Remarks on Prefrattini Subgroups of a finite Solvable Group

Remarks on Prefrattini subgroups of a finite solvable group

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1. The Prefrattini subgroups of a finite solvable group were introduced by W. Gaschütz in [3]. These are a characteristic conjugacy class of subgroups which avoid every complemented chief factor and cover every Frattini chief factor. W. Gaschütz [3, Satz 6.4] showed that if W is a Prefrattini subgroup of a finite solvable group G and N is a normal subgroup of G, then WN/N is a Prefrattini subgroup of G/N.

By using Sylow systems, T. Hawkes [4] introduced a characteristic conjugacy class of subgroups of a finite solvable group corresponding to a saturated formation $\mathfrak{F}$, called the $\mathfrak{F}-$Prefrattini subgroups which avoid every complemented $\mathfrak{F}-$eccentric chief factor and cover the rest. T. Hawkes [4, Theorem 4.1] showed that an $\mathfrak{F}-$Prefrattini subgroups of a finite solvable group can be expressed as a Product of a Prefrattini subgroup and an $\mathfrak{F}-$normalizer of the group.

It is described in [4] that the Prefrattini subgroup of a finite solvable group is not selfnormalizing, and this fact can be proved by using the above theorem. In this note, we give an alternative proof by using the concept of the Prefrattini residual introduced by H. Bechtell in [1] and consider the nilpotency of the Prefrattini subgroups of a finite solvable group.

All groups considered are finite and solvable. Our notation and terminology is standard, and can be found in [2].

2. Let $\mathfrak{S}$ be a Sylow system of a finite solvable group G and S^p the Sylow $p-$complement of G contained in $\mathfrak{S}$. As in [4], we let

$$M^p(S^p) = \cap\{M;\ M \text{ is an maximal subgroup of } G \text{ containing } S^p\}.$$

T. Hawkes showed in [4] that the subgroups $W(\mathfrak{S}) = \cap\{M^p(S^p);\ S^p \in \mathfrak{S}\}$ coincide with the Prefrattini subgroups formulated by W. Gaschütz in [3], and we call it the Prefrattini subgroups of G corresponding to $\mathfrak{S}$. The system normalizers of G corresponding $\mathfrak{S}$ were introduced by P. Hall. They are the subgroups $D(\mathfrak{F}) = \cap\{N_G(S^p);\ S^p \in \mathfrak{S}\}$ which cover every central chief factor and avoid every eccentric chief factor.

LEMMA 1 [4, p.149]. *Let D be a system normalizer of G and W be a Prefrattini subgroup of G corresponding to $\mathfrak{S}$. Then D normalizes W.*

PROOF. Let g be any element of D. Then by the definition of D, g normalizes

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any Sylow p-complement S^p in $\mathfrak{S}$. Therefore, since conjugate subgroups of a maximal subgroup of G are also maximal, g normalizes $M^p(S^p)$ for each S^p in $\mathfrak{S}$. Thus g normalizes W and so D normalizes W.

LEMMA 2. *Let D be a system normalizer of G. Then G is the normal closure of D in G.*

PROOF. Let D^G be the normal closure of D in G and suppose that $G \neq D^G$ Then there is a maximal normal subgroup M of G containing D^G since D^G is a normal subgroup of G. Now G/M is a central chief factor of G. Therefore D covers G/M and so D is not contained in M, which is a contradiction. Thus $G=D^G$

A nonempty class $\mathfrak{F}$ of a finite solvable group is a formation if :

(i) $\mathfrak{F}$ contains all homomorphic images of groups in $\mathfrak{F}$, and

(ii) if G/M and G/N are in $\mathfrak{F}$ then $G/(M \cap N)$ is in $\mathfrak{F}$, for any normal subgroups M, N of G.

From the definition of a formation $\mathfrak{F}$, it follows that every group G possesses the smallest normal subgroup $G_{\mathfrak{F}}$, called $\mathfrak{F}$-residual of G, such that $G/G_{\mathfrak{F}}$ is in $\mathfrak{F}$.

PROPOSITION 3. *Let $\mathfrak{F}$ be a class of finite solvable groups whose the Prefrattini subgroup is trivial. Then $\mathfrak{F}$ is a formation.*

PROOF. Suppose that G is in $\mathfrak{F}$ and let N be any normal subgroup of G. Then, since the Prefrattini subgroup of G is trivial, the Prefrattini subgroup of G/N is also trivial by [3, Satz 6.4]. Thus G/N is in $\mathfrak{F}$. Now let M and N be tow normal subgroups of G and suppose that G/M and G/N are in $\mathfrak{F}$. Then, by [3, Satz 6.4], the Prefrattini subgroup W of G is contained in M and N, and so in $M \cap N$. Therefore the Prefrattine subgroup $W(M \cap N)/(M \cap N)$ of $G/(M \cap N)$ is trivial and hence $G/(M \cap N)$ is in $\mathfrak{F}$. The proof is now complete.

LEMMA 4. [1, (1. 1)]. *Let $G_{\mathfrak{F}}$ be the $\mathfrak{F}$-residual of G for the above formation $\mathfrak{F}$ and W is a Prefrattini subgroup of G. Then $G_{\mathfrak{F}}$ is the normal closure of W in G.*

PROOF. Let W^G be the normal closure of W in G. Since $G/G_{\mathfrak{F}}$ is in $\mathfrak{F}$, we have $W \subseteq G_{\mathfrak{F}}$ by [3, Satz 6.4] and the construction of $\mathfrak{F}$. Hence $W^G \subseteq G_{\mathfrak{F}}$ since $G_{\mathfrak{F}}$ is a normal subgroup of G. On the other hand, the Prefrattini subgroup of G/W^G is trivial. Therefore G/W^G is in $\mathfrak{F}$ and hence $G_{\mathfrak{F}} \subseteq W^G$ by the definition of $\mathfrak{F}$-residual. The proof is now complete.

LEMMA 5. *Let $G_{\mathfrak{F}}$ be the $\mathfrak{F}$-residual of G for the above formation $\mathfrak{F}$. Then $G_{\mathfrak{F}}$ is a proper subgroup of G.*

PROOF. Let M be a maximal normal subgroup of G. Now G/M is a group of order p for some prime p. Therefore $\Phi(G/M)=1$ and hence G/M is a complemented chief factor of G. By the covering-avoidance property of the Prefrattini subgroup,

G/M is in $\mathfrak{F}$ and hence we obtain $G_{\mathfrak{F}} \subseteq M$, which implies that $G_{\mathfrak{F}}$ is a proper subgroup of G.

PROPOSITION 6. *The Prefrattini subgroup of G is not selfnormalizing.*

RROOF. Let W be a Prefrattini subgroup of G, D a system normalizer of G and D^G, W^G the normal closure of D, W, respectively. Suppose that $N_G(W) = W$. Then we have $D \subseteq W$ by lemma 1 and hence $D^G \subseteq W^G$. Therefore by lemma 2 and lemma 4 we obtain $G = G_{\mathfrak{F}}$ which contradicts lemma 5. Hence $N_G(W) \neq W$.

3. We shall make use of the well known facts that the Fitting subgroup $F(G)$ of a finite solvable group G is the intersection of the centralizers of all the chief factors of G, and the greatest normal p-nilpotent subgroup $O_{p'p}(G)$ is the intersection of the centralizers of all the p-chief factors of G. (see [5, VI, Satz 5.4]).

PROPOSITION 7. *Let W be a Prefrattini subgroup of G. Then W is nilpotent if and only if, for any Frattini chief factor H/K of G if H/K is a p-chief factor, then $W/C_W(H/K)$ is a p-group.*

PROOF. Let H/K be any Frattini chief factor of G. Since W covers H/K, H is a subgroup of WK. Therefore as W-groups, H/K is isomorphic to $(W \cap H)/(W \cap K)$, so that $C_W(H/K) = C_W((W \cap H)/(W \cap K))$. Now, suppose that W is nilpotent. Then we have

$$W = W_{p_1} \times \cdots \times W_{p_n}$$

where each W_{p_i} is the Sylow p_i-subgroup of W and hence

$$W/(W \cap K) = W_{p_1}(W \cap K)/(W \cap K) \times \cdots \times W_{p_n}(W \cap K)/(W \cap K)$$

If H/K is a p-group, say $p = p_1$, then $(W \cap H)/(W \cap K)$ is p_1-subgroup of $W/(W \cap K)$ and hence $(W \cap H)/(W \cap K)$ is contained in $W_{p_1}(W \cap K)/(W \cap K)$.

Therefore we have for each p_i $(i = 2, \cdots, n)$ $(i = 2, \cdots, n)$

$$W_{p_i}(W \cap K)/(W \cap K) \subseteq C_{W/(W \cap K)}((W \cap H)/(W \cap K))$$

and hence

$$W_{p_i} \subseteq C_W((W \cap H)/(W \cap K)) = C_W(H/K)$$

Thus $C_W(H/K)$ contains every p'-element of W and so $W/C_W(H/K)$ is a p-group as required.

Let H/K be any Frattini p-chief factor of G such that every chief factor below K is complemented chief factor of G. Then we have $H \subseteq WK$ and $W \cap K = 1$, by the covering-avoidance property of the Prefrattini subgroup of G.

Suppose that for p-chief factor H/K of G, $W/C_W(H/K)$ is a p-group. Now by induction on group order, we prove that W is nilpotent. Assume first that $K = 1$. Then H is a minimal normal p-subgroup of G such that $H \subseteq \Phi(G)$, and H is a subgroup of W. By induction, it follow that W/H is nilpotent. Let

$$1 = A_0 < \cdots < A_r = H < \cdots < A_n = W$$

be a chief series of W through H. Since W/H is nilpotent, it follows that

$$W/H = C_{W/H}((A_i/H)/(A_{i-1}/H))$$

for each $i \geq r+1$. Thus we have $W = C_W(A_i/A_{i-1})$ for each $i \geq r+1$.

On the other hand, since $W/C_W(H)$ is a p-group and $C_W(H) \subseteq C_W(A_i/A_{i-1})$ for each $1 \leq i \leq r$, $W/C_W(A_i/A_{i-1})$ is a p-group for each $1 \leq i \leq r$. Therefore we have

$$O_{p'p}(W) = \cap \{C_W(A_i/A_{i-1});\ p \mid \ |A_i/A_{i-1}|\} = \cap \{C_W(A_i/A_{i-1});\ 1 \leq i \leq r\}$$

and hence $W/O_{p'p}(W)$ is a p-group. It is known that this implies $W = O_{p'p}(W)$. Thus we have $W = C_W(A_i/A_{i-1})$ for each $1 \leq i \leq r$. It follows that $W = F(W)$ and W is nilpotent as required.

Thus we may assume that $K \neq 1$. Now we consider the factor group G/K. Then WK/K is a Prefrattini subgroup of G/K and H/K is a minimal normal p-subgroup of G/K such that $H/K \subseteq \Phi(G/K)$ Therefore it follows from the above case that WK/K is nilpotent and since $W \cap K = 1$, W is nilpotent as required.

References

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