Versal family of holomorphic vector bundles on strongly pseudoconvex manifolds.

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By Takao AKAHORI *

Introduction

The main purpose of the present paper is to establish the existence of a versal family of holomorphic vector bundles on a compact strongly pseudo convex manifold.

M. Kurahishi develops a deformation theory of isolated singularities (V, x) via deformation theory of the complex subbundle induced by the tangent bundle of the differentiable manifold $M = V \cap S_\epsilon$([2]). We shall here apply Kuranishi's method to defor ations of holomorphic vector bundles on compact strongly pseudoconvex manifolds in the sense of Tanaka.

In the case of a complex manifold, the operator which maps ξ in A^0(M. End E) to $H\xi + \partial^* G(\omega_0 \xi)$ in A^0(M. End E) is Banach isomorphic at 0. But this is not true in our case, for the harmonic operator H is not defined as in the case of complex manifolds. Thus, we cannot use Banach inverse mapping theorem, and have to rely on the inverse mapping theorem of Nash-Moser. But the inverse mapping theorem of Nash-Moser does not preserve the analyticity in the parameter space. Therefore, our discussion of the completeness is not complete. The author hopes to improve this point in the near future.

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§1. Partially complex manifolds and holomorphic vector budles.

Let M' be a complex manifold, and S' the subbundle of CTM' consisting of all tangent vectors of type (1, 0) to M'. Then S' satisfies the conditions:

(C. 1) $\quad CTM' = S' + \bar{S}'$ (direct sum);

(C. 2) $\quad [\Gamma(S'), \Gamma(S')] \subset \Gamma(S')$,

where $\Gamma(S')$ denotes the space of global differentiable sections of S'.

Let M be a real submanifold of M'. For each $x \in M$, we define a subspace S_x of CTM_x by

$$S_x = S'_x \cap CTM_x$$

and assume that $\dim_c S_x$ is constant all $x \in M$. Then the union $S = \bigcup_{x \in M} S_x$ forms a

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* Dept. of Mathematics, Sci. & Engr Div., Univ. of the Ryukus

subbundle of CTM, and by (C. 1) and (C. 2) we have

(PC. 1) $\quad S\cap\bar{S}=0$,

(PC. 2) $\quad [\Gamma(S),\ \Gamma(S)]\subset(S)$,

Let M be a real manifold and S a subbundle of $C\otimes TM$. Then S is called a partially complex structure if S satisfies (PC. 1) and (PC. 2), and the manifold M together with the structure S is called a partially complex manifold. Clearly the notion of a partially complex manifold generalizes that of a complex manifold.

A complex vector bundle E over M is said to be holomorphic if there is given a differential operator

$$\bar{\partial}_E:\Gamma(E)\longrightarrow\Gamma(E\otimes\bar{S}^*)$$

satisfying the following conditions:

(HV. 1) $\quad \bar{X}(fu)=\bar{X}f\cdot u+f\bar{X}u$,
(HV. 2) $\quad [\bar{X},\ \bar{Y}]u=\bar{X}\cdot\bar{Y}u-\bar{Y}\cdot\bar{X}u$,

where $u\in\Gamma(E)$, $f\in CF(M)$ i. e. f is a C-valued differentiable function nn M, X, Y $Y\in\Gamma(S)$, and we put $\bar{Z}\cdot u=\bar{\partial}_E u(\bar{Z})$, $Z\in\Gamma(S)$.

More properly, $(E,\ \bar{\partial}_E)$ is said to be a holomorphic structure.

We put

$$C^q(M,\ E)=\Gamma(M,\ E\otimes\overset{q}{\wedge}(\bar{S})^*)$$

and define differential operators

$$\bar{\partial}_E^q:c^q(M,\ E)\longrightarrow c^{q+1}(M,\ E)$$

by

$$(\bar{\partial}_E^q\varphi)\ (\bar{X}_1,\cdots,\bar{X}_{q+1})=\sum_i(-1)^{i+1}\bar{X}_i(\varphi\ (\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\bar{X}_{q+1}))+\sum_{i<j}(-1)_{i+j}\varphi\ ([\bar{X}_i,\ \bar{X}_j],\ \bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\hat{\bar{X}}_j,\cdots\bar{X}_{q+1}),$$

for all $\varphi\in C^q(M,\ E)$ and $X_1,\cdots X_{q+1}\in\Gamma(S)$. Thus the collection $\{c^q(M,\ E),\bar{\partial}_E^q\}$ gives a complex and we denote by $H^q(M,\ E)$ the cohomology groups of this complex.

As for holomorphic vector bundles, we have the notions such as homomorphisms, isomorphisms, the tensor products etc, which are all defined in natural manners. For example, let E and F be two holomorphic vector bundles, then a bundle homomorphism $\varphi:E\longrightarrow F$ is called holomorphic if

$$\bar{X}(\varphi(u))=\varphi(\bar{X}\cdot u)\ \text{ for }\ u\in\Gamma(E),\ X\in\Gamma(S),$$

and the tensor product $E\otimes F$ becomes a holomorphic vector bundle by the rule

$$\bar{X}(u\otimes v)=\bar{X}u\otimes v+u\otimes\bar{X}v \text{ for } u\in\Gamma(E),\ v\in\Gamma(F),\ x\in\Gamma(S).$$

Now we shall formulate a deformation of a holomorphic vector bundle $(E,\ \bar{\partial}_E)$.

The trio $(R,\ E,\ \bar{\partial}_E(r))$ is said to be a deformation of the holomorphic vector bundle $(E,\ \bar{\partial}_E)$ if $(R,\ E,\ \bar{\partial}_E(r))$ satisfies the following——1) and 2).

1) R is, a germ of analytic set at the origin.
2) $\bar{\partial}_E(r)$ is a family of operators from $\Gamma(E)$ to $\Gamma(E\otimes(S)^*)$ parametrized analytically by the analytic space R, and $(E,\ \bar{\partial}_E(r))$ is a holomorphic structure of E for each r.

Then, we can prove the following Theorem.

Theorem 1. 1. Let $(R,\ E,\ \bar{\partial}_E(r))$ be a deformation of $(E,\ \bar{\partial}_E)$. Then corresponding to this family, we can determine an element $\omega_E(r)\in\Gamma(EndE\otimes(\bar{S})^*($ such that $\omega_E(r)$ is holomorphic in R, $\omega_E(0)=0$, and $\omega_E(r)$ satisfies the condition

$$\bar{\partial}^{(1)}_{EndE}\,\omega_E(r)+\omega_E(r)\wedge\omega_E(r)=0\,.$$

Conversely, for any $\omega_E(r)$ which is holomorphic in R and satisfies $\omega_E(0)=0$ and $\bar{\partial}^{(1)}_{EndE}\,\omega_E(r)+\omega_E(r)\wedge\omega_E(r)=0$, we can construct uniquely a deformation $(R,\ E,\ \bar{\partial}_E(r))$ of $(E,\ \bar{\partial}_E)$ associated with $\omega_E(r)$. (Here we fix the decomposition of $C\otimes TM=S\oplus\bar{S}\oplus F$.)

Proof) Let $(R,\ E,\ \bar{\partial}_E(r))$ be a deformation of the holomorphic vector bundle $(E,\ \bar{\partial}_E)$. Then from assumptions (HV. 1) and (HV. 2), we can get the following relations (HV. 1)' and (HV. 2)'.

(HV. 1)' $\quad \bar{X}(fu)=\bar{X}f\cdot u+f\bar{X}u,$

(HV. 2)' $\quad [\bar{X},\ \bar{Y}]u=\bar{X}\bar{Y}u-\bar{Y}\bar{X}u,$

where $x\in\Gamma(S)$, $u\in\Gamma(E)$ and we put $\bar{X}\cdot u=\bar{\partial}_E(r)u(\bar{X})$. We put $\omega_E(r)=\bar{\partial}_E(r)-\bar{\partial}_E$, then $\omega_E(r)$ is an operator which maps $\Gamma(EndE)$ into $\Gamma(EndE\otimes(\bar{S})^*)$. Then, from the assumption (HV. 1)'

(1. 1) $\quad \bar{\partial}_E(r)(fu)(\bar{X})=\bar{X}f\cdot u+f\bar{\partial}_E(r)u(\bar{X}),$

(1. 2) $\quad \bar{\partial}_E(fu)(\bar{X})=\bar{X}f\cdot u+f\bar{\partial}_E u(\bar{X}).$

And so from (1. 1) and (1. 2)

(1. 3) $\quad (\bar{\partial}_E(r)-\bar{\partial}_E)(fu)(\bar{X})=f(\bar{\partial}_E(r)-\bar{\partial}_E)u(\bar{X}).$

Hence

(1. 4) $\quad \omega_E(r)(fu)(\bar{X})=f\omega_E(r)\,u(\bar{X}).$

Thus we see that $\omega_E(r)$ is an element in $\Gamma(EndE\otimes(\bar{S})^*)$, and it is clear that it satisfies the relation $\omega_E(0)=0$.

Now, we shall prove that $\omega_E(r)$ satisfies the differential equation $\bar{\partial}^{(1)}_{EndE}\,\omega_E(r)\wedge$

$\omega_E(r)=0$. The operator $\bar\partial_E(r)=\bar\partial_E+\omega_E(r)$ satisfies the condition (HV. 2)' :

(HV. 2)'
$$[\bar X, \bar Y]u=\bar X\cdot\bar Y u-\bar Y\bar X u$$
where $X, Y\in\Gamma(S)$, $\bar Z u=\bar\partial_E(r)\,u(\bar Z)$, $u\in\Gamma(E)$.

The left hand side of (HV. 2)' becomes

$$\bar\partial_E u([\bar X, \bar Y])+\omega_E(r)\,u([\bar X, \bar Y]), \tag{1. 5}$$

when we write it down according to the definition, and the right hand side of (HV. 2)' becomes

$$\begin{aligned}&\bar\partial_E(\bar\partial_E u(\bar Y))(\bar X)+\omega_E(r)(\bar\partial_E u(\bar X)+\bar\partial_E(\omega_E(r)\,u(\bar Y))(\bar X)\\&+\omega_E(r)(\ \ _E(r)\,u(\bar Y))(\bar X)-\bar\partial_E(\bar\partial_E u(\bar X))(\bar Y)-\omega_E(r)(\bar\partial_E u(\bar X)\ \bar Y)\\&-\bar\partial_E(\omega_E(r)\,u(\bar X))(\bar Y)-\bar\omega_E(r)(\omega_E(r)\,u(\bar X))(\bar Y).\end{aligned} \tag{1. 6}$$

This can be rewritten as

$$\begin{aligned}&\bar\partial_E(\bar\partial_E u(\bar Y))(\bar X)-\bar\partial_E(\bar\partial_E u(\bar X))(\bar Y)+\omega_E(r)(\bar\partial_E u(\bar Y))(\bar X)\\&-\omega_E(r)(\bar\partial_E u(\bar X))(\bar Y)+\bar\partial_E(\omega_E(r)\,u(\bar Y))(\bar X)-\bar\partial_E(\omega_E(r)\,u(\bar X))(\bar Y)\\&+\omega_E(r)(\omega_E(r)\,u(\bar Y))(\bar X)-\omega_E(r)(\omega_E(r)\,u(\bar X))(\bar Y).\end{aligned} \tag{1. 7}$$

