STRUCTURE OF GROUP C*-ALGEBRAS OF LIE SEMI-DIRECT PRODUCTS OF $ {\Bbb R} $ OR $ {\Bbb C} $ BY LIE GROUPS

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STRUCTURE OF GROUP C^*-ALGEBRAS OF LIE SEMI-DIRECT PRODUCTS OF $\mathbb{R}$ OR $\mathbb{C}$ BY LIE GROUPS

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ABSTRACT. This paper is devoted to the case by case study on structure of group C^*-algebras of the Lie semi-direct products of the 1-dimensional real or complex Lie groups by connected Lie groups, in particular, by commutative Lie groups or the generalized Heisenberg Lie groups. As corollaries, we determine existence and nonexistence of projections of these special group C^*-algebras in terms of groups, and obtain some generalized results for more general C^*-algebras by another methods.

§1. INTRODUCTION

We first recall that the $(2n+1)$-dimensional, generalized Heisenberg Lie group H_{2n+1} consists of all $(n+2)\times(n+2)$ matrices as follows:

$$(c,b,a)=\begin{pmatrix} 1 & a_1 & \cdots & a_n & c \\ & 1 & & & b_1 \\ & & \ddots & & \vdots \\ & & & 1 & b_n \\ 0 & & & & 1 \end{pmatrix} \quad a_i,b_i,c\in\mathbb{R}(1\le i\le n)$$

with $a=(a_i), b=(b_i)\in\mathbb{R}^n$. Then H_{2n+1} is a simply connected nilpotent Lie group, and isomorphic to the Lie semi-direct product $\mathbb{R}^{n+1}\rtimes_\alpha\mathbb{R}^n$ with the action α of $\mathbb{R}^n$ on $\mathbb{R}^{n+1}$ defined by $\alpha_a(c,b)=(c+\sum_{i=1}^n a_ib_i, b)$.

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We now set up some notations. For a locally compact Hausdorff space X, we denote by $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity, and by $C(X)$ it when X is compact. We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. For a Lie group G, we denote by $C^*(G)$ its group C^*-algebra (cf.[Dx]).

We next review the structure of $C^*(H_{2n+1})$. Since H_{2n+1} is isomorphic to $\mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^n$, we have that $C^*(H_{2n+1})$ is isomorphic to the crossed product $C^*(\mathbb{R}^{n+1}) \rtimes_\alpha \mathbb{R}^n$. Then the Fourier transform induces the isomorphism: $C^*(\mathbb{R}^{n+1}) \rtimes_\alpha \mathbb{R}^n \cong C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{R}^n$, where the action $\hat{\alpha}$ is defined by $\hat{\alpha}_a(c', b') = (c', (b'_i + a_i c'))$ for $a_i, b'_i, c' \in \mathbb{R}$ with $b' = (b'_i) \in \mathbb{R}^n$. Since the action $\hat{\alpha}$ is trivial on $\{0\} \times \mathbb{R}^n$, and free and wandering on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ (See [Gr1] for the precise definition of "wandering"), we get the following exact sequence:

$$0 \to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes \mathbb{R}^n \to C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{R}^n \to C_0(\mathbb{R}^{2n}) \to 0,$$

and $C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K}$ by Green's result [Gr1, Corollary 15]. Refer to [Gr1], [Gr2], [Rs], [Sd1,2,3] and [Zp] for more other cases.

In this paper we first consider structure of group C^*-algebras of the (solvable) Lie semi-direct products of $\mathbb{R}$ by $\mathbb{R}^n$ or H_{2n+1} in details. By the same way, we next analyze the cases of the (solvable) Lie semi-direct products of $\mathbb{C}$ by connected commutative Lie groups or H_{2n+1}. As corollaries, we determine the conditions that these group C^*-algebras have nontrivial projections or does not have in terms of groups. For analysis of structure of basic building blocks of these group C^*-algebras, Green's imprimitivity theorem [Gr2, Corollary 2.10] serves a crucial role. More general cases for the Lie semi-direct products of commutative Lie groups $\mathbb{R}^u \times \mathbb{C}^v$ by connected Lie groups with diagonal actions are investigated in [Sd3] on the similar manner. This research gives the more exact description for the cases where $u = 0$ or $v = 0$. About the projection problem for more general (group) C^*-algebras, some results are obtained in each chapter.

§2. Lie semi-direct products of $\mathbb{R}$ by $\mathbb{R}^n$ or H_{2n+1}

Lie semi-direct products of $\mathbb{R}$ by $\mathbb{R}^n$.

