On the existence of a versal family of deformations of solutions of a differential equation

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By

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Introduction

Let E, F C^∞−vector bundles over compact complex manifold M and D : Γ(E) Γ(F) an analytic polynomial-differential operator satisfying D(0)= 0. Let s(t) be a parametrized family of cross sections of E, where t moves in an analytic space T.

We say that s(t) is a deformation of the solution 0 if s(0)=0 and D(s(t))=0. In the present paper, we will show the existence of a versal family of deformations of the solution by the method of Kuranishi, in case the linearized operator L of D at 0 is elliptic. (Theorem. I)

Next, we shall assume, in addition to the above conditions, that,

1) D satisfies the condition(A) to be stated in §2,

and

2) $H^1(M, Ⓗ)=0$, where Ⓗ is the solution sheaf of the equation L(s)=0

Then, we can prove that the parameter space of the versal family is non-singular at 0. (Thorem. 2)

This is our main theorem. Kiso(Publ. RIMS. Kyoto Univ 10(1975). 763-776)has given a similar results. It seems to the author that our condition is simpler than his.

§1. The construction of a versal family of deformations

Let π : E→M and ρ : F→M be C^∞−vector bundles over a compact complex manifold M and D; Γ(E)→Γ(F) a differential operator of order k, not mecessarly linear with D(0)=0 and whose linearization L at 0 is a differential operator of the same order k.

Let s(t) be a parametrized family of cross sections of E, where t moves in an analytic space T and s(t) depends holomorphically on t. We say that s(t) is a deformation of the solution 0 if s(0) and D(s(t))=0.

Then, we can prove the following Theorem I,which is a simple variant of a theorem due to Kuranishi (2).

Theorem 1. Assume the following,

1) D is a polynomial operator of degree n and the linearization L of D at 0 is a polynomial operator of degree n.

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2) L is elliptic.

Then, there exists a versal family of deformation of 0.

(Proof) We take a Hermitean metric on M and ones along fibers of E and F respectively, into pre hilbert spaces.
We take Sobolev k norms on $C^\infty(E)$ (resp. $C^\infty(F)$), and denote by $\Gamma_k(E)$ and $\Gamma_k(F)$ the hibert spaces obtained by completing $C^\infty(E)$ and $C^\infty(F)$ respectively, by Sobolev k no rms. We shall also introduce the adjoint operator L^* of L, and consider the self adjoint strongly elliptic operator L^*L. As is usual in the theory of harmonic forms, we have the operator H of the projection onto harmonic part and the Green s operator G.

We recall the following fundametal fact. (See for the details(2))

Lemma 1. Let L^*L be a self adjoint linear strongry elliptic operator.

Then, we can prove the following.

1) There exists a complete ortho normal system $\{e_h : e_h \in \Gamma(E)\ h=1,2,\cdots\cdots\}$ such that e_h satisfie 's the relation $L^*Le_h = \lambda_h e_h$. where λ_h is a real number.

2) The harmonic operator H can be expressed as $H\psi = \sum_{\lambda_h=0}(\psi, e_h)e_h$, and $\sum$ is a finite The Green pperator G can be expressed as $G\psi = \sum_{\lambda_h \neq 0}(\psi, e^h)/\lambda^h e^h$ and G is a continuous operator $H_j \to H_{j+2k}$ (where 2k is the order of L^*L, and $H_j = \Gamma_j(E)$ is the Sobolev 1 normed space.) And for any $\psi \in \Gamma(E)$, we get the relations $L^*LG\psi + H\psi = \psi$ and $L^*LG = GL^*L$.

By virtue of Lemma I, we can prove the following lemmas. (Lemma 2 and Lemma 4)

Lemma 2. Every element in Ker L is perpendicular to $L\Gamma(E)$.

(Proof) For any in f Ker L^*, we have the relations $L^*f=0$
And so f satisfies the relation (L^*f, g) 0 for any element g in $\Gamma(E)$.
In other words $(f, Lg)=0$ for any g. QED.

Lemma 3. $H\Gamma_{k'}(E) = H^\circ(M, \textcircled{H})$ (where H is the Hamonic operator)

(Proof) For any element g in $H\Gamma_{k'}(E)$, we get the relations $L^*Lf=0$. And so Lf is perpendicular to $L\Gamma(E)$. Hence $Lf=0$. QED.

Lemma 4. $HL^*\Gamma(E)=0$

(Proof) For any element g in $\Gamma(F)$, we have the relations $(HLg, f)=(HLg, Hf)=(Lg, Hf)$ for any element f in $\Gamma(E)$. From Lemma 3, we get Lemma 4.
Now, we prove Theorem 1. QED.
The proof will be divided into four steps.
Step. 1 The solution φ of $D\varphi=0$ is in $\Phi = \{\varphi : \varphi \in \Gamma(E)$ and $\varphi + GL^*(D-L)\varphi \in H\Gamma(E)$
In fact, for any φ of the solution of $D\varphi=0$, satisfies

(1) $L\varphi + (D-L)\varphi = 0$

From this, we get the relations

(2) $L^*L\varphi + L^*(D-L)\varphi = 0$

(3) $\varphi + GL^*(D-L)\varphi = H\varphi \in H\Gamma(E)$. QED.

Step. 2 "The neighbourhood of 0 in $H^\circ(M, \textcircled{H})$ parametrizes a sufficiently small neighbourhood V of 0 in $\Omega = \{\varphi : \varphi E\Gamma_{k'}(E)$ and $\varphi + GL^*(P-L)\varphi \in H\Gamma(E)\}$"

To prove this, we will define a Banach analytic map D^1 from $\Gamma_{k'}(E)$ to $\Gamma_{k'}(E)$ by $D^1\varphi=\varphi+GL^*(D-L)\varphi$, then D^1 is isomorphic at 0 from the inverse mapping theorem for Banach analytic spaces. Denote by ϕ the restriction to $H\Gamma(E)$ of the map $(D^1)^{-1}$ the ϕ gives an isomorphism from a neighbourhood of 0 in $H^\circ(M,Ⓗ)$ onto the neighbourhood V in $\Phi=\{\varphi: \varphi\in\Gamma_{k'}(E)$ and $\varphi+GL^*(D-L)\varphi\in H\Gamma_{k'}(E)\}$ QED.

Step. 3 "$\phi(t)$ satisfies the relation $D\phi(t)=0$ if $((D-L)\phi(t), f_i)=0$ for an orthonormal system $\{f_i\}$ of Ker L^* and for any i."

In fact from the relation $\phi(t)+GL^*(D-L)\phi(t)=t$ for any t in $H\Gamma(E)$, we get the relation

$$(4)\qquad L\phi(t)+LGL^*(D-L)\phi(t)=0$$

Since $HL^*(D-L)\varphi=0$ by Lemma 4, we see $L^*LGL^*(D-L)\varphi$ and hence

$$(5)\qquad LGL^*(D-L)\phi(t)-(D-L)\phi(t)=-\sum_i((D-L)\phi(t), f_i)f_i$$ QED.

Step. 4 "$\phi(t)$ is in $\Gamma(E)$."

In fact from step 3, $\phi(t)$ satisfies the elliptic differential equation

$$(6)\qquad D\phi(t)=\sum_i((D-L)\phi(t), f_i)f_i$$

And so $\phi(t)$ is in $\Gamma(E)$. (See for the details theorem 5 in (3)) Therefore the analytic space $T=\{t\in H^\circ(M,Ⓗ)$, t is sufficiently small and $((D-L)\phi(t), f_i)=0$ for any $f_i\in$ Ker L^* parametrizes the solutions of $D\varphi=0$ which is sufficiently small.

§2. Smoothness

We shall define differential operators of type (A).

Definition 2. 1 By a differential operator D, from E to F of type (A), we mean the following.

There is a C vector bundle G and a differential operator D from F to G such that

1) $D_1D=0 \quad E\xrightarrow{D}F\xrightarrow{D_1}G$

2) $E\xrightarrow{L}F\xrightarrow{L_1}G$ is exact

Where L_1 is the linearized operator of D

Now, we can state and prove Theorem 2.

Theorem 2. In addition to the conditions in theorem 1,

Assume the followings.

1) $H^1(M.Ⓗ)=0$

2) D is a differential operator of type (A).

Then the parameter space of the versal family constructed in sectin 1 is non-singular at o.

(Proof) If the point o is isolated in the parameter space, there is nothing to prove. Let us suppose the contrary. The proof will be divided into two steps.

Step. 1 From the condition $H^1(M,Ⓗ)=0$, we see that L maps $\Gamma(M, E)$ onto $\Gamma(M, LE)$. Hence $L^*|\Gamma(M, LE)$ is injective.

Step. 2 The assumption that some of the equation $((D-L)\phi(t).f_i)$ 0 are non trivial leads to a contradiction, because of the relation $D\phi(t)=\sum_i((D-L)\phi(t),f_i)f_i$. Consider a curve through o in the neighbourhood of $H^\circ(M,\text{Ⓗ})$ and expand $((D-L)\phi(t),f_i)$ as $((D-L)\phi(t),f_i)=\sum a_{i_\mu}t_{i_\mu}$ where $i_1<i_2<$. and $a_{i_1}\neq 0$, t being the parameter of the curve. We define $V=\min\{i_1:i_1=1,2,\cdots\}$

Then (6) $D\phi(t)=a_{i_1}t^V f_i+$ terms of higher order in t.

We operate D on both sides of the equation (6), then from the assumption 2), we get the relation.

(7) $0=L_1(a_{i_1}f_i t^V)+$ terms of higher order in t.

We can conclude the relation $L_1(a_{i_1}f_i)$. And fron the condition that $E\xrightarrow{L}F\xrightarrow{L_1}G$ is exact, we see that $\sum_{i_1=V}a_{i_1}f_i$ is in $\Gamma(M,LE)$. But, f is in Ker L^*, and L^* is injective on $\Gamma(M,LE)$ by the step. 1.

Hence we have arrived at a contradiction.

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