$(E, \bar\partial_E)$ being a holomorphic vector bundle, the first term of (1. 7) is equal to

$$\bar\partial_E(\bar\partial_E u(\bar Y))(\bar X)-\bar\partial_E(\bar\partial_E u(\bar X))(\bar Y)=\bar\partial_E u([\bar X, \bar Y]), \tag{1. 8}$$

the second term

$$\omega_E(r)\wedge\bar\partial_E u(\bar X, \bar Y), \tag{1. 9}$$

the third term

$$\bar\partial_E(\omega_E(r)\,u)(\bar X, \bar Y)+\omega_E(r)\,u([\bar X, \bar Y]), \tag{1. 10}$$

and the fourth term

$$\omega_E(r)(\omega_E(r)\,u)(\bar X, \bar Y). \tag{1. 11}$$

And from the definiton of the holomorphic structure on EndE, we have the relation:

$$(\bar\partial^{(1)}_{EndE}\ \omega_E(r))\,u=\bar\partial^{(1)}_E(\omega_E(r)\,u)+\omega_E(r)(\bar\partial_E u). \tag{1. 12}$$

Thus we can rewrite the expression (1. 7) as follows:

$$\begin{aligned}&\bar\partial_E u([\bar X, \bar Y])+(\bar\partial^{(1)}_{EndE}\ \omega_E(r))\,u(\bar X, \bar Y)\\&+(\omega_E(r)\wedge\omega_E(r))\,u(\bar X, \bar Y)+\omega_E(r)\,u([\bar X, \bar Y]).\end{aligned} \tag{1. 13}$$

From (1. 5) and (1. 13)

$$(\bar\partial^{(1)}_{EndE}\ \omega_E(r))\,u(\bar X, \bar Y)+(\omega_E(r)\wedge\omega_E(r))\,u(\bar X, \bar Y)=0 \tag{1, 14}$$

for any $u \in \Gamma(E)$, X, $Y \in \Gamma(S)$.

From (1. 14) we have

(1. 15) $$\bar{\partial}^{(1)}_{EndE}\, \dot{\omega}_E(r) + \omega_E(r) \wedge \omega_E(r) = 0\,.$$

Conversely, we will define $\bar{\partial}_E(r)$ to be equal to $\bar{\partial}_E + \omega_E(r)$. It is clear that it satisfies the relation (HV. 1) and (HV. 2). Q.E.D.

§ 2. Deformation and induced family.

Let $(E, \bar{\partial}_E)$ and $(E', \bar{\partial}_{E'})$ be two holomorphic vector bundles on a partially complex manifold M. $(E, \bar{\partial}_E)$ and $(E', \bar{\partial}_{E'})$ is said to be equivalent if the following condition is satisfied.

There exists a C^∞-bundle isomorphism $\varphi : E \to E'$ such that the relation

$$\bar{X} \cdot \varphi(u) = \varphi(\bar{X} \cdot u) \text{ for } X \in \Gamma(S) \text{ and } u \in \Gamma(E)$$

holds.

Now, we will define the induced family. Let $(R, E, \bar{\partial}_E(r))$ be a deformation of $(E, \bar{\partial}_E)$ and $(R', E', \bar{\partial}_{E'}(r'))$ another deformation of $(E, \bar{\partial}_E)$, Let h be a holomorphic map from R to R' such that $h(0) = 0'$. We say that the map h induces $(R, E, \bar{\partial}_E(r))$ from $(R', E', \bar{\partial}_{E'}(r'))$ if it satisfies the following conditions:

1) There exists a family φ_r of C^∞-bundle isomorphisms from E to E' which is analytic in r and satisfies $\varphi_0 = id$.

2) $\bar{X} \cdot \varphi_r(u) = \varphi_r(\bar{X} \cdot u)$ for $X \in \Gamma(S)$ and $u \in \Gamma(E)$.

Theorem 2. 1 Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle, and let $(R, E, \bar{\partial}_E(r))$, $(R', E', \bar{\partial}_{E'}(r'))$ be deformations of $(E, \bar{\partial}_E)$. Let $\omega_E(r)$ and $\omega_{E'}(r)$ be the elements of $\Gamma(EndE \otimes (\bar{S})^*)$ which are associated to $(R, E, \bar{\partial}_E(r))$ and $(R', E', \bar{\partial}_{E'}(r'))$ respectively.

Then the following statements 1) and 2) are equivalent to each other :

1) There exists a map h from R to R' and h induces $(R, E, \bar{\partial}_E(r))$ form $(R', E', \bar{\partial}_{E'}(r))$.

2) There exist a map h from R to R' and C^∞-bundle isomorphism φ_r from E to E' which depends holomorphically in r such that the following relations hold:

$$\omega_{E'}(h(r)) = \varphi_r^{-1}\, \omega_E(r)\, \varphi_r + \varphi_r^{-1}\, \bar{\partial}_{EndE}\, \varphi_r\,,$$

$$\varphi_0 = id.$$

Proof) We shall prove that the condition 1) implies the condition 2).

From the assumption, we have

$$\bar{\partial}_{E'}(r') = \bar{\partial}_{E'} + \omega_{E'}(r')\,,$$

$$\bar{\partial}_E(r) = \bar{\partial}_E + \omega_E(r),$$

and there exist φ_r which is holomorphic in r and a holomorphic map h from R to R' such that the following diagram is commutative:

$$\begin{array}{ccc}
\Gamma(E) & \xrightarrow{\quad \bar{\partial}_E(r) \quad} & \Gamma(E \otimes (\bar{S})^*) \\
\uparrow \varphi_r & & \uparrow \varphi_r \\
\Gamma(E') & \xrightarrow{\quad \bar{\partial}_{E'}(h(r)) \quad} & \Gamma(E' \otimes (\bar{S})^*)
\end{array}$$

Therefore, we get the following relation.

(2. 1) $$(\bar{\partial}_E + \omega_E(r))\,\varphi_r(u)(\bar{X}) = \varphi_r(\bar{\partial}_{E'} + \omega_{E'}(h(r))\,u(\bar{X})$$

$$u \in \Gamma(E), \quad X \in \Gamma(S).$$

From (2. 1), it follows that

(2. 2) $$\varphi_r^{-1}\,\bar{\partial}_E\,\varphi_r(u)(\bar{X}) + \varphi_r^{-1}\,\omega_r(u)(\bar{X}) - \bar{\partial}_E u(\bar{X})$$

$$= \omega_{E'}(h(r))\,u(\bar{X}),$$

and

(2. 3) $$(\bar{\partial}_{EndE}\,\varphi_r)\,u(\bar{X}) = \bar{\partial}_E(\varphi_r(u))(\bar{X}) - \varphi_r\,\bar{\partial}_E u(\bar{X}).$$

From (2. 3) and (2. 2), it follows that

(2. 4) $$\omega_{E'}(h(r))\,u(\bar{X}) = (\varphi_r^{-1}\,\bar{\partial}_{EndE}\,\varphi_r)\,u(\bar{X}) + \varphi_r^{-1}\,\omega_E(r)\,\varphi_r(u)(\bar{X}),$$

hence

(2. 5) $$\omega_{E'}(h(r)) = \varphi_r^{-1}\,\omega_E(r)\,\varphi_r + \varphi_r^{-1}\,\bar{\partial}_{EndE}\,\varphi_r.$$

Conversely, it is clear that we can conclude (2. 1) from the relation (2. 5). Hence 2) implies 1).

Q. E. D.

§ 3. Construction of a complete family

In this seetion and the next, we shall assume that M is a compact strongly pseudo-convex manifold and $dim_R M = 2n-1 \geqq 5$. Then we can use Kohn's harmonic theory. (See Tanaka [1] for the details.)

We consider the set $\Phi = \{\omega \in \Gamma(M, EndE \otimes (\bar{S})^*) : \bar{\partial}^{(1)}_{EndE}\,\omega + \omega \wedge \omega = 0 \;\; \bar{\partial}^*_{EndE}\,\omega = 0\}$ and try to prove that Φ forms an effective and complete family. We shall show that a suitable neighborhood of 0 in Φ is parametrized by an analytic space. We will first show the next proposition.

Proposition 3. 1 If we denote by Φ_k the set of $EndE \otimes (\bar{S})^*$-valued forms ω of class C^k with $\bar{\partial}^{(1)}_{EndE}\, \omega + \omega \wedge \omega = 0$ and $\bar{\partial}^*_{EndE}\, \omega = 0$, then $\Phi_k = \Phi$ for $k \geqq 2$.

Proof) We will abbreviate $\bar{\partial}_{EndE}$ as $\bar{\partial}$. Any element ω in Φ_k satisfies the condition $(\bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*)\omega + \bar{\partial}^*(\omega \wedge \omega) = 0$. As $\square = \bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*$ is hypo-elliptic, ω is of class C^∞. Hence $\omega \in \Phi$.

Q. E. D.

Now, we come to the main part of this section. First, we consider the set $\Psi = \{\omega \in \Gamma(M, EndE \otimes (\bar{S})^*);\ \omega \bar{\partial}^* N(\omega \wedge \omega) \in \boldsymbol{H}^{0,1}\}$, where N is the Neumann operator for $\square = \bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*$ and $\boldsymbol{H}^{0,1}$ is the space of harmonic forms, then $\Phi \subsetneq \Psi$. In fact, any ω in Φ satisfies the condition $\bar{\partial}\omega + \omega \wedge \omega = 0$ and $\bar{\partial}^* \omega = 0$. Then $(\bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*)\,\omega + \bar{\partial}^*(\omega \wedge \omega) = 0$. Therefore it satisfies the relation $\omega - \boldsymbol{H}\omega + \bar{\partial}^* N(\omega \wedge \omega) = 0$, where $\boldsymbol{H}$ is the harmonic projection. And so $\omega + \bar{\partial}^* N(\omega \wedge \omega) = \boldsymbol{H}\omega \in \boldsymbol{H}^{0,1}$, therefore $\Phi \subsetneq \Psi$. Next we will show that Ψ is parametrized by a suitable neighborhood of 0 in $\boldsymbol{H}^{0,1}$. In fact consider the map F_k;

$$F_k : \Gamma_k(M, EndE \otimes (\bar{S})^*) \ni \omega \longrightarrow \omega + \bar{\partial}^* N(\omega \wedge \omega) \in \Gamma_k(M, EndE \otimes (\bar{S})^*).$$

Then it is clear that F_k is complex Banach-analytic. The differential of F_k at the origin is clearly the identity map. Hence, by the inverse mapping theorem in Banach manifolds, when we set $W = \{s \in \boldsymbol{H}^{0,1},\ \|s\|_k < \varepsilon\}$ for a sufficiently small ε, there is a complex Banach-analytic map $\phi_k : W \to \Gamma_k(M, EndE \otimes (\bar{S})^*)$ such that

$$F_k(\phi_k(S\bar{S})) = s \text{ for any } s \text{ in } W.$$

By sobolev's lemma and proposition 3. 1, $\phi_k(s)$ is of class C^∞. We abbreviate $\phi_k(s)$ as $\phi(s)$. We recall we have the equality:

$$\text{(3. 1)} \qquad \phi(s) + \bar{\partial}^* N(\phi(s) \wedge \phi(s)) = s \text{ for any } s \text{ in } W.$$

By the construction, it is clear the image of ϕ covers a neighborhood of 0 in Ψ in the topology of $\Gamma_k(M, EndE \otimes (\bar{S})^*)$.

We now try to find a necessary and sufficient condition on s for $\phi(s)$ being in Φ. From the relation (3. 1), $\phi(s)$ satisfies the following relation.

$$\text{(3. 2)} \qquad \bar{\partial}\phi(s) + \bar{\partial}\bar{\partial}^* N(\phi(s) \wedge \phi(s)) = 0.$$

Then, we have by Hodge decomposition theorem that

$$\begin{aligned} \text{(3. 3)} \qquad \bar{\partial}\phi(s) + \phi(s) \wedge \phi(s) &= -\bar{\partial}\bar{\partial}^* N(\phi(s) \wedge \phi(s)) + \phi(s) \wedge \phi(s) \\ &= \bar{\partial}^* \bar{\partial} N(\phi(s) \wedge \phi(s)) + \boldsymbol{H}(\phi(s) \wedge \phi(s)) \\ &= \bar{\partial}^* N \bar{\partial}(\phi(s) \wedge \phi(s)) + \boldsymbol{H}(\phi(s) \wedge \phi(s)). \end{aligned}$$

Since the images of $\bar{\partial}^* N$ and $\boldsymbol{H}$ are orthogonal, the above shnws that $\bar{\partial}\phi(s) + \phi(s) \wedge \phi(s) = 0$ if and only if $\boldsymbol{H}(\phi(s) \wedge \phi(s)) = 0$ and $\bar{\partial}^* N \bar{\partial}(\phi(s) \wedge \phi(s)) = 0$. We claim that the condition $\boldsymbol{H}(\phi(s)\wedge\phi(s))=0$ implies $\bar{\partial}\phi(s)+\phi(s)\wedge\phi(s)=0$.

In fact if we set $\xi(s)=\bar{\partial}\phi(s)+\phi(s)\wedge\phi(s)$, we have

$$(3.\ 4)\qquad \begin{aligned}\xi(s)&=\bar{\partial}^{*}N(\bar{\partial}\phi(s)\wedge\phi(s)-\phi(s)\wedge\bar{\partial}\phi(s))\\ &=\bar{\partial}^{*}N((\xi(s)-\phi(s)\wedge\phi(s))\wedge\phi(s)-\phi(s)\wedge(\xi(s)-\phi(s)))\\ &=\bar{\partial}^{*}N(\xi(s)\wedge\phi(s)-\phi(s)\wedge\xi(s)).\end{aligned}$$

Therefore we have the following relation.

$$(3.\ 5)\qquad |\xi(s)|_k\leqq 2C|\xi(s)|_k|\phi(s)|_k.$$

Hence, if we choose ε so small that $|\phi(s)|_k<(4C)^{-1}$ for all s in W, we obtain $\xi(s)=0$.. Therefore $\boldsymbol{H}(\phi(s)\wedge\phi(s))=0$ is the necessary and sufficient condition for $\phi(s)$ being integrable, and the analytic space $S=\{s\in W;\ \boldsymbol{H}(\phi(s)\wedge\phi(s))=0\}$ parametrizes Φ.

§ 4. Completeness

Now, we will prove $\Phi=\{\omega:\ \omega\in\Gamma(M,\ EndE\otimes(\bar{S})^{*}),\ \bar{\partial}^{*}\omega=0\ \ \bar{\partial}\omega+\omega\wedge\omega=0\}$ is complete in the C^{∞} category. (What we actually want is the completeness in the category of complex analiticity. In this sence the present discussion is not complete. The author hopes to improve this point in the near future.) For this purpose, we recall the inverse mapping .theorem of Nash-Moser.

The inverse mapping theorem of Nash-Moser

Let E and F be Frechet spaces, and let the topology of E and F be defined by fundamental systems of seminorms on E and F bo h denoted by $\|\ \|_s$ for $s\in\boldsymbol{Z}^{+}$. Let A be a map from E to F, then if A satisfies the following 1), 2), 3), A has the inverse.

1) $A(0)=0$

2) There exist linear maps

$$A'(x):E\to F$$

for x in a sufficiently small neighborhood of 0 in E with the following estimates:

There exist $k\in\boldsymbol{Z}^{+}$ and positive constants C_s depending on $s\in\boldsymbol{Z}^{+}$ such that

$$2-1)\qquad \|A(x+y)-A(x)-A'(x)y\|_s\leqq C_s(\|y\|_{k+s}+\|x\|_{k+s}\|y\|_k)\|y\|_k.$$

$$2-2)\qquad \|A'(x)y\|_s\leqq C_s(\|y\|_{k+s}+\|x\|_{k+s}\|y\|_k).$$

for x and y in $E(s_1,\ a)=\{x\in E:\|x\|_{s_1}<a\}$, s_1 being independent of s.

3) $A'(x)$ has the inverse $D(x)$ such that:

$$\|D(x)y\|_s\leqq C_s'(\|y\|_{k+s}+\|x\|_{k+s}\|y\|_k).$$

To prove that Φ is complete in the C^∞ category, we must prove the following.

For any analytic space T and $\omega(t)\in\Gamma(M, EndE\otimes(\bar{S})^*)$ which is holomorphic in T and satisfies the relations $\omega(0)=0$ and $\bar{\partial}\omega(t)+\omega(t)\wedge\omega(t)=0$, there exists $\xi(t)$ $\Gamma(M, EndE)$ which is holomorphic in T and satisfies the relations $\xi(0)=0$ and $\bar{\partial}^*((1+\xi(t)^{-1}\omega(t)(1+\xi(t))+(1+\xi(t))^{-1}\bar{\partial}\xi(t))=0$.

In order to prove the above statement, we will take $\bar{\partial}^*\Gamma(M, EndE\otimes(\bar{S})^*)$ as E. And we will put $F=\bar{\partial}\Gamma(M, EndE)$. Then E and F are Frechet spaces and E is isomorphic to F.

In fact define L and R bv

$$L: \xi\in E=\bar{\partial}^*\Gamma(M, EndE\otimes(\bar{S})^*)\longrightarrow\bar{\partial}\xi\in F=\bar{\partial}\Gamma(M, EndE),$$

$$R: \gamma\in F=\bar{\partial}\Gamma(M, EndE)\longrightarrow\bar{\partial}^*N\gamma\in E=\bar{\partial}^*\Gamma(M, EndE\otimes(\bar{S})^*).,$$

then $R\cdot L=id_E$ and $L\cdot R=id_F$.

Now, we prove that Φ is complete in the C^∞ category. To prove this, it is sufficient to show that the map $\tilde{A}$ from $E\times T$ to $F\times T$ is isomorphic at (0, 0), where $\tilde{A}: E\times T\ni(\xi, t)\longrightarrow(A(\xi, t), t)\in F\times T$ is defined by

$$A(\xi, t)=\bar{\partial}\bar{\partial}^*N((1+\xi)^{-1}\omega(t)(1+\xi)+(1+\xi)^{-1}\bar{\partial}\xi).$$

In fact, we restrict the inverse ϕ of $\tilde{A}$ to $0\times T$.

$$F\times T\supset 0\times T\ni(0, t)\longrightarrow\phi(0, t)=(\xi(t), t)\in E\times T.$$

Then $\xi(t)$ satisfies the relation

$$\bar{\partial}\bar{\partial}^*N((1+\xi(t))^{-1}\omega(t)(1+\xi(t))+(1+\xi(t))^{-1}\bar{\partial}\xi(t))=0.$$

And so

$$\bar{\partial}^*N((1+\xi(t))^{-1}\omega(t)(1+\xi(t))+(1+\xi(t))=0.$$

We show that $\tilde{A}$ satisfies the condition 1), 2), 3) of Nash-Moser.

It is clear that $\tilde{A}$ satisfies the relation 1). We shall prove that $\tilde{A}$ satisfies the condition 2−1), 2−2), 3). We shall compute $A'(x, t)$.

$$A(x+y, t)-A(x, t)=\bar{\partial}\bar{\partial}^*N((1+x+y)^{-1}\omega(t)(1+x+y)+(1+x+y)^{-1}\bar{\partial}(x+y))-\bar{\partial}\bar{\partial}^*N((1+x)^{-1}\omega(t)(1+x)+(1+x)^{-1}\bar{\partial}_x). \tag{4. 1}$$

From $(1+x+y)^{-1}=\{1+(1+x)^{-1}y\}^{-1}(1+x)^{-1}$, we have the following relation.

$$A'(x, t)y=\bar{\partial}\bar{\partial}^*N(-(1+x)^{-1}y(1+x)^{-1}\omega(t)(1+x)+(1+x)^{-1}\omega(t)y-(1+x)^{-1}y(1+x)^{-1}\bar{\partial}x+(1+x)^{-1}\bar{\partial}y). \tag{4. 2}$$

Lemma 4. 1 For any $\xi\in\Gamma(M, EndE\otimes\overset{q}{\wedge}(S)^*)$, $\eta\in\Gamma(M, EndE\otimes\overset{q}{\wedge}(\bar{S})^*)$ and $Q(\xi)\in\Gamma(M, End(End\otimes\overset{q}{\wedge}(\bar{S})^*))$ which depends on ξ and $j'(\xi)$, we have $\|Q(\xi)\eta\|_s\leq C_s\|\eta\|_s+C_s'\|\xi\|_{m_1+s}\|\eta\|_{m_1}$ for any $s\in\boldsymbol{Z}^+$, where m_1 is a natural number independent of s.

For the proof see Kuranishi [2].

Lemma 4. 2 For any C^∞ functions a and f with compact supports, there exist a constant c and an integer m such that

$$\|af\|_s \leqq C\|a\|_s \|f\|_m + C_s \|a\|_m \|f\|_{m+s},$$

where $\| \quad , \quad \|_s$ denotes the Sobolev s-norm.

For the proof see Kuranishi [2].

Lemma 4. 3 For any $\xi,\ \eta \in \Gamma(M,\ EndE \otimes \overset{p}{\wedge}(\bar{S})^*)$, we have

$$\|Q(\xi,\ \eta)\eta\|_s \leqq C_s \|\eta\|_{m+s} + C_s' \|\xi\|_{m+s} \|\eta\|_m,$$

where $Q(\xi,\ \eta)$ depends C^∞ on $\xi,\ \eta$ in $\Gamma(End(EndE \otimes \overset{p}{\wedge}(\bar{S})^*))$.

For the proof see Kuranishi [2].

Now, we shall prove the relation 2−1). From (4. 1) and (4. 2), we have

$$(4.\ 3) \qquad A(x+y,\ t) - A(x,\ t) - A'(x,\ t)y$$
$$= \bar{\partial}\bar{\partial}^* N(1+x+y)^{-1}\omega(t)(1+x+y) + (1+x+y)^{-1}\bar{\partial}(x+y) - (1+x)^{-1}\omega(t)$$
$$(1+x) - (1+x)^{-1}\bar{\partial}x + (1+x)^{-1}y(1+x)^{-1}\omega(t)(1+x) - (1+x)^{-1}\omega(t)y +$$
$$(1+x)^{-1}y(1+x)^{-1}\bar{\partial}x - (1+x)^{-1}\bar{\partial}y).$$

In order to estimate this, we will divide the right hand side of (4. 3) into two parts. One part is:

$$(4.\ 4) \qquad \bar{\partial}\bar{\partial}^* N((1+x+y)^{-1}\omega(t)(1+x+y) - (1+x)^{-1}\omega(t)(1+x)$$
$$+ (1+x)^{-1}y(1+x)^{-1}\omega(t)(1+x) - (1+x)^{-1}\omega(t)y).$$

The other is:

$$(4.\ 5) \qquad \bar{\partial}\bar{\partial}^* N((1+x+y)^{-1}\bar{\partial}(x+y) - (1+x)^{-1}\bar{\partial}x + (1+x)^{-1}y(1+x)^{-1}\bar{\partial}x - (1+x)^{-1}\bar{\partial}y).$$

(4. 4) becomes

$$(4.\ 6) \qquad \bar{\partial}\bar{\partial}^* N((1+x+y)^{-1}\omega(t)(1+x+y) - (1+x)^{-1}\omega(t)(1+x)$$
$$+ (1+x)^{-1}y(1+x)^{-1}\omega(t)(1+x)^{-1}(1+x) - \omega(t)y)$$
$$= \bar{\partial}\bar{\partial}^* N(\{(1+x+y)^{-1}\omega(t) - (1+x)^{-1}\omega(t) + (1+x)^{-1}y(1+x)^{-1}\omega(t)\}(1+x)$$
$$+ \{(1+x+y)^{-1}\omega(t) - (1+x)^{-1}\omega(t)\}y)$$
$$= \bar{\partial}\bar{\partial}^* N((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)$$
$$- (1+x+y)^{-1}y(1+x)^{-1}\omega(t)y).$$

(4. 5) becomes

(4. 7) $$\bar{\partial}\bar{\partial}^* N((1+x+y)^{-1}\bar{\partial}(x+y)-(1+x)^{-1}\bar{\partial}x+(1+x)^{-1}y(1+x)^{-1}\bar{\partial}x-(1+x)^{-1}\bar{\partial}y)$$
$$=\bar{\partial}\bar{\partial}^* N(\{(1+x+y)^{-1}-(1+x)^{-1}+(1+x)^{-1}y(1+x)^{-1}\}\bar{\partial}x$$
$$+\{(1+x+y)^{-1}-(1+x)^{-1}\}\bar{\partial}y)$$
$$=\bar{\partial}\bar{\partial}^* N((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\bar{\partial}x-(1+x+y)^{-1}y(1+x)^{-1}\bar{\partial}y).$$

From (4. 6) and (4. 7) we shall derive the inequality estimate 2−1).
By the first term of (4. 7), we have

(4. 8) $$\|\bar{\partial}\bar{\partial}^* N((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_s$$
$$\leqq C_s\|((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_s$$
$$\leqq C_s'\|(1+x)^{-1}y\|_s\|(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_m$$
$$+C_s'\|(1+x)^{-1}y\|_m\|(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_{m+s},$$

(By lemma 4. 2).
While by lemma 4. 1, we have

(4. 9) $$\|(1+x)^{-1}y\|_s\leqq C_s\|y\|_s+C_s'\|x\|_{m+s}\|y\|_m.$$

By lemma 4. 3, we have

(4. 10) $$\|(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_s$$
$$\leqq C_s\|y\|_{m+s}+C_s'\|x\|_{m+s}\|y\|_m.$$

And so by (4. 9) and (4. 10), we have

(4. 11) $$\|\bar{\partial}\bar{\partial}^* N((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\omega(t)(1+x)\|_s$$
$$\leqq C_s(C_s\|y\|_s+C_s'\|x\|_{m+s}\|y\|_m)(C_m\|y\|_{2m}+C_m'\|x\|_{2m}\|y\|_m)$$
$$+C_s(C_m\|y\|_m+C_m'\|x\|_{2m}\|y\|_m)(C_{m+s}\|y\|_{2m+s}+C_{m+s}'\|x\|_{2m+s}\|y\|_m)$$
$$\leqq C_s\|y\|_{s+2m}\|y\|_{2m}+C_s'\|x\|_{s+2m}\|y\|_{2m}^2.$$

Therefore we have the inequality estimate 2−1). Similarly, we can derive that the second term of (4. 6) satisfies the inequality estimate 2−1). We shall derive the quantity (4. 7) satisfies the estimate 2−1). The first term of (4. 7) becomes

(4. 12) $$\|\bar{\partial}\bar{\partial}^* N((1+x)^{-1}y(1+x+y)^{-1}y(1+x)^{-1}\bar{\partial}x)\|_s$$
$$\leqq C\|(1+x)^{-1}y(1+x+y)^{-1}\|_s\|y(1+x)^{-1}\bar{\partial}x\|_m$$
$$+C_s\|(1+x)^{-1}y(1+x+y)^{-1}\|_m\|y(1+x)^{-1}\bar{\partial}x\|_{m+s}.$$

While $\|(1+x)^{-1}y(1+x+y)^{-1}\|_s$ has the following estimate.

(4. 13) $$\|(1+x)^{-1}y(1+x+y)\|_s$$

$$\leqq C_s \| y \|_{m+s} + C_s' \| x \|_{m+s} \| y \|_m .$$

And

$$(4.\ 14) \qquad \| y(1+x)^{-1} \bar{\partial} x \|_s \leqq C_s \| y \|_s + C_s' \| x \|_{m+s} \| y \|_m .$$

Therefore we have the following inequality by (4. 11), (4. 12), (4. 13) and (4. 14).

$$(4.\ 15) \qquad \| \bar{\partial}\bar{\partial}^* N((1+x)^{-1} y(1+x+y)^{-1} y(1+x)^{-1} \bar{\partial} x) \|_s$$

$$\leqq C_s (C_s \| y \|_{m+s} + C_s' \| x \|_{m+s} \| y \|_m) (C_m \| y \|_m + C_m' \| x \|_{2m} \| y \|_m)$$

$$+ C_s (C_m \| y \|_{2m} + C_m' \| x \|_{2m} \| y \|_m) (C_{m+s} \| y \|_{m+s} \| x \|_{2m+s} \| y \|_m)$$

$$\leqq C_s \| y \|_{2m+s} \| y \|_{2m} + C_s' \| x \|_{2m+s} \| y \|_{2m}^2 .$$

Therefore we have the inequality estimate $2-1)$ for the equation (4. 11).

Similarly we can prove that the second term of (4. 7) satisfies estimate $2-1)$.

Now, we shall derive that A satisfies estimate $2-2)$. We shall study the inequality estimate $2-2)$. From (4. 1), we have

$$(4.\ 16) \qquad A'(x,\ t) y = \bar{\partial}\bar{\partial}^* N(-(1+x)^{-1} y(1+x)^{-1} \omega(t) (1+x) + (1+x)^{-1} \omega(t) y$$

$$-(1+x)^{-1} y(1+x)^{-1} \bar{\partial} x + (1+x)^{-1} \bar{\partial} y)$$

The first term of (4. 16) have the following inequality.

$$\| \bar{\partial}\bar{\partial}^* N(-(1+x)^{-1} y(1+x)\, \omega(t) (1+x)) \|_s$$

$$\leqq C_s \| y \|_s + C_s \| x \|_{m-s} \| y \|_m \quad \text{(By lemma 4. 1).}$$

Therefore we have that the first term of (4. 16) satisfies estimate $2-2)$. Similarly we can prove that the second term and the third term of (4. 16) satisfies the estimate $2-2)$. The fourth term of (4. 16) have the follwoing estimate.

$$\| (1+x)^{-1} \bar{\partial} y \|_s$$

$$\leqq C_s \| \bar{\partial} y \|_s + C_s' \| x \|_{m+s} \| \bar{\partial} y \|_m$$

$$\leqq C_s \| y \|_{s+1} + C_s' \| x \|_{m+s} \| y \|_{m+1} .$$

Therefore we have that A satisfies the inequality estimate $2-2)$.

Now, we shall prove that A satisfies estimate 3). From (4. 16), we have

$$(4.\ 17) \qquad A'(x,\ t) y = \bar{\partial}\bar{\partial}^* N(-(1+x)^{-1} y(1+x)^{-1} \omega(t) (1+x) + (1+x)^{-1} \omega(t) y$$

$$-(1+x)^{-1} y(1+x)^{-1} \bar{\partial} x + (1+x)^{-1} \bar{\partial} y)$$

$$= \bar{\partial}\bar{\partial}^* N(\bar{\partial}((1+x)^{-1} y) + ((1+x)^{-1} \omega(t) (1+x) +$$

$$+ (1+x)^{-1} \bar{\partial} x) (1+x)^{-1} y - (1+x)^{-1} y(1+x)^{-1}$$

$$\times\,\omega(t)\,(1+x)+(1+x)^{-1}\bar{\partial}x)).$$

Now, we will define the operator $K(x)$ from $EndE$ to $EndE$ as $K(x)y=(1+x)y$.

(4. 18) $$A'(x,\ t)\cdot K(x)y=\bar{\partial}\bar{\partial}^{*}N(\bar{\partial}y+(\omega\circ x)\,y-y(\omega\circ x)),$$

where $\omega\circ x=(1+x)^{-1}\omega(t)\,(1+x)+(1+x)^{-1}\bar{\partial}x$.

And we shall put the operator L and R as follows,

$$E=\bar{\partial}^{*}\Gamma(EndE\otimes(\bar{S})^{*})\ni y\xrightarrow{\ L\ }\bar{\partial}y\in F=\bar{\partial}\Gamma(EndE),$$

$$F=\bar{\partial}\Gamma(EndE)\ni Z\xrightarrow{\ R\ }\bar{\partial}^{*}NZ\in E=\bar{\partial}^{*}\Gamma\,(EndE\otimes(\bar{S})^{*}).$$

Then, by (4. 17) we have:

(4. 19) $$(A'(x,\ t)\circ K(x)\circ R)\,(Z)=Z+\bar{\partial}\bar{\partial}^{*}N(\omega\circ x\wedge\bar{\partial}^{*}NZ-\bar{\partial}^{*}NZ\wedge\omega\circ x).$$

Then we have that the operator $(A'(x,\ t)\circ K(x)\circ R)\,(Z)$ from $F=\bar{\partial}\Gamma(EndE)$ to $F=\bar{\partial}\Gamma$ $(EndE)$ is isomorphic at 0. In fact, putting S_xZ as $Z+\bar{\partial}\bar{\partial}^{*}N(\omega\circ x\wedge\bar{\partial}^{*}NZ\wedge\omega\circ x)$, we have the following relation:

(4. 19) $$\|(S_x-1)Z\|_s\leqq 2C\|\omega\circ x\wedge\bar{\partial}^{*}NZ\|_s$$

$$\leqq C'\|\omega\circ x\|_m\|\bar{\partial}^{*}NZ\|_s+C_s'\|\omega\circ x\|_{m+s}\|\bar{\partial}^{*}NZ\|_m\ .$$

While

(4. 20) $$\|\omega\circ x\|_m=\|(1+x)^{-1}\omega(t)\,(1+x)+(1+x)^{-1}\bar{\partial}x\|_m$$

$$\leqq C_m\|x\|_m+C'_m\|\omega(t)\|_m\ .$$

So, $\|x\|_m$ and $\|\omega(t)\|_m$ being so small, S_x satisfies the condition of Proposition 8. 1 in *M.* Kuranishi [2]. Therefore S_x has the inverse S_x^{-1}.

$$A'(x)\circ K(x)\circ R\circ S_x^{-1}=id_F.$$

Therefore, $A'(x)$ has the inverse $K(x)\circ R\circ S_x^{-1}$. So by lemma 4. 1, we have that A satisfies estimate 3).

References

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