leqq i \leqq 2$.

Then for $v \in <v^1, v^2, v^3>$ *we have* $d_m v = 0$, $m < n$, *and*

$$(d_n v \cdot \bar{v}) \in <(d_n v^1\ \bar{v}^1), \begin{pmatrix} v^2 & 0 \\ d_n v^2 & \bar{v}^2 \end{pmatrix}, \begin{pmatrix} v^3 & 0 \\ d_n v^3 & \bar{v}^3 \end{pmatrix} > .$$

The relation between the algebraic Steenrod powers acting $Ext_{A_p}(Z_p, Z_p)$ and the differentials in the Adams spectral sequence was studied by R.J.Milgram [7].

THEOREM 2. 10. (R.J.Milgram) *Let* $a \in Ext_{A_p}(Z_p, Z_p)$ *then there are operations* $\mathcal{P}^i$, $\beta\mathcal{P}^i$, *in* $Ext_{A_p}^{**}(Z_p, Z_p)$ *and* $d_2(\mathcal{P}^i(a)) = a_0 \cdot \beta\mathcal{P}^i(a)$ *for* p *odd prime.*

§ 3. Differentals in the range 78 ≦ t-s ≦ 104

In this section we will consider the differentils in the range $78 \leqq t\text{-}s \leqq 104$. From now on we will write $(\tilde{E}_r, \delta_r)$ and (E_r, d_r) for the May spectral sequence and the Adams', respectively.

PROPOSITION 3. 1. $d_3(x) = h_0ka_2$.

PROOF . Since $kb_{01}h_0 = 0$ and $\delta_3(h_0b_{01}a_2) = \delta_3(h_0a_1b_{02}) = h_0b_{01}a_1h_2$ in $\tilde{E}_3$, Massey product $<k, b_{01}h_2, h_0a_1>$ is defined and equal to $h_0b_{01}a_2k$ in $\tilde{E}_4$. Since $k \cdot b_{01}h_2 = 0$ and $b_{01}h_3 . h_0a_1 = 0$ in $H^{**}(A_3)$, we have the relation $h_0b_{01}.a_2k = <k, b_{01}h_2, h_0a_1>$ in $H^{**}(A_3)$, by Theorem 2.6. k, $b_{01}h_2$ and h_0a_1 are permanent cycles in the Adams spectral sequence and converge to β_2, φ and α_4, respectively. Since $<\beta_2, \varepsilon_1, \alpha_1> \in \Pi^s_{68}(S^0:3) = \{\lambda\}$ and $\lambda\alpha_1 = 0$ (S. Oka [10]) and $\varphi \in <\varepsilon_1, \alpha_1, \alpha_1>$(H.Toda [14]), we have $\beta_2\varphi \in \beta_2<\varepsilon_1, \alpha_1, \alpha_1> = <\beta_2, \varepsilon_1, \alpha_1>\alpha_1 = 0$. Since $\Pi^s_{60}(S^0:3) = 0$, we have $\varphi\alpha_4 = 0$. Then Toda bracket $<\beta_2, \varphi, \alpha_4>$ is defined. It is easy to check that Massey product $<k, b_{01}h_2, h_0a_1>$ satisfies the condition (**) of Theorem 2.8. Then we have the relation $\alpha_1\beta_1\beta_5 = <\beta_2, \varphi, \alpha_4>$, since $h_0b_{01}ka_2$ converges to $\alpha_1\beta_1\beta_5$. Since $<\varphi, \alpha_4, \beta_1> \in \Pi^s_{71}(S^0:3)/(\varphi\Pi^s_*(S^0:3) + \beta_1\Pi^s_*(S^0:3))$, $\Pi^s_{71}(S^0:3) = \{\alpha''_{18}\}$ and $\beta_2\alpha''_{18} = 0$, we have $\alpha_1\beta_1^2\beta_5 = \beta_1<\beta_2, \varphi, \alpha_4> = \beta_2<\varphi, \alpha_4, \beta_1> = 0$, up to sign. Then $h_2b_{01}^2ka_2$ must be killed by some differetial. For dimensional reasons, there is only one possible differential $d_3(b_{01}^2x) = h_0b_{01}^2ka_2$. Then $d_r(x) \neq 0$ for some r such that $2 \leqq r \leqq 3$. There is only one possible differential $d_3(x) = h_0ka_2$.

PROPOSITION 3. 2. $d_2(m) = b_{01}ub_{02}$.

PROOF . Since $h_0a_1a_0 = 0$ and $\delta_3(w) = -a_0h_1c$ in $\tilde{E}_3$, Massey product $<h_0a_1, a_0, h_1c>$ is defined and equal to h_0a_1w in $\tilde{E}_4$. Since $h_0a_1.a_0 = 0$ and $a_0.h_1c = 0$ in $H^{**}(A_3)$, we have $h_0a_1w = <h_0a_1, a_0, h_1c>$ by Theorem 2. 6. By Theorem 2. 7, $d_2(h_0a_1w) = d_2<h_0a_1, a_0, h_1c> = <h_0a_1, a_0, d_2(h_1c)> = <h_0a_1, a_0, a_0b_{11}^2>$. Since $\delta_5(ub_{02}) = a_0^2b_{11}^2$ in $\tilde{E}_5$. Massey product $<h_0a_1, a_0, a_0b_{11}^2>$ is equal to $h_0a_1ub_{02}$ in $\tilde{E}_5$. By Theorem 2.6. it is easy to see that $h_0a_1ub_{02} = <h_0a_1, a_0, a_0b_{11}^2>$. Then $d_2(h_0m) = d_2(h_0a_1w) = h_0a_1ub_{02}$. For the dimensional reasons, $d_2(m) = a_1ub_{02}$.

PROPOSITION 3. 2. $d_2(d) = a_2h_2b_{02}$.

PROOF . We first consider the actions of $\mathcal{P}^0$ and $\beta\mathcal{P}^0$ to $k \in H^{2,28}(A_3)$. The definition of $\mathcal{P}^0$ and $\beta\mathcal{P}^0$ is given in [4]. By the direct calculation, we have

$$\Delta_4([\alpha_1 | \alpha_2] \otimes [\beta_1 | \beta_2] \otimes [\gamma_1 | \gamma_2])$$
$$= -[\alpha_1\beta_1\gamma_1 | \alpha_2\beta_2\gamma_2]$$

and

$$\Delta_3 ([\alpha_1 | \alpha_2] \otimes [\beta_1 | \beta_2] \otimes [\gamma_1 | \gamma_2])$$
$$= [\alpha_1\beta_1\gamma_1' | \alpha_2\beta_2\gamma_1'' | \gamma_2] + [\alpha_1'\beta_1\gamma_1' | \alpha_1''\beta_2\gamma_1'' | \alpha_2\gamma_2]$$
$$- [\alpha_1\gamma_1 | \alpha_2\beta_1\gamma_2' | \beta_2\gamma_2''] - [\alpha_1'\gamma_1 | \alpha_1''\beta_1\gamma_2' | \alpha_2\beta_2\gamma_2'']$$

where $\alpha_i, \beta_i, \gamma_i \in A_3^*$ $(i=1,2)$ and let $\varphi^*(\alpha) = \alpha' \otimes \alpha''$ instead of $\varphi^*(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$ for coproduct of α in A_3^*. It is easy to see that k in $H^{2,28}(A_3)$ is represented by the cocycle

$[\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1]$ in the cobar construction $F^*(A_3^*)$. Then $\mathcal{P}^0(k)$ is represented in $F^*(A_3^*)$ by

$$- \Delta_4 \{([\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1]) \otimes ([\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1])$$
$$\otimes ([\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1])\}$$
$$= [\xi_1^9 | \xi_2^3] - [\xi_1^{18} | \xi_1^3].$$ Since $[\xi_1^9 | \xi_2^3] - [\xi_1^{18} | \xi_1^3]$

is a representative of $d \in H^{2,84}(A_3)$, we have $\mathcal{P}^0(k) = d$. Next, $\beta\mathcal{P}^0(k)$ is represented in $F^*(A_3^*)$ by

$$\Delta_3 \{([\xi_1^3 | \xi^2] - [\xi_1^6 | \xi_1]) \otimes ([\xi_1^3 | \xi_2] - [\xi_1^9 | \xi])$$
$$\otimes ([\xi_1^3 | \xi_2] - [\xi_1^6 | \xi_1])$$
$$= [\xi_1^9] \cdot \overline{b_{02}} - [\xi_1^{18}] \cdot \overline{b_{01}} - [\xi_2^3] \cdot \overline{b_{11}} + [\xi_1^9 | \xi_1^6 | \xi_1^6]$$
$$+ \delta (- [\xi_1^6\xi_2^3 | \xi_1^3] - [\xi_1^3\xi_2^3 | \xi_1^6] - [\xi_1^3 | \xi_1^6\xi_2^3] - [\xi_1^6 | \xi_1^3\xi_2^3]$$
$$- [\xi_1^{12} | \xi_1\xi_2^2] + [\xi_1^{15} | \xi_1^2\xi_2] - [\xi_1^{15} | \xi_1^6] - [\xi_1^3 | \xi_1^{18}]).$$

Since $[\xi_1^9] \cdot \overline{b_{02}} - [\xi_1^{18}] \cdot \overline{b_{01}} - [\xi_2^3] \cdot \overline{b_{11}} + [\xi_1^9 | \xi_1^6 | \xi_1^6]$ is a represenative of $h_2b_{02} \in H^{3,84}(A_3)$, we have $\beta\mathcal{P}^0(k) = h_2b_{02}$. By Theorem 2.10, we have $d_2(d) = d_2(\mathcal{P}^0(k) = a_2 \cdot \beta\mathcal{P}^0(k) = a_0h_2b_{02}$.

PROPOSITION 3. 4. $d_2(v_0) = g_1h_2b_{02}$

PROOF. Since $h_0v_0 = - b_{01}d$, we have $h_0d_2(v_0) = d_2(b_{01}d) = h_0 \cdot g_1h_2b_{02}$. For the dimensional reasons, we have $d_2(v_0) = g_1h_2b_{02}$.

PROPOSITION 3. 5. $d_2(a_1^2w) = a_1^2ub_{02}$.

PROOF. Since $h_0d_2(a_1^2w) = d_2(h_0a_1 \cdot m) = h_0a_1^2ub_{02}$, we have $d_2(a_1^2w) = a_1^2ub_{02}$ for the dimensional reasons.

PROPOSITION 3. 6. a). $d_4(h_1x) = b_{01}^2 b_{11}^2$.

b). $d_4(g_2x) = h_0b_{01}^2kb_{02}$.

PROOF. Since h_0kb_{02} in $H^{5,80}(A_3)$ is a permanent cycle and $d_6(e_1) = b_{01}^6$, $h_0b_{01}^6kb_{02}$ must be killed by some differential. It is easy to see that $d_2(b_{01}^2$ (乙)) $= d_2(f^2) = h_0b_{01}^5b_{11}a_2$, $d_2(g_1ua_2$ (甲)) $= g_1a_1$ (乙) and $b_{01}^2b_{11}^2a_2 = - b_{01}k^3a_2$ is a permanent cycle. Then, for the dimensional reasons, there is only one possible differential $d_4(b_{01}^4 g_2 x) = h_0 b_{01}^6 k b_{02}$ and therefore, we have $d_r(g_2 x) \neq 0$ for $r=3$ or 4. Let $\bar{x}$ be a cocycle in $F^*(A_3^*)$ which represents $x \in H^{3,78}(A_3)$. Then $[\xi_1^3] \cdot \overline{b_{01}} \cdot \bar{x}$ and $([\xi_2 | \xi_1] - [\xi_1^3 | \xi_1^2]) \cdot \overline{b_{01}} \cdot \bar{x}$ are representatives of $h_1 b_{01} x \in H^{6,105}(A_3)$ and$g_2 b_{01} x \in H^{7,113}(A_3)$, respectively. Since $\delta [\xi_1^2] = [\xi_1 | \xi_1]$,

$\delta([\xi_2] - [\xi_1^4]) \cdot \overline{b_{01}} \cdot \bar{x} = [\xi_1 | \xi_1^3] \cdot \overline{b_{01}} \cdot \bar{x}$ and

$([\xi_1^2 | \xi_1^3] + [\xi_1 | \xi_2] - [\xi_1 | \xi_1^4]) \cdot \overline{b_{01}} \cdot \bar{x}$

$= -([\xi_2 | \xi_1] - [\xi_1^3 | \xi_1^2]) \cdot \overline{b_{01}} \cdot \bar{x} - \delta [\xi_1 \xi_2] + [\xi_1^5]) \cdot \overline{b_{01}} \cdot \bar{x}$,

Massey product $\langle h_0, h_0, h_1 b_{01} x \rangle$ is defined and equal to $- g_2 b_{01} x$ in E_2. Since $d_3(h_0) = 0$ and $d_3(h_1 x) = 0$ for the dimensional reasons, we have $d_3(g_2 b_{01} x) = 0$ by Theorem 2.9. For dimensional reasons, we have $d_3(g_2 x) = 0$. Then we have $d_4(g_2 x) = h_0 b_{01}^2 k b_{02}$ and therefore we have $d_4(h_1 x) = b_{01}^2 b_{11}^2$ for dimensional reasons.

PROPOSITION 3. 7. $d_2(g_1 G) = g_1 a_1 h_2 b_{02}$.

PROOF. Since $\delta_3(a_1) = - a_0^3 h_1$ and $a_0 v_0 = 0$ in $\tilde{E}_3$, Massey product $\langle v_0, a_0, a_0^2 h_1 \rangle$ is defined and equal to $g_1 G = a_1 v_0$ in $\tilde{E}_4$. Since the May's weight of $a_0 v_0$ and $h_1 x$ are 2 and 4, respectively, we have a relation $a_0 v_0 = 0$ in $H^{**}(A_3)$. Then it is easy to see that $g_1 G = \langle v_0, a_0, a_0^2 h_1 \rangle$ in $H^{**}(A_3)$ by Theorem 2.6. By Theorem 2.7, we have $d_2(g_1 G) = d_2(\langle v_0, a_0, a_0^2 h_1 \rangle) = \langle g_1 h_2 b_{02}, a_0, a_0^2 h_1 \rangle$. It is easy to see that $g_1 a_1 h_2 b_{02} = \langle g_1 h_2 b_{02}, a_0, a_0^2 h_1 \rangle$. Then we have $d_2(g_1 G) = g_1 a_1 h_2 b_{02}$.

PROPOSITION 3. 8. $d_2(a_0 h_1 b_{02} a_2) = a_0^2 a_1 b_{02}^2$.

PROOF. Since $a_0 h_1 b_{02} a_2 = \langle a_0, h_0, g_1 G \rangle$ in $H^{**}(A_3)$ by Theorem 2.6, we have $d_2(a_0 h_1 b_{02} a_2) = d_2 \langle a_0, h_0, g_1 G \rangle = \langle a_0, h_0, g_1 h_2 b_{01} a_2 \rangle$ by Theorem 2. 7. Since $\langle a_0, h_0, g_1 h_2 b_{01} a_2 \rangle = a_0^2 a_1 b_{02}^2$ by Theorem 2.6, we have $d_2(a_0 h_1 b_{02} a_2) = a_0^2 a_1 b_{02}^2$.

PROPOSITION 3. 9. $d_2(h_0 G) = h_0 b_{01} h_2 a_2$.

PROOF. Sicce $a_0 h_1 b_{02} a_2 = \langle a_0, g_1, h_0 G \rangle$ in $H^{**}(A_3)$ by Theorem 2. 6, we have $a_0^2 a_1 b_{02}^2 = d_2(a_0 h_1 b_{02} a_2) = d_2 \langle a_0, g_1, h_0 G \rangle = \langle a_0, g_1, d_2(h_0 G) \rangle$ by Theorem 2.7. Then there is only one possible differential $d_2(h_0 G) = h_0 b_{01} h_2 a_2$.

PROPOSITION 3. 10. $d_2(h_2 w) = b_{01} G$.

PROOF. Since $d_2(h_0 h_2 w) = a_0 b_{11} \cdot h_0 w + h_2 \cdot h_0 u b_{02} = h_0 b_{01} G$ by Proposition 1.1. f), we have $d_2(h_2 w) = b_{01} G$ for the dimensional reasons.

All other non-zero differentials in the range $t\text{-}s \leqq 104$ are easily determined by the differentials listed in §2 , above propositions and the fact that E_r is a differential algebra.

Next we consider some Toda brackets.

LEMMA 3. 11. *h_0x converges to $<\alpha_1, \alpha_1, \beta_5>$.*

PROOF . Since $d_3(x) = h_0ka_2$ and $h_0h_0 = 0$, Massey product $<h_0, h_0, ka_2>$ is defined and equal to h_0x in E_4. It is easy to check the conditions of Theorem 2.8 for this product. Then the result follows.

LEMMA 3. 12. *Let h_2b_{02} converge to an element $\widetilde{h_2b_{02}}$ of $\Pi^s_{81}(S^0 ; 3)$. Then we have $3.\widetilde{h_2b_{02}} = 0$.*

PROOF . If not, we have $3.\widetilde{h_2b_{02}} = \pm <\alpha_1, \alpha_1, \beta_5>$. Then we have $0 \neq \beta_1\beta_5 = \pm <\alpha_1, \alpha_1, \alpha_1> \beta_5 = \pm \alpha_1 <\alpha_1, \alpha_1, \beta_5> = \pm \alpha_1 \cdot 3.\widetilde{h_2b_{02}} = 0$. This is a contraction.

LEMMA 3. 13. *$h_0b_{11}a_2$ converges to $<\alpha_1, 3\iota, \beta_5>$.*

PROOF . Since $d_2(h_1) = a_0b_{01}$ and $h_2b_{11}a_2 = -h_1ka_2$, Massey product $<b_{01}, a_0, ka_2>$ is defined and equal to $h_0b_{11}a_2$ in E_3. Then we have the result by Theorem 2.8.

LEMMA 3.14. *Let a_0^2d converge to an element $\widetilde{a_0^2d}$ of $\Pi^s_{82}(S^0 : 3)$.* Then *$h_0h_2a_2$ converges to $<\alpha_1, 3\iota, \widetilde{a_0^2d}>$.*

PROOF . Since $\delta_5(h_2a_2) = a_0^3d$ in the May spectral sequence, Massey product $<h_0, a_0, a_0^2d>$ is defined and equal to $h_0h_2a_2$ in $\tilde{E}_6$. By Theorem 2. 6, we have a relation $h_0h_2a_2 = <h_0, a_0, a_0^2d>$ in $H^{**}(A_3)$. It is easy to check the condition of Theorem 2.8 for this product. Then the result follows.

LEMMA 3. 16. *$g_1h_2a_2$ converges to $<\alpha_1, 3\iota, \rho_1>$, where $\rho_1 = <\alpha_1, 3\iota, \widetilde{a_0^2d}>$.*

PROOF . Let $\overline{a_2}$ be a cochain in the cobar construction $F^*(A_3^*)$:

$$\begin{aligned}
\overline{a_2} = {} & [\tau_2 | \tau_2 | \tau_2] + [\tau_2 | \xi_2\tau_2 | \tau_0] - [\tau_2 | \xi_2 | \tau_2\tau_0] \\
& - [\tau_2\xi_2 | \tau_0 | \tau_2] + [\xi_2 | \tau_2\tau_0 | \tau_2] + [\tau_2 | \xi_1^3\tau_2 | \tau_1] \\
& - [\tau_2 | \xi_1^3 | \tau_2\tau_1] - [\xi_1^3\tau_2 | \tau_1 | \tau_2] + [\xi_1^3 | \tau_2\tau_1 | \tau_2] \\
& - [\xi_2^2 | \tau_0 | \tau_2\tau_0] - [\xi_2^2 | \tau_2\tau_0 | \tau_0] + [\xi_2^2\tau_2 | \tau_0 | \tau_0] \\
& + [\xi_2 | \xi_2\tau_0 | \tau_2\tau_0] - [\xi_2\tau_2 | \xi_2\tau_0 | \tau_0] + [\xi_2 | \xi_2\tau_2\tau_0 | \tau_0] \\
& - [\xi_2\tau_2 | \xi_1^3\tau_0 | \tau_1] + [\xi_2\tau_2 | \xi_1^3 | \tau_0\tau_1] + [\tau_2 | \xi_1^3\xi_2 | \tau_1\tau_0] \\
& - [\xi_1^3\xi_2\tau_2 | \tau_1 | \tau_0] - [\xi_1^3\tau_2 | \xi_2\tau_1 | \tau_0] + [\xi_1^3 | \xi_2\tau_2\tau_1 | \tau_0] \\
& + [\xi_1^3\xi_2 | \tau_2\tau_1 | \tau_0] + [\xi_1^3\xi_2 | \tau_1 | \tau_2\tau_0] - [\xi_1^3\xi_2 | \tau_1\tau_0 | \tau_2]
\end{aligned}$$

$$+ [\xi_1^3 | \xi_2\tau_1 | \tau_2\tau_0] + [\xi_2 | \xi_1^3\tau_2\tau_0 | \tau_1] + [\xi_2 | \xi_1^3\tau_0 | \tau_2\tau_1]$$
$$+ [\xi_2 | \xi_1^3\tau_2 | \tau_1\tau_0] - [\xi_2 | \xi_1^3 | \tau_2\tau_1\tau_0] - [\xi_1^3\tau_2 | \xi_1^3\tau_1 | \tau_1]$$
$$+ [\xi_1^6\tau_2 | \tau_1 | \tau_1] + [\xi_1^3 | \xi_1^3\tau_2\tau_1 | \tau_1] + [\xi_1^3 | \xi_1^3\tau_1 | \tau_2\tau_1]$$
$$- [\xi_1^6 | \tau_1 | \tau_2\tau_1] - [\xi_1^6 | \tau_2\tau_1 | \tau_1] - [\xi_1^3\xi_2^2 | \tau_1\tau_0 | \tau_0]$$
$$- [\xi_1^3\xi_2 | \xi_2\tau_1\tau_0 | \tau_0] - [\xi_2 | \xi_1^3\xi_2\tau_0 | \tau_1\tau_0] - [\xi_1^3\xi_2 | \xi_1^3\tau_1\tau_0 | \tau_1]$$
$$+ [\xi_1^6 | \xi_2\tau_1 | \tau_1\tau_0] - [\xi_1^3 | \xi_1^3\xi_2\tau_1 | \tau_1\tau_0] - [\xi_1^3\xi_2 | \xi_1^3\tau_1 | \tau_1\tau_0]$$
$$+ [\xi_1^6\xi_2 | \tau_1 | \tau_1\tau_0] + [\xi_1^6\xi_2 | \tau_1\tau_0 | \tau_1] + [\xi_1^6\tau_2 | \xi_1\tau_1 | \tau_0]$$
$$- [\xi_1^6 | \xi_1\tau_2\tau_1 | \tau_0] - [\xi_1^6 | \xi_1\tau_1 | \tau_2\tau_0] + [\xi_1^6\tau_2 | \xi_1 | \tau_1\tau_0]$$
$$- [\xi_1^3\tau_2 | \xi_1^4 | \tau_1\tau_0] + [\xi_1^3 | \xi_1^4\tau_2 | \tau_1\tau_0] - [\xi_1^6 | \xi_1\tau_2 | \tau_1\tau_0]$$
$$- [\xi_1^3 | \xi_1^4 | \tau_2\tau_1\tau_0] + [\xi_1^6 | \xi_1 | \tau_2\tau_1\tau_0] + [\xi_1^6\xi_2 | \xi_1\tau_1\tau_0 | \tau_0]$$
$$- [\xi_1^6\xi_2 | \xi_1\tau_0 | \tau_1\tau_0] + [\xi_1^3\xi_2 | \xi_1^4\tau_0 | \tau_1\tau_0] + [\xi_1^9 | \xi_1\tau_1 | \tau_1\tau_0].$$

Let $\overline{a_1}$ and α be

$$\overline{a_1} = [\tau_1 | \tau_1 | \tau_1] + [\tau_1 | \xi_1\tau_1 | \tau_0] - [\tau_1 | \xi_1 | \tau_1\tau_0]$$
$$- [\xi_1\tau_1 | \tau_0 | \tau_1] + [\xi_1 | \tau_1\tau_0 | \tau_1] - [\xi_1^2 | \tau_1\tau_0 | \tau_0]$$
$$- [\xi_1\tau_1 | \xi_1\tau_0 | \tau_0] + [\xi_1^2\tau_1 | \tau_0\tau_0] + [\xi_1 | \tau_1\tau_0\xi_1 | \tau_0]$$
$$- [\xi_1^2 | \tau_0 | \tau_1\tau_0] + [\xi_1 | \xi_1\tau_0 | \tau_1\tau_0]$$

and

$$\alpha = [\xi_1^9 | \xi_3] - [\xi_1^9 | \xi_1\xi_3^2] - [\xi_1^{18} | \xi_2] - [\xi_1^{10} | \xi_2^3] + [\xi_1^{18} | \xi_1^4]$$
$$+ [\xi_1^{19} | \xi_1^3].$$

By the tedious but routine calculations, we see that $h_0h_2a_2$ is represented by

$$[\xi_1] \cdot [\xi_1^9] \cdot \overline{a_2} - [\xi_1] \cdot [\xi_1^{18}] \cdot \overline{a_1} + \alpha . [\tau_0 | \tau_0 | \tau_0].$$

Let β be

$$[\tau_0 | \xi_1^9 | \tau_3] - [\tau_0 | \xi_1^{18} | \tau_2] + [\xi_1^9\tau_1 | \xi_2^3 | \tau_0]$$
$$- [\xi_1^9 | \tau_1\xi_2^3 | \tau_0] + [\xi_1^9 | \xi_2^3 | \tau_1\tau_0] + [\xi_1^9 | \xi_2^3\tau_0 | \tau_1]$$
$$- [\xi_1^9\tau_0 | \xi_2^3 | \tau_1] - [\xi_1^{18}\tau_1 | \xi_1^3 | \tau_0] + [\xi_1^{18} | \xi_1^3\tau_1 | \tau_0]$$
$$- [\xi_1^{18} | \xi_1^3 | \tau_1\tau_0] - [\xi_1^{18} | \xi_1^3\tau_0 | \tau_1] + [\xi_1^{18}\tau_0 | \xi_1^3 | \tau_1]$$
$$- [\xi_1^9\tau_0 | \xi_1\xi_2^3 | \tau_0] + [\xi_1^9 | \xi_1\xi_2^3\tau_0 | \tau_0] - [\xi_1^{10}\tau_0 | \xi_2^3 | \tau_0]$$
$$- [\xi_1^{18} | \xi_1^4\tau_0 | \tau_0] + [\xi_1^{18}\tau_0 | \xi_1^4 | \tau_0] + [\xi_1^{19}\tau_0 | \xi_1^3 | \tau_0].$$

Since Massey product $\langle h_0, a_0, h_0h_2a_2 \rangle$ is represented by

$$([\xi_1 | \xi_1\tau_0] - [\xi_1 | \tau_1]) \cdot ([\xi_1^9] \cdot \overline{a_2} - [\xi_1^{18}] \cdot \overline{a_1})$$
$$+ [\xi_1] \cdot \beta \cdot [\tau_0 | \tau_0] - [\tau_1] \cdot \alpha \cdot [\tau_0 | \tau_0 | \tau_0]$$
$$- [\tau_1 | \xi_1] \cdot ([\xi_1^9] \cdot \overline{a_2} - [\xi_1^{18}] \cdot \overline{a_1})$$
$$= ([\xi_1 | \tau_1] - [\xi_1^2 | \tau_0]) \cdot ([\xi_1^9] \cdot \overline{a_2} - [\xi_1^{18}] \cdot \overline{a_1})$$

$$+ [\xi_1]\cdot\beta\cdot[\tau_0|\tau_0] - [\tau_1]\cdot\alpha\cdot[\tau_0|\tau_0|\tau_0]$$
$$+ [\xi_1\tau_1]\cdot([\xi_1^9|\xi_2^3] - [\xi_1^{18}|\xi_1^3])\cdot[\tau_0|\tau_0|\tau_0]$$
$$- \delta[\xi_1\tau_1]\cdot([\xi_1^9]\cdot\overline{a_2} - [\xi_1^{18}]\cdot\overline{a_1}),$$

we have $<h_0, a_0, h_0h_2a_2> = g_1h_2a_2$ in $H^{**}(A_3)$. Then, by Theorem 2.8. we have the result.

LEMMA 3. 16. *$b_{01}h_2a_2$ converges to $<\alpha_1, \alpha_1, \rho_1>$.*

PROOF. We make use the same notation of the proof of Lemma 3. 15. Let μ and ν be

$$[\xi_1|\xi_1^9|\tau_3] - [\xi_1|\xi_1^{18}|\tau_2] - \alpha\cdot[\tau_1]$$
$$+ [\xi_1^{10}|\xi_2^3|\tau_1] + [\xi_1^9|\xi_1\xi_2^3|\tau_1] - [\xi_1^{19}|\xi_1^3|\tau_1]$$
$$- [\xi_1^{18}|\xi_1^4|\tau_1] - [\xi_1^9|\xi_3|\tau_1] + [\xi_1^{18}|\xi_2|\tau_1]$$

and

$$[\xi_1^{10}|\xi_1\xi_2^3] - [\xi_1^{11}|\xi_2^3] - [\xi_1^{19}|\xi_1^4] + [\xi_1^{20}|\xi_1^3] - [\xi_1^9|\xi_1^2\xi_2^3]$$
$$+ [\xi_1^{18}|\xi_1^5] + [\xi_1^{10}|\xi_3] + [\xi_1^9|\xi_1\xi_3] - [\xi_1^{19}|\xi_2] - [\xi_1^{18}|\xi_1\xi_2].$$

Since Massey product $<h_0, h_0, h_0h_2a_2>$ is represented by

$$([\xi_1|\xi_1^2] + [\xi_1^2|\xi_1])([\xi_1^9]\cdot\overline{a_2} - [\xi_1^{18}]\cdot\overline{a_1})$$
$$+ [\xi_1]\cdot\mu\cdot[\tau_0|\tau_0] + [\xi_1]\cdot\nu\cdot[\tau_0|\tau_0|\tau_0]$$
$$+ [\xi_1^2]\cdot\alpha\cdot[\tau_0|\tau_0|\tau_0],$$

we have $<h_0, h_0, h_0h_2a_2> = b_{01}h_2a_2$ in $H^{**}(A_3)$. Then, by Theorem 2. 8. we have the result.

LEMMA 3. 17. *$b_{01}b_{11}a_2$ converges to $<\beta_1, \alpha_2, \beta_5>$.*

PROOF. Since $d_2(g_2) = b_{01}g_1$ and $b_{01}b_{11}a_2 = g_2ka_2$, Massey produet $<b_{01}, g_1, ka_2>$ is defined and equal to $b_{01}b_{11}a_2$ in E_3. Then, by Theorem 2.8. we have the result.

LEMMA 3. 18. *$b_{01}v_0$ converges to $<\beta_1, \alpha_2, \widetilde{h_2b_{02}}>$.*

PROOF. By Theorem 2. 1. $d_2(h_1) = \tilde{\alpha}\cdot a_0b_{01}$, where $\tilde{\alpha}$ is a non-zero (constant) coefficient. Since $0 = d_2(h_1h_2) = \tilde{\alpha}\cdot a_0b_{01}h_2 - h_1\cdot d_2(h_2)$, $\delta_3(a_0b_{02}) = a_0h_1b_{11} - a_0b_{01}h_2$ and $0 \neq a_0b_{01}h_2 \in H^{4,49}(A_3)$, we have $d_2(h_2) = \tilde{\alpha}\cdot a_0b_{11}$. Since $[\xi_1^9|\xi_2^3] - [\xi_1^{18}|\xi_1^3]$ is a representative of $d \in H^{2,84}(A_3)$, it is easy to see that Massey product $<h_2, -h_2, h_1>$ is defined and equal to d in $H^{**}(A_3)$. By Theorem 2.7. we have

$$d_2<h_2, -h_2, h_1> \in -<(\tilde{\alpha}\cdot a_0b_{11}\ h_2), \begin{pmatrix} -h_2 & 0 \\ \tilde{\alpha}\cdot a_0b_{11} & -h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ \tilde{\alpha}\cdot a_0b_{01} \end{pmatrix}>.$$

This matric Massey product is represented by

$- \tilde{\alpha}\ ([\xi_1^9]\cdot\overline{b_{02}}\cdot[\tau_0] - [\xi_1^{18}]\cdot\overline{b_{01}}\cdot[\tau_0] + [\xi_1^9]\cdot\overline{b_{11}}\cdot[\xi_1^3\tau_0]$

$- \overline{b_{11}}\cdot[\tau_0 | \xi_2^3] - \overline{b_{11}}\cdot[\xi_1^9\tau_0 | \xi_1^3] - [\xi_1^{15} | \xi_1^3 | \tau_0 | \xi_1^3]$

$- [\xi_1^{12} | \xi_1^6 | \tau_0 | \xi_1^3] - [\xi_1^6 | \xi_1^{12} | \tau_0 | \xi_1^3] - [\xi_1^3 | \xi_1^{15} | \tau_0 | \xi_1^3])$

$= -\tilde{\alpha}([\xi_1^9]\cdot\overline{b_{02}}\cdot[\tau_0] - [\xi_1^{18}]\cdot\overline{b_{01}}\cdot[\tau_0] - [\xi_3^2]\cdot\overline{b_{11}}\cdot[\tau_0]$

$+ [\xi_1^9 | \xi_1^6 | \xi_1^6 | \tau_0])$

$+ \tilde{\alpha}.\delta\ (\overline{b_{11}}\cdot[\xi_2^3\tau_0] + [\xi_1^{15} | \xi_1^3 | \xi_1^3\tau_0] + [\xi_1^{12} | \xi_1^6 | \xi_1^3\tau_0]$

$+ [\xi_1^6 | \xi_1^{12} | \xi_1^3\tau_0] + [\xi_1^3 | \xi_1^{15} | \xi_1^3\tau_0] + [\xi_1^6 | \xi_1^3\xi_2^3 | \tau_0]$

$+ [\xi_1^3 | \xi_1^6\xi_2^3 | \tau_0] + [\xi_1^6\xi_2^3 | \xi_1^3 | \tau_0] + [\xi_1^3\xi_2^3 | \xi_1^6 | \tau_0]$

$+ [\xi_1^{15} | \xi_1^6 | \tau_0] + [\xi_1^9 | \xi_1^{12} | \tau_0] + [\xi_1^{12} | \xi_1^9 | \tau_0])$.

Then we have $d_2(d) = -\tilde{\alpha}.a_0h_2b_{02}$. Since $-h_0\cdot d_2(v_0) = d_2(h_0v_0) = d_2(-b_{01}d) = \tilde{\alpha}.b_{01}\cdot a_0h_2b_{02} = -\tilde{\alpha}.h_0\cdot g_1h_2b_{02}$ and $-h_0\cdot d_2(g_2) = d_2(h_0g_2) = d(-h_1b_{01}) = -\tilde{\alpha}.a_0b_{01}^2 = \tilde{\alpha}.h_0g_1b_{01}$, we have $d_2(v_0) = \tilde{\alpha}.g_1h_2b_{02}$ and $d_2(g_2) = -\tilde{\alpha}.g_1b_{01}$. Then Massey product $\langle b_{01}, g_1, h_2b_{02}\rangle$ is defined and equal to $-\tilde{\alpha}.g_2\cdot h_2b_{02} - \tilde{\alpha}.b_{01}\cdot v_0 = \tilde{\alpha}.b_{01}v_0$ (by proposition 1.1. e)) in E_3. By Theorem 2.8, it is easy to see that $b_{01}v_0$ converges to $\langle\beta_1, \alpha_2, \widetilde{h_2b_{02}}\rangle$ (up to sign).

We summarize the above propositions in the following theorem for E_∞ term in the range $77 \leqq t\text{-}s \leqq 103$.

THEOREM 3.19. *E_∞ for $77 \leqq t\text{-}s \leqq 103$ is given in Table B.*

Table B.

t-s	$\pi_*^s(S^0{:}3)$	survivor (corresponding generator)
77	0	
78	Z_3	$b_{01}b_{11}^2(\beta_1\lambda)$
79	Z_3	$g_1a_1^6(\alpha_{20})$
80	0	
81	Z_3+Z_3	h_2b_{02}, h_0x ($\langle\alpha_1, \alpha_1, \beta_5\rangle$)
82	Z_3	a_0^2d
83	Z_9	$a_0^2a_1^5u$, $a_0^3a_1^5u$ (α'_{21})
84	Z_3+Z_3	$h_0h_2b_{02}$, bka_2 ($\beta_1\beta_5$)
85	Z_3+Z_3	$h_0b_{11}a_2$ ($\langle\beta_1, 3\iota, \beta_5\rangle$), $h_0b_{01}kb_{02}$ ($\beta_1\mu$)
86	Z_3	$h_0h_2a_2$ ($\langle\alpha_1, 3\iota, a_0^2d\rangle = \rho_1$)
87	Z_3	$h_0a_1^7$ (α_{22})
88	0	
89	0	
90	Z_3	$g_1h_2a_2$ ($\langle\alpha_1, 3\iota, \rho_1\rangle$)

Table B (continued)

t-s	$\pi_*^s(S^0{:}3)$	survivor (corresponding generater)
91	$Z_3+Z_3+Z_3$	$b_{01}h_2b_{02}$, $h_0b_{01}x$ $(\beta_1 < \alpha_1, \alpha_1, \beta_5 >)$, $g_1a_1^7$ (α_{23})
92	Z_3+Z_3	$b_{11}c \pm h_0v_0$, $b_{01}b_{11}a_2$ $(< \beta_1, \alpha_2, \beta_5 >)$
93	Z_9	$b_{01}h_2a_2$, $a_0b_{01}h_2a_2$ $(< \alpha_1, \alpha_1, \rho_1 >)$
94	Z_3+Z_3	$kb_{11}^2(\beta_2\lambda)$, $b_{01}^2ka_2$ $(\beta_1^2\beta_5)$
95	Z_3+Z_9	$h_0b_{01}b_{11}a_2$ $(\alpha_1 < \beta_1, \alpha_2, \beta_5 >)$, $a_0^2a_1^6u$, $a_0^3a_1^6u$ (α'_{24})
96	0	
97	0	
98	0	
99	Z_3+Z_3	$b_{01}v_0$ $(< \beta_1, \alpha_2, \widetilde{h_2b_{02}} >)$, $h_0a_1^8$ (α_{25})
100	Z_3	$g_2b_{11}a_2$ $(\beta_2\beta_5)$
101	Z_3+Z_3	$b_{01}^2h_2b_{02}$, $h_0b_{01}^2x$ $(\beta_1^2 < \alpha_1, \alpha_1, \beta_5 >)$
102	Z_3+Z_3	$h_0b_{01}v_0$ $(\alpha_1 < \beta_1, \alpha_2, \widetilde{h_2b_{02}} >)$, $b_{01}^2b_{11}a_2(\beta_1 < \beta_1, \alpha_2, \beta_5 >)$
103	Z_3	$g_1a_1^8$ (α_{26})

REMARK. Recently, $\pi_*^s(S^0{:}3)$ in this range was announced by M. C. Tangora.

Appendix

Display chart of $E^0Ext_{A_3}(Z_3, Z_3)$ for t-$s \leqq 136$

Vertical and slanting lines indicate multiplication by a_0 and h_0, respectively.

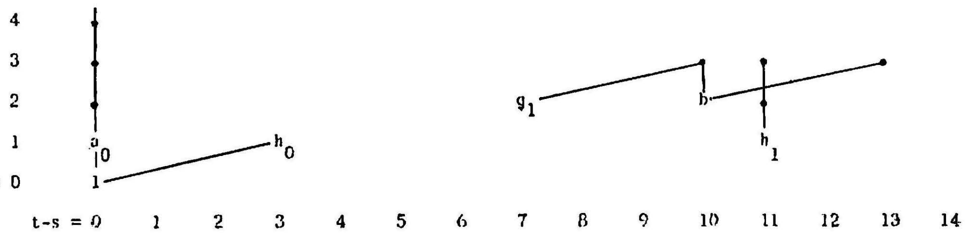

Fig. 1

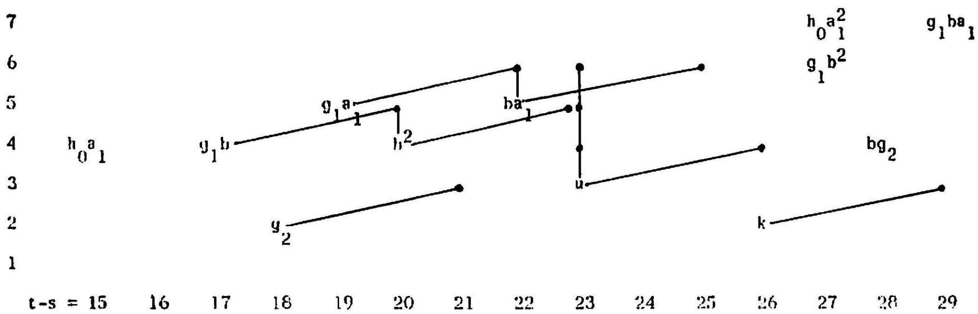

Fig. 2

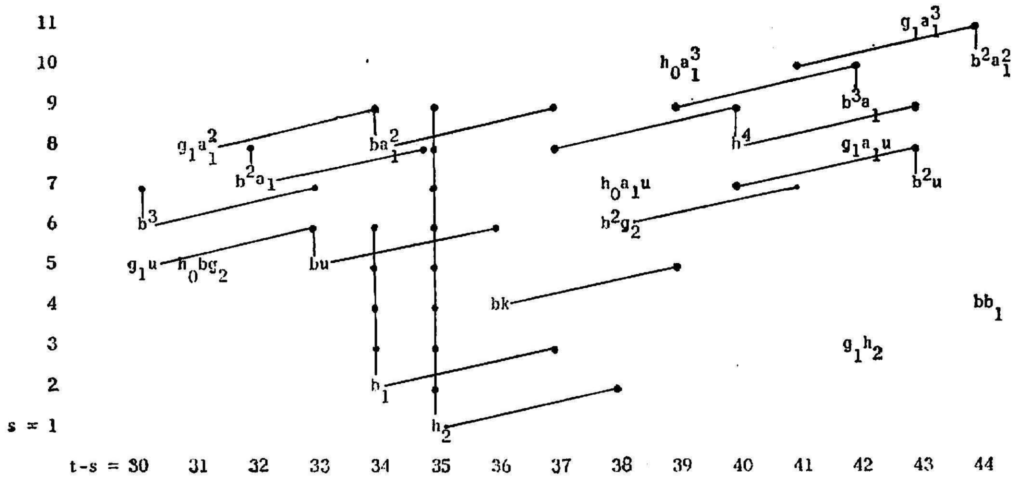
s = 1
2
3
4
5
6
7
8
9
10
11
t-s = 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44
h_2
h_1
bk
g_1u
h_0bg_2
bu
b^3
b^2a_1
$g_1a_1^2$
ba_1^2
h_0a_1u
b^2g_2
$h_0a_1^3$
b^4
b^3a_1
g_1a_1u
b^2u
$g_1a_1^3$
$b^2a_1^2$
bb_1
g_1h_2

Fig. 3

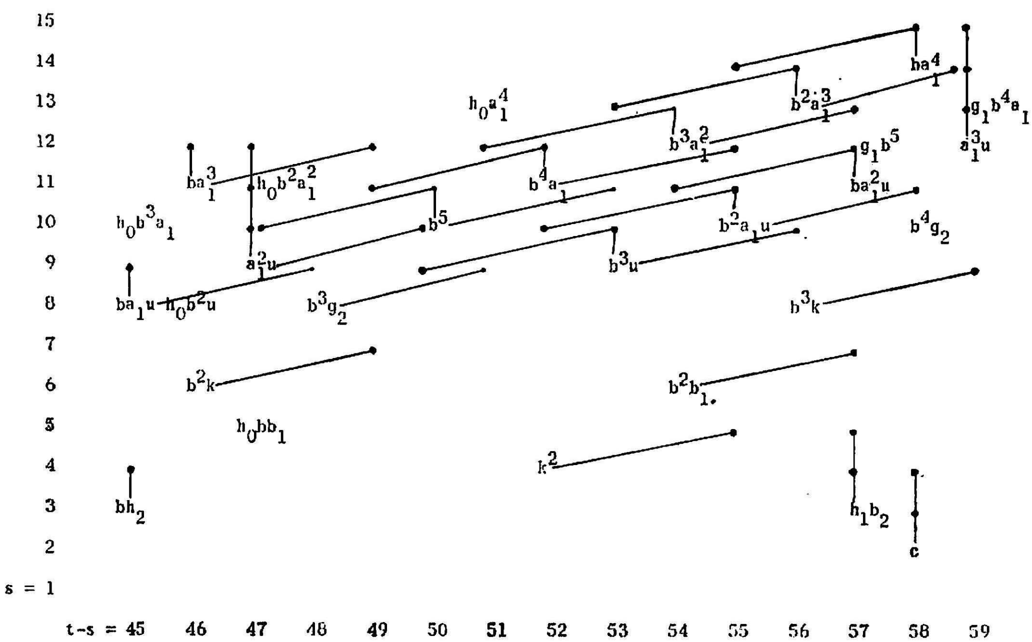
s = 1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
t-s = 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59
bh_2
h_0bb_1
b^2k
ba_1u
h_0b^2u
$h_0b^3a_1$
ba_1^3
$h_0b^2a_1^2$
a_1^2u
b^3g_2
b^5
$h_0a_1^4$
b^4a_1
k^2
b^3u
$b^3a_1^2$
b^2b_1
b^2a_1u
$b^2a_1^3$
b^3k
g_1b^5
ba_1^2u
ba_1^4
b^4g_2
h_1b_2
c
$g_1b^4a_1$
a_1^3u

Fig. 4

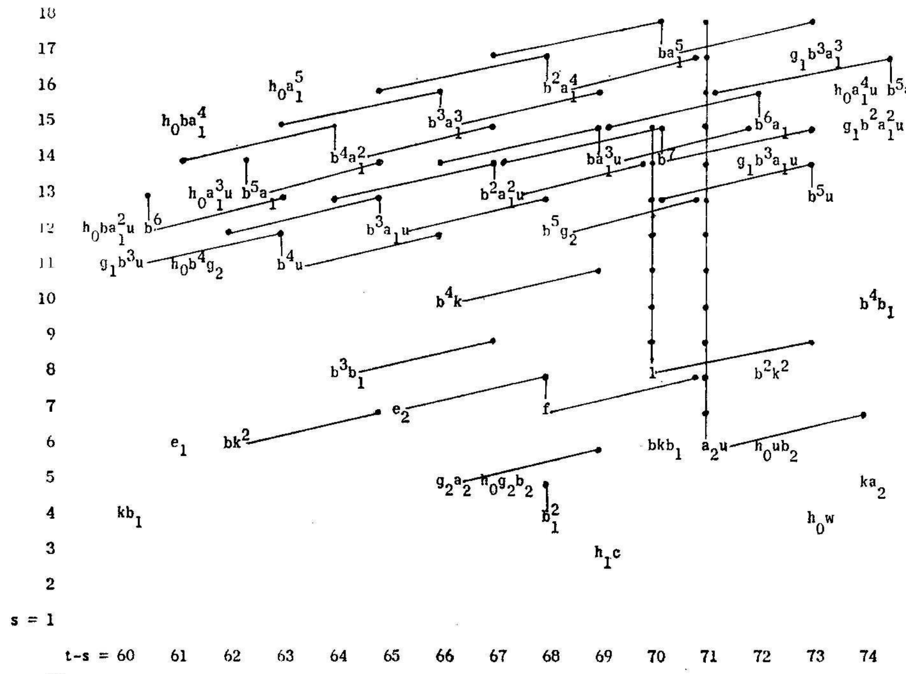

Fig. 5

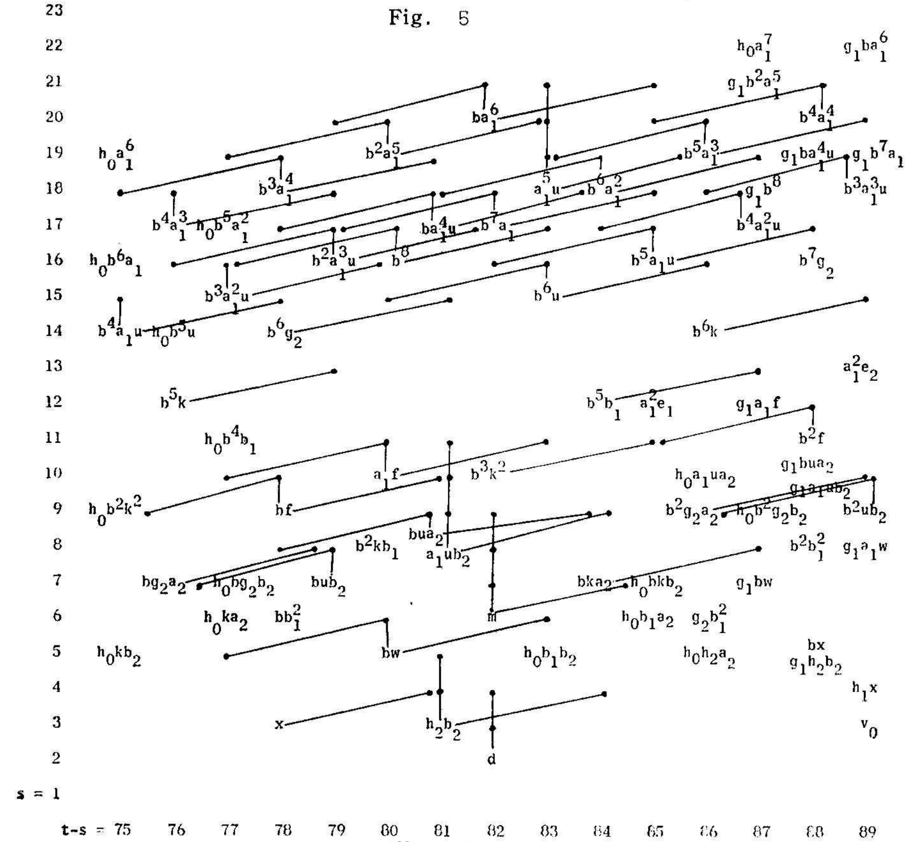

Fig. 6

REMARK · In this appendix we make use following notations;
$h_0 = g_{1,0}, g_1 = g_{1,1}, b = b_{01}, g_2 = g_{2,0}, k = k_{1,0}, b_1 = b_{11}, b_2 = b_{02}$,
$n = a_1G + a_0^2ca_2, p = b_1^2a_2 + a_0^2wb_2, q = bg_2a_2^2 - g_1ub_2a_2, r = h_0xa_2 + a_0h_2b_2a_2$
and $s = bxa_2 - g_1h_2b_2a_2$.

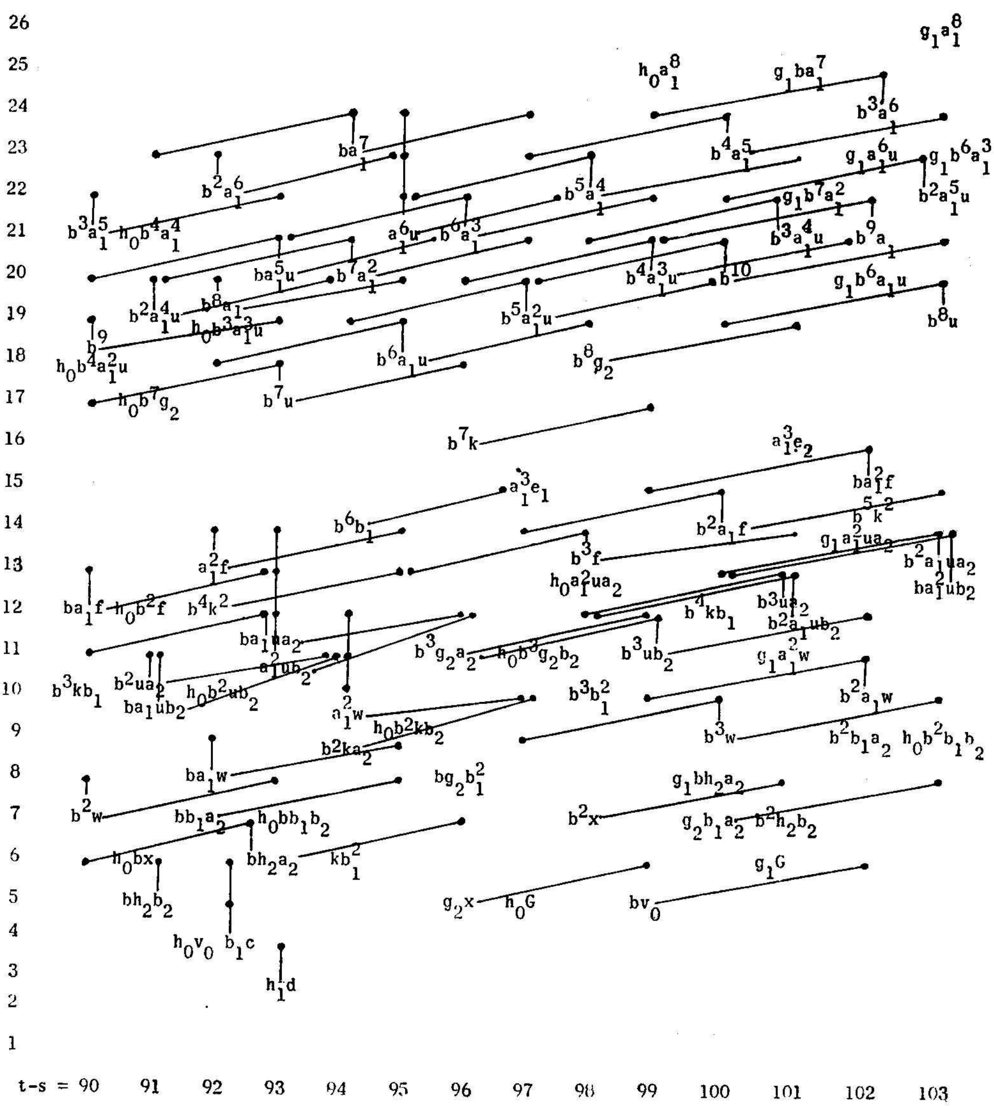

Fig. 7

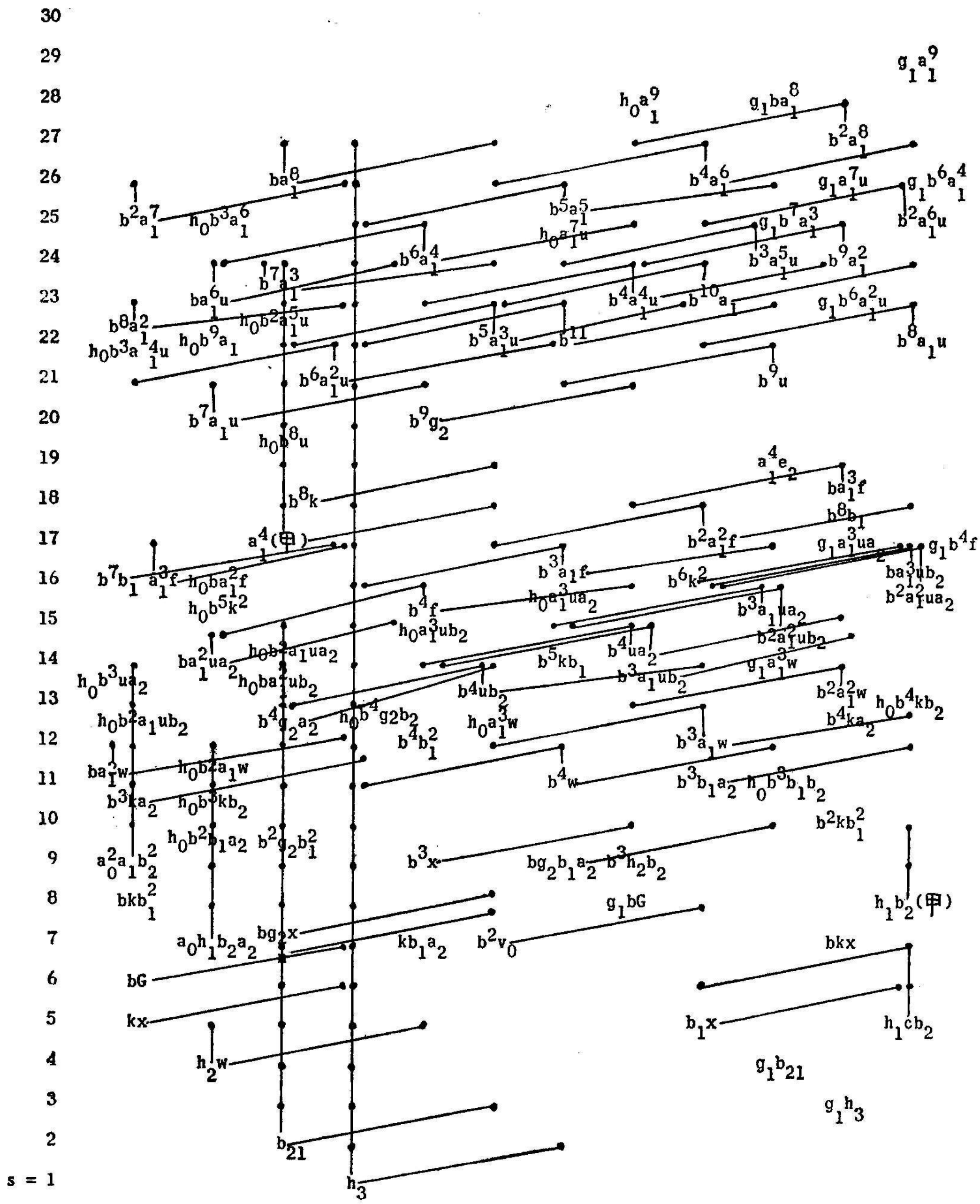
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
s = 1
t-s = 104
105
106
107
108
109
110
111
112
113
114
115

Fig. 8

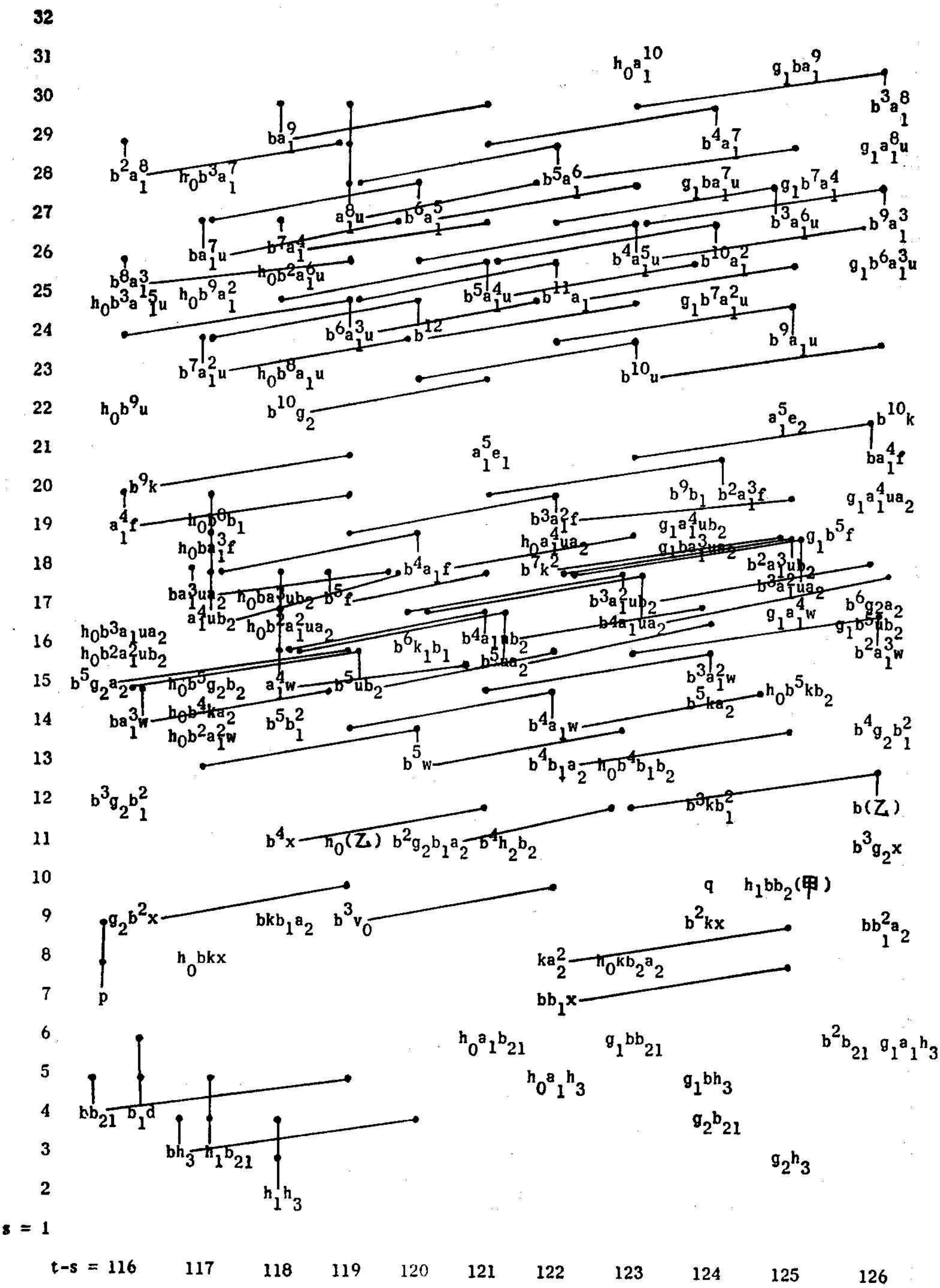

Fig. 9

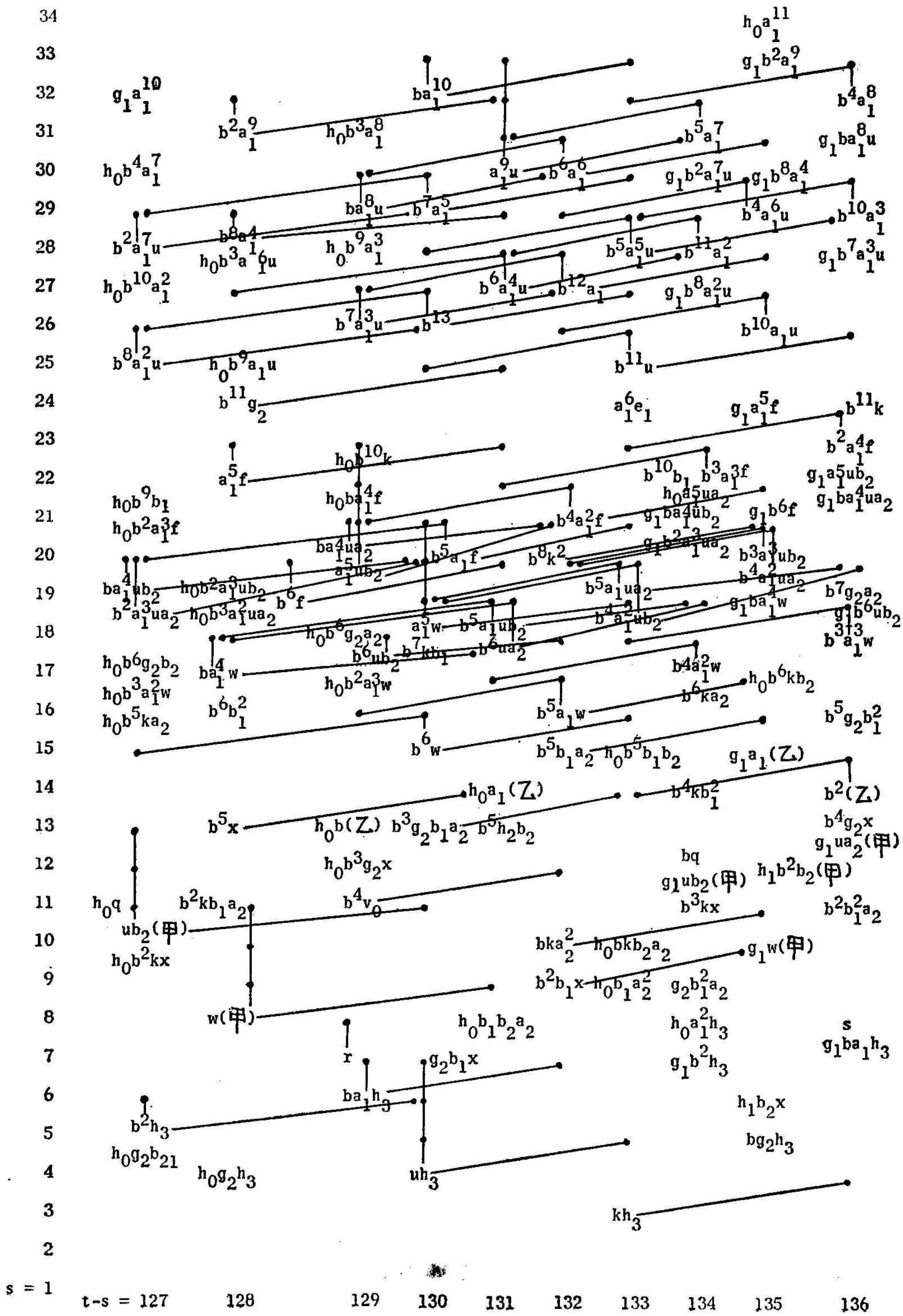
s = 1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
t-s = 127
128
129
130
131
132
133
134
135
136

Fig. 10

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