In this subsection we analyze structure of group C^*-algebras of the Lie semi-direct products $G = \mathbb{R} \rtimes_\alpha \mathbb{R}^n$. Then we may assume that the action α of $\mathbb{R}^n$ on $\mathbb{R}$ is given by

$$\alpha_g(t) = e^{(\sum_{k=1}^{u} g_{i_k})}t, \quad g = (g_i) \in \mathbb{R}^n, t \in \mathbb{R}$$

where $\{i_k\}_{k=1}^{u}$ is a subset of $\{i\}_{i=1}^{n}$. If α is trivial, then we have $C^*(G) \cong C_0(\mathbb{R}^{n+1})$. We assume that α is nontrivial in the following. By the Fourier transform, $C^*(G) \cong C_0(\mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ where $\hat{\alpha} = \alpha$ in this case. Since the origin $\{0\}$ of $\mathbb{R}$ is fixed under $\hat{\alpha}$, we have that

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \rtimes \mathbb{R}^n \to C_0(\mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}^n \to C_0(\mathbb{R}^n) \to 0.$$

As the subspace $\mathbb{R}_+$ of all positive real numbers of $\mathbb{R} \setminus \{0\}$ is invariant under α, the above ideal is isomorphic to the 2-direct sum $\oplus^2(C_0(\mathbb{R}_+) \rtimes \mathbb{R}^n)$. Since the action of $\mathbb{R}^n$ on $\mathbb{R}_+$ is transitive, by Green's imprimitivity theorem [Gr2, Corollary 2.10], each direct summand is isomorphic to the following C^*-tensor product:

$$C_0(\mathbb{R}_+) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R}^n/(\mathbb{R}^n)_1) \rtimes \mathbb{R}^n \cong C^*((\mathbb{R}^n)_1) \otimes \mathbb{K}$$

where $(\mathbb{R}^n)_1$ is the stabilizer of $1 \in \mathbb{R}_+$, which is isomorphic to the product group of $\mathbb{R}^{n-u}$ and the closed subgroup of $\mathbb{R}^u$ defined by the hyperplane

$$\{(g_{i_k})_{k=1}^{u} \in \mathbb{R}^u : \sum_{k=1}^{u} g_{i_k} = 0\}.$$

Hence $(\mathbb{R}^n)_1 \cong \mathbb{R}^{n-u} \times \mathbb{R}^{u-1} \cong \mathbb{R}^{n-1}$.

Summing up we get the following:

Theorem 2.1. *Let G be a nontrivial Lie semi-direct product of $\mathbb{R}$ by $\mathbb{R}^n$. Then $C^*(G)$ has the following exact sequence:*

$$0 \to \oplus^2(C_0(\mathbb{R}^{n-1}) \otimes \mathbb{K}) \to C^*(G) \to C_0(\mathbb{R}^n) \to 0$$

where the stabilizer of $1 \in \mathbb{R}_+$ under the action of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^{n-1}$.

Remark. When $n = 1$, G is the proper $ax + b$ group $\mathbb{R} \rtimes \mathbb{R}$ (cf.[Zp]).

Corollary 2.2. *Let G be a nontrivial Lie semi-direct product of $\mathbb{R}$ by $\mathbb{R}^n$. Then we have that*

$$C^*(G) \text{ has } \begin{cases} \text{nontrivial projections when } n = 1, \\ \text{no nontrivial projections if } n \geq 2. \end{cases}$$

Proof. We remark that for any noncompact, connected locally compact space X and any C^*-algebra $\mathfrak{A}$, both $C_0(X)$ and $C_0(X) \otimes \mathfrak{A}$ have no nontrivial projections.

If $n = 1$, the ideal $\mathbb{K} \oplus \mathbb{K}$ has nontrivial projections. Then so does $C^*(G)$.

If $n \geq 2$, the ideal and quotient in the above exact sequence have no nontrivial projections. Hence so does $C^*(G)$.

□

Remark. We notice that any automorphic action of $\mathbb{T}$ on $\mathbb{R}$ is trivial. So if G is a Lie semi-direct product $\mathbb{R} \rtimes (\mathbb{R}^n \times \mathbb{T}^m)$, then we have that

$$C^*(G) \cong C^*(\mathbb{R} \rtimes \mathbb{R}^n) \otimes C_0(\mathbb{Z}^m).$$

Therefore, we have that $C^*(G)$ has no nontrivial projections if and only if $n = 1$, that is, G is isomorphic to $(\mathbb{R} \rtimes \mathbb{R}) \times \mathbb{T}^m$ $(m \geq 0)$.

Lie semi-direct products of $\mathbb{R}$ by H_{2n+1}.

In this subsection we consider structure of group C^*-algebras of the Lie semi-direct products $G = \mathbb{R} \rtimes_\alpha H_{2n+1}$. Then we may assume that the action α of H_{2n+1} on $\mathbb{R}$ is given by

$$\alpha_{(c,b,a)}(t) = e^{(\sum_{k=1}^{u} b_{i_k} + \sum_{k=1}^{v} a_{j_k})}t \quad (c, b, a) \in H_{2n+1}, t \in \mathbb{R}$$

for some $0 \leq u, v \leq n$. Denote by w the cardinal number of the union of $\{i_k\}_{k=1}^{u}$ and $\{j_k\}_{k=1}^{v}$. We assume that α is nontrivial. By the Fourier transform, we have $C^*(G) \cong C_0(\mathbb{R}) \rtimes_{\hat{\alpha}} H_{2n+1}$, where $\hat{\alpha} = \alpha$ in this case. Since the origin $\{0\}$ of $\mathbb{R}$ is fixed under $\hat{\alpha}$, the following exact sequence is obtained:

$$\begin{aligned} 0 \to C_0(\mathbb{R} \setminus \{0\}) \rtimes H_{2n+1} &\cong \oplus^2 (C_0(\mathbb{R}_+) \rtimes H_{2n+1}) \\ \to C_0(\mathbb{R}) \rtimes H_{2n+1} &\to C^*(H_{2n+1}) \to 0. \end{aligned}$$

Moreover, the action of H_{2n+1} on $\mathbb{R}_+$ is transitive. So by Green's imprimitivity theorem [Gr2],

$$\begin{aligned} C_0(\mathbb{R}_+) \rtimes H_{2n+1} &\cong C_0(H_{2n+1}/(H_{2n+1})_1) \rtimes H_{2n+1} \\ &\cong C^*((H_{2n+1})_1) \otimes \mathbb{K} \end{aligned}$$

where $(H_{2n+1})_1$ is the stabilizer of $1 \in \mathbb{R}_+$, which is isomorphic to

$$\begin{cases} (\mathbb{R} \times \mathbb{R}^{n-1}) \rtimes_\gamma \mathbb{R}^n & \text{if } u \neq 0, \\ \mathbb{R}^{n+1} \rtimes_\rho \mathbb{R}^{n-1} & \text{if } u = 0 \end{cases}$$

with the actions γ, ρ defined by

$$\gamma_a(c, (b_i)_{i\neq i_1}^n) = (c + \sum_{k\neq i_1}^{n} a_k b_k - a_{i_1} \sum_{k=2}^{u} b_{i_k}, (b_i)_{i\neq i_1}^n),$$

$$\rho_{(a_k)_{k\neq j_1}^n}(c, b) = (c + \sum_{k\neq j_1}^{n} a_k b_k - (\sum_{k=j_1}^{v} a_{j_k}) b_{j_1}, b).$$

Then by the same methods as the analysis of $C^*(H_{2n+1})$ in the introduction, each $C^*((H_{2n+1})_1)$ is decomposed into the following:

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*((H_{2n+1})_1) \to C_0(\mathbb{R}^{2n-1}) \to 0.$$

From the above analysis we get the following:

Theorem 2.3. *Let G be a nontrivial Lie semi-direct product of $\mathbb{R}$ by H_{2n+1}. Then $C^*(G)$ decomposes into the following exact sequence:*

$$0 \to \oplus^2(C^*((H_{2n+1})_1) \otimes \mathbb{K}) \to C^*(G) \to C^*(H_{2n+1}) \to 0.$$

Moreover, we have that

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*((H_{2n+1})_1) \to C_0(\mathbb{R}^{2n-1}) \to 0.$$

Combining the above theorem with the structure of $C^*(H_{2n+1})$ given in the introduction, we obtain that

Corollary 2.4. *If G is a nontrivial Lie semi-direct product of $\mathbb{R}$ by H_{2n+1}, then there exists a composition series $\{\mathfrak{I}_j\}_{j=1}^4$ of $C^*(G)$ such that*

$$\mathfrak{I}_3/\mathfrak{I}_2 \cong C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \qquad \mathfrak{I}_4/\mathfrak{I}_3 \cong C_0(\mathbb{R}^{2n})$$
$$\mathfrak{I}_1 \cong \oplus^4(C_0(\mathbb{R}) \otimes \mathbb{K}) \qquad \mathfrak{I}_2/\mathfrak{I}_1 \cong \oplus^2(C_0(\mathbb{R}^{2n-1}) \otimes \mathbb{K})$$

Moreover, we deduce from the same reason with Corollary 2.2 that

Corollary 2.5. *If G is a (solvable) Lie semi-direct product of $\mathbb{R}$ by H_{2n+1}, then $C^*(G)$ has no nontrivial projections.*

Remark. We have that $C^*(H_{2n+1})$ itself has no nontrivial projections. More generally, it is shown by [Sd1] that the group C^*-algebras of simply connected nilpotent Lie groups have no nontrivial projections.

In a more general situation, we have that

Proposition 2.6. *Let G be a nontrivial Lie semi-direct product of $\mathbb{R}$ by a connected Lie group N. Then $C^*(G)$ has the following exact sequence:*

$$0 \to \oplus^2(C^*(N_1) \otimes \mathbb{K}) \to C^*(G) \to C^*(N) \to 0$$

where N_1 means the stabilizer of $1 \in \mathbb{R}_+$.

Proof. We notice that the quotient group $N/[N,N]$ by the commutator $[N,N]$ of N is isomorphic to $\mathbb{R}^{n-m} \times \mathbb{T}^m$ for some $n \geq 1$, $m \geq 0$. Since $\mathbb{R}$ is 1-dimensional, it is clear that the action of N on $\mathbb{R}$ is reduced to that of $N/[N,N]$. Moreover, the action of $\mathbb{T}^m$ on $\mathbb{R}$ is trivial. Therefore, we have the same conclusion of the first exact sequence of Theorem 2.3.

□

Theorem 2.7. *Let G be a Lie semi-direct product of $\mathbb{R}$ by a simply connected nilpotent Lie group N. Then*

$$C^*(G) \text{ has} \begin{cases} \text{nontrivial projections if } N = \mathbb{R}, \\ \text{no nontrivial projections if otherwise.} \end{cases}$$

Proof. By Proposition 2.6, if $C^*(N)$ and $C^*(N_1) \otimes \mathbb{K}$ have no nontrivial projections, then so does $C^*(G)$. We note that $N/N_1 \cong \mathbb{R}$.

Thus $N \cong N_1 \rtimes \mathbb{R}$. Therefore, N_1 is a simply connected nilpotent Lie group. By [Sd1], $C^*(N)$ and $C^*(N_1)$ have no nontrivial projections. Tensoring $\mathbb{K}$ with the structure of $C^*(N_1)$ obtained in [Sd1], we deduce that $C^*(N_1) \otimes \mathbb{K}$ also has no nontrivial projections if N_1 is nontrivial.

$\square$

§3. Lie semi-direct products of $\mathbb{C}$ by commutative Lie groups or H_{2n+1}

Lie semi-direct products of $\mathbb{C}$ by commutative Lie groups.

In this subsection we consider structure of group C^*-algebras of the Lie semi-direct products of $\mathbb{C}$ by connected commutative Lie groups. It is well known that any connected commutative Lie group is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m$ for some $n, m \geq 0$..

First of all, we consider the case $G = \mathbb{C} \rtimes_\alpha \mathbb{R}^n$. Then the automorphic action α is given by

$$\alpha_g(z) = e^{(\sum_{k=1}^{u} z_{i_k} g_{i_k})} z \quad g = (g_i) \in \mathbb{R}^n, z, z_{i_k} \in \mathbb{C}$$

for $0 \leq u \leq n$. Then $C^*(G) \cong C_0(\mathbb{C}) \rtimes_{\hat{\alpha}} \mathbb{R}^n$ by the Fourier transform, where $\hat{\alpha}_g$ is given by the complex conjugate of α_g. Since the origin $\{0\}$ of $\mathbb{C}$ is fixed under $\hat{\alpha}$, we obtain that

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^n \to C_0(\mathbb{C}) \rtimes \mathbb{R}^n \to C_0(\mathbb{R}^n) \to 0.$$

First suppose that all $(z_{i_k})_{k=1}^u$ are purely imaginary. Then the action on the radius direction of $\mathbb{C}$ is trivial, so that

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes \mathbb{R}^n).$$

Since the nontrivial action of $\mathbb{R}^n$ on $\mathbb{T}$ is transitive, Green's imprimitivity theorem [Gr2] implies that

$$C(\mathbb{T}) \rtimes \mathbb{R}^n \cong C(\mathbb{R}^n/(\mathbb{R}^n)_1) \rtimes \mathbb{R}^n \cong C^*((\mathbb{R}^n)_1) \otimes \mathbb{K}$$

where the stabilizer $(\mathbb{R}^n)_1$ of $1 \in \mathbb{T}$ is isomorphic to the product group of $\mathbb{R}^{n-u}$ and the closed subgroup of $\mathbb{R}^u$ defined by the union $\cup_{t\in\mathbb{Z}} P_t$ of the hyperplanes

$$P_t = \{(g_{i_1}, \cdots, g_{i_u}) \in \mathbb{R}^u : \sum_{k=1}^{u} z_{i_k} g_{i_k} = 2t\pi i\}, \quad t \in \mathbb{Z}.$$

Therefore, we get that $(\mathbb{R}^n)_1 \cong \mathbb{Z} \times \mathbb{R}^{u-1} \times \mathbb{R}^{n-u}$. Hence we have $C^*((\mathbb{R}^n)_1) \cong C_0(\mathbb{T} \times \mathbb{R}^{n-1})$.

Next suppose that for some $0 < l < u$, $(z_{i_k})_{k=1}^{l}$ are purely imaginary and $(z_{i_k})_{k=l+1}^{u}$ are not purely imaginary. Then we may assume that the nontrivial action of $\mathbb{R}^n$ on $\mathbb{C} \setminus \{0\}$ is transitive. By Green's imprimitivity theorem,

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R}^n/(\mathbb{R}^n)_1) \rtimes \mathbb{R}^n \cong C^*((\mathbb{R}^n)_1) \otimes \mathbb{K}$$

where $(\mathbb{R}^n)_1$ is equal to the product group of $\mathbb{R}^{n-u}$ and the closed subgroup of $\mathbb{R}^u$ defined by the union $\cup_{t\in\mathbb{Z}}(P \cap Q_t)$ of the intersections $P \cap Q_t$ of the hyperplanes defined by

$$P = \{(g_{i_1}, \cdots, g_{i_u}) \in \mathbb{R}^u : \sum_{k=l+1}^{u} \mathrm{Re}(z_{i_k}) g_{i_k} = 0\}$$

and the hyperplanes defined by

$$Q_t = \{(g_{i_1}, \cdots, g_{i_u}) \in \mathbb{R}^u : \sum_{k=1}^{u} \mathrm{Im}(z_{i_k}) g_{i_k} = 2t\pi\}$$

for $t \in \mathbb{Z}$, where $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ mean the real, imaginary parts of z respectively. Thus we obtain that $(\mathbb{R}^n)_1 \cong \mathbb{Z} \times \mathbb{R}^{u-2} \times \mathbb{R}^{n-u}$ so that $C^*((\mathbb{R}^n)_1) \cong C_0(\mathbb{T} \times \mathbb{R}^{n-2})$.

Finally suppose that all $(z_{i_k})_{k=1}^{u}$ are not purely imaginary. We first note that the crossed product $C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^n$ is isomorphic to $C_0(\mathbb{R}^{n-u}) \otimes (C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^u)$. Then the action of $\mathbb{R}^u$ on $\mathbb{C} \setminus \{0\}$ is not transitive if and only if all $(z_{i_k})_{k=1}^{u}$ are linearly dependent as a vector of $\mathbb{R}^2$. In this case we may assume that all $(z_{i_k})_{k=1}^{u}$ are real. Hence we get that

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^u \cong C(\mathbb{T}) \otimes (C_0(\mathbb{R}_+) \rtimes \mathbb{R}^u).$$

Since the action of $\mathbb{R}^u$ on $\mathbb{R}_+$ is transitive, by Green's imprimitivity theorem [Gr2],

$$C_0(\mathbb{R}_+) \rtimes \mathbb{R}^u \cong C_0(\mathbb{R}^u/(\mathbb{R}^u)_1) \rtimes \mathbb{R}^u \cong C^*((\mathbb{R}^u)_1) \otimes \mathbb{K}$$

where the stabilizer $(\mathbb{R}^u)_1$ is isomorphic to the closed subgroup of $\mathbb{R}^u$ defined by

$$\{(g_{i_k})_{k=1}^u \in \mathbb{R}^u : \sum_{k=1}^u z_{i_k} g_{i_k} = 0\}.$$

Hence $(\mathbb{R}^u)_1 \cong \mathbb{R}^{u-1}$. Thus $C^*((\mathbb{R}^u)_1) \cong C_0(\mathbb{R}^{u-1})$.

If the action of $\mathbb{R}^u$ on $\mathbb{C} \setminus \{0\}$ is transitive, then Green's imprimitivity theorem gives that

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^u \cong C_0(\mathbb{R}^u/(\mathbb{R}^u)_1) \rtimes \mathbb{R}^u \cong C^*((\mathbb{R}^u)_1)) \otimes \mathbb{K},$$

where the stabilizer $(\mathbb{R}^u)_1$ is given by the union $\cup_{t\in\mathbb{Z}}(P' \cap Q'_t)$ of the intersections $P' \cap Q'_t$ of the hyperplane

$$P' = \{(g_{i_k})_{k=1}^u \in \mathbb{R}^u : \sum_{k=1}^u \mathrm{Re}(z_{i_k}) g_{i_k} = 0\}$$

and the hyperplanes $(t \in \mathbb{Z})$

$$Q'_t = \{(g_{i_k})_{k=1}^u \in \mathbb{R}^u : \sum_{k=1}^u \mathrm{Im}(z_{i_k}) g_{i_k} = 2t\pi\}.$$

Thus $(\mathbb{R}^u)_1 \cong \mathbb{Z} \times \mathbb{R}^{u-2}$. Hence $C^*((\mathbb{R}^u)_1) \cong C_0(\mathbb{T} \times \mathbb{R}^{u-2})$.

Summing up, we get that

Theorem 3.1. *Let G be a nontrivial Lie semi-direct product of $\mathbb{C}$ by $\mathbb{R}^n$ with the action α. Then $C^*(G)$ has the following exact sequence:*

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^n \to C^*(G) \to C_0(\mathbb{R}^n) \to 0.$$

Moreover, the above ideal is isomorphic to

$$\begin{cases} C_0(\mathbb{T} \times \mathbb{R}^n) \otimes \mathbb{K} & \text{if } O(z) \approx \mathbb{T} \text{ for any } z \in \mathbb{C} \setminus \{0\}, \\ C_0(\mathbb{T} \times \mathbb{R}^{n-1}) \otimes \mathbb{K} & \text{if } O(z) \approx \mathbb{R}_+ \text{ for any } z \in \mathbb{C} \setminus \{0\}, \\ C_0(\mathbb{T} \times \mathbb{R}^{n-2}) \otimes \mathbb{K} & \text{if } \alpha \text{ is transitive on } \mathbb{C} \setminus \{0\} \text{ and } n \geq 2 \end{cases}$$

where $O(z)$ means the orbit of z under the action $\hat{\alpha}$.

As a corollary, we obtain that

Corollary 3.2. *Let G be a Lie semi-direct product of $\mathbb{C}$ by $\mathbb{R}^n$. Then $C^*(G)$ has nontrivial projections if and only if*

$$G \cong \begin{cases} \mathbb{C} \rtimes_\alpha \mathbb{R} \text{ with } \mathbb{R}_1 \cong \{0\}, \text{ or} \\ \mathbb{C} \rtimes_\alpha \mathbb{R}^2 \text{ with } (\mathbb{R}^2)_1 \cong \mathbb{Z}, \end{cases}$$

where $\mathbb{R}_1, (\mathbb{R}^2)_1$ means the stabilizers of the unit of $\mathbb{C}$ under $\hat{\alpha}$.

Proof. The first case of G corresponds to the second case with $n = 1$ in the latter part of Theorem 3.1. And the second case of G corresponds to the third case with $n = 2$ in it.

□

Remark. The complex $ax + b$ group $\mathbb{C} \rtimes_\alpha \mathbb{C}$ with $\alpha_z(w) = zw$ for $z, w \in \mathbb{C}$ (cf.[Rs]) is a special case of $\mathbb{C} \rtimes_\alpha \mathbb{R}^2$ with $(\mathbb{R}^2)_1 \cong \mathbb{Z}$.

In a slightly general situation, we have the following:

Theorem 3.3. *Let G be a nontrivial Lie semi-direct product of $\mathbb{C}$ by the direct product $\mathbb{R}^n \times \mathbb{T}^m$ with the action α. Then $C^*(G)$ has the following exact sequence:*

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes (\mathbb{R}^n \times \mathbb{T}^m) \to C^*(G) \to C_0(\mathbb{R}^n \times \mathbb{Z}^m) \to 0.$$

Moreover, the above ideal is isomorphic to

$$C_0(\mathbb{R}^{n+1} \times \mathbb{Z}^{m-1}) \otimes \mathbb{K} \quad \text{if the action of } \mathbb{R}^n \text{ is trivial,}$$

and otherwise,

$$\begin{cases} C_0(\mathbb{T} \times \mathbb{R}^n \times \mathbb{Z}^m) \otimes \mathbb{K} & \text{if } O(z) \approx \mathbb{T} \text{ for any } z \in \mathbb{C} \setminus \{0\}, \\ C_0(\mathbb{T} \times \mathbb{R}^{n-1} \times \mathbb{Z}^m) \otimes \mathbb{K} & \text{if } O(z) \approx \mathbb{R}_+ \text{ for any } z \in \mathbb{C} \setminus \{0\}, \\ C_0(\mathbb{T} \times \mathbb{R}^{n-2} \times \mathbb{Z}^m) \otimes \mathbb{K} & \text{if } \alpha \text{ is transitive on } \mathbb{C} \setminus \{0\}. \end{cases}$$

Proof. We first note that the action of $\mathbb{T}$ on $\mathbb{C}$ is trivial or the rotation by multiplication. So we may have the decomposition $\mathbb{T}^m = \mathbb{T}^{m_0} \times \mathbb{T}^{m-m_0}$ where $\mathbb{T}^{m_0}$ acts on $\mathbb{C}$ trivially and each direct factor of $\mathbb{T}^{m-m_0}$ acts on $\mathbb{C}$ by the rotation. Then, we have that

$$\begin{aligned} C^*(\mathbb{C} \rtimes (\mathbb{R}^n \times \mathbb{T}^m)) &\cong C_0(\mathbb{C}) \rtimes (\mathbb{R}^n \times \mathbb{T}^m) \\ &\cong C_0(\mathbb{Z}^{m_0}) \otimes (C_0(\mathbb{C}) \rtimes (\mathbb{R}^n \times \mathbb{T}^{m-m_0})). \end{aligned}$$

Moreover, we have the following exact sequence:

$$\begin{aligned} 0 &\to C_0(\mathbb{C} \setminus \{0\}) \rtimes (\mathbb{R}^n \times \mathbb{T}^{m-m_0}) \\ &\to C_0(\mathbb{C}) \rtimes (\mathbb{R}^n \times \mathbb{T}^{m-m_0}) \to C^*(\mathbb{R}^n \times \mathbb{T}^{m-m_0}) \to 0. \end{aligned}$$

Now suppose that any orbit in $\mathbb{C}\setminus\{0\}$ is homeomorphic to $\mathbb{T}$. Then the above ideal is isomorphic to $C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes (\mathbb{R}^n \times \mathbb{T}^{m-m_0}))$. Since the action of $\mathbb{R}^{n-m} \times \mathbb{T}^{m-m_0}$ on $\mathbb{T}$ is transitive, we have that

$$\begin{aligned} &C(\mathbb{T}) \rtimes (\mathbb{R}^n \times \mathbb{T}^{m-m_0}) \\ &\cong C((\mathbb{R}^n \times \mathbb{T}^{m-m_0})/(\mathbb{R}^n \times \mathbb{T}^{m-m_0})_1) \otimes \mathbb{K} \\ &\cong C^*((\mathbb{R}^n \times \mathbb{T}^{m-m_0})_1) \otimes \mathbb{K}. \end{aligned}$$

If the restriction of α to $\mathbb{R}^n$ is trivial, the stabilizer $(\mathbb{R}^n \times \mathbb{T}^{m-m_0})_1$ is given by

$$\mathbb{R}^n \times \{(w_i)_{i=1}^{m-m_0} \in \mathbb{T}^{m-m_0} : \Pi_{i=1}^{m-m_0} w_i = 1\},$$

so that it is isomorphic to $\mathbb{R}^n \times \mathbb{T}^{m-m_0-1}$. If the restriction of α to $\mathbb{T}^{m-m_0}$ is trivial, the stabilizer is isomorphic to $\mathbb{Z}\times\mathbb{R}^{n-1}\times\mathbb{T}^{m-m_0}$ by the argument before Theorem 3.1. In the case where both restrictions of α to $\mathbb{R}^n$ and $\mathbb{T}^{m-m_0}$ are nontrivial, the stabilizer has the fiber structure on the base space $\mathbb{T}$ parameterized by $\mathbb{T}^{m-m_0-1}$ such that $w = \Pi_{j=1}^{m-m_0} w_j \in \mathbb{T}$ with fibers isomorphic to $\mathbb{Z} \times \mathbb{R}^{n-1}$, which is induced by the equation

$$e^{(\sum_{i=1}^n z_i g_i)} \Pi_{j=1}^{m-m_0} w_j = 2t\pi i, \quad z_i, g_i \in \mathbb{R}, t \in \mathbb{Z}.$$

By the local triviality of this fiber structure, we have in fact that the stabilizer is isomorphic to $\mathbb{R}^n \times \mathbb{T}^{n-m_0-1}$.

For other cases, we note that the action of $\mathbb{T}^m$ on the radius direction of $\mathbb{C}\setminus\{0\}$ is trivial. Thus we can repeat the similar argument as before Theorem 3.1.

□

Theorem 3.4. *Let G be a Lie semi-direct product of $\mathbb{C}$ by the direct product $\mathbb{R}^n \times \mathbb{T}^m$ with the action α. Then $C^*(G)$ has nontrivial projections if and only if*

$$G \cong \begin{cases} \mathbb{C} \rtimes (\mathbb{R} \times \mathbb{T}^m) \text{ with } (\mathbb{R} \times \mathbb{T}^m)_1 \cong \mathbb{T}^m, \text{ or} \\ \mathbb{C} \rtimes (\mathbb{R}^2 \times \mathbb{T}^m) \text{ with } (\mathbb{R}^2 \times \mathbb{T}^m)_1 \cong \mathbb{Z} \times \mathbb{T}^m \end{cases}$$

where $(\mathbb{R} \times \mathbb{T}^m)_1$ *and* $(\mathbb{R}^2 \times \mathbb{T}^m)_1$ *are the stabilizers of the unit of* $\mathbb{C}$ *under* $\hat{\alpha}$.

Proof. Theorem 3.3 tells us that if $\mathbb{R}^n$ acts on $\mathbb{C}$ nontrivially, $C^*(G)$ has nontrivial projections if and only if the second case with $n = 1$ and the third one with $n = 2$ among the three cases, that is, G is isomorphic to the two cases in this statement.

If $\mathbb{R}^n$ $(n \geq 1)$ acts on $\mathbb{C}$ trivially, we deduce from Theorem 3.3 that $C^*(G)$ has no nontrivial projections.

If $n = 0$, we note that $C^*(G)$ is regarded as the C^*-algebra of continuous fields over $\{0\} \cup \mathbb{R}_+$ vanishing at infinity with fibers $\mathfrak{A}_t$ given by

$$\mathfrak{A}_0 = C_0(\mathbb{Z}^m), \quad \mathfrak{A}_t = C_0(\mathbb{Z}^{m-1}) \otimes \mathbb{K} \text{ for } t \in \mathbb{R}_+$$

(cf.[Dx]). Now suppose that $C^*(G)$ has a nontrivial projection p. Then the function $\hat{p} : t \mapsto \|p(t)\|$ with $p(t) \in \mathfrak{A}_t$ must be continuous and vanishing at infinity. Since $p(t)$ is also a projection of $\mathfrak{A}_t$, $\|p(t)\| = 0$ or 1. Since $\hat{p}$ is nontrivial and continuous, $\hat{p}$ must be the constant function 1 over $\{0\} \cup \mathbb{R}_+$, which is the contradiction.

□

The same methods as the latter part of the proof of Theorem 3.4 implies that

Proposition 3.5. *Let* $\mathfrak{A}$ *be a* C^**-algebra of continuous fields over a noncompact connected locally compact space. Then* $\mathfrak{A}$ *has no nontrivial projections.*

Remark. The group C^*-algebra $C^*(H_{2n+1})$ is regarded as the C^*-algebra of continuous fields over $\mathbb{R}$ with fibers $C_0(\mathbb{R}^{2n})$ at $0 \in \mathbb{R}$ and $\mathbb{K}$ over $\mathbb{R} \setminus \{0\}$.

As a corollary, we get that (cf.[Sd1])

Corollary 3.6. *Let* $\mathfrak{A}$ *be a (liminal)* C^**-algebra with its spectrum a noncompact and connected Hausdorff space. Then* $\mathfrak{A}$ *has no nontrivial projections.*

Proof. It is known by [Dx, Theorem 10.5.4] that any (liminal) C^*-algebra with its spectrum a Hausdorff space is isomorphic to the C^*-algebra of continuous fields over the spectrum.

□

Lie semi-direct products of $\mathbb{C}$ by H_{2n+1}.

In this subsection we analyze structure of group C^*-algebras of the Lie semi-direct products $G = \mathbb{C} \rtimes_\beta H_{2n+1}$. Then the action β of H_{2n+1} on $\mathbb{C}$ may be defined by

$$\beta_{(c,b,a)}(z) = e^{(\sum_{k=1}^{u} z_{i_k} b_{i_k} + \sum_{k=1}^{u} w_{i_k} a_{i_k})} z$$

with $(c,b,a) \in H_{2n+1}$, $z, z_{i_k}, w_{i_k} \in \mathbb{C}$ for some $0 \leq u, v \leq n$. Then $C^*(G)$ is isomorphic to $C_0(\mathbb{C}) \rtimes_{\hat{\beta}} H_{2n+1}$, where $\hat{\beta}$ is given by the complex conjugate of β. Since the origin of $\mathbb{C}$ is fixed under $\hat{\beta}$, we have that

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes H_{2n+1} \to C_0(\mathbb{C}) \rtimes H_{2n+1} \to C^*(H_{2n+1}) \to 0.$$

Moreover, we analyze the above ideal using the analysis given in the above subsection. We note that $H_{2n+1}/[H_{2n+1}, H_{2n+1}] \cong \mathbb{R}^{2n}$ and the action $\hat{\alpha}$ is induced from than of $\mathbb{R}^{2n}$.

First suppose that all orbits under $\hat{\beta}$ in $\mathbb{C} \setminus \{0\}$ are homeomorphic to $\mathbb{T}$. Then $C_0(\mathbb{C} \setminus \{0\}) \rtimes H_{2n+1}$ splits into the tensor product $C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes H_{2n+1})$. Moreover, by Green's imprimitivity theorem,

$$\begin{aligned} C(\mathbb{T}) \rtimes H_{2n+1} &\cong C(H_{2n+1}/(H_{2n+1})_1) \rtimes H_{2n+1} \\ &\cong C^*((H_{2n+1})_1) \otimes \mathbb{K}. \end{aligned}$$

And the stabilizer of $1 \in \mathbb{T}$ under $\mathbb{R}^{2n}$ is isomorphic to $\mathbb{Z} \times \mathbb{R}^{2n-1}$. Hence, $(H_{2n+1})_1$ is isomorphic to either

$$\text{(C1)} \qquad (\mathbb{R} \times \mathbb{Z} \times \mathbb{R}^{n-1}) \rtimes_\alpha \mathbb{R}^n \quad \text{or} \quad \mathbb{R}^{n+1} \rtimes_\alpha (\mathbb{Z} \times \mathbb{R}^{n-1}).$$

Then we have the next exact sequence respectively:

$$\begin{aligned} 0 \to &\begin{cases} C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{T} \times \mathbb{R}^{n-1}) \rtimes_{\hat{\alpha}} \mathbb{R}^n, \\ C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes_{\hat{\alpha}} (\mathbb{Z} \times \mathbb{R}^{n-1}) \end{cases} \\ &\to C^*((H_{2n+1})_1) \to C_0(\mathbb{T} \times \mathbb{R}^{2n-1}) \to 0. \end{aligned}$$

In the first case, $\hat{\alpha}$ is transitive, and in the second case, it is free and wandering. Hence by [Gr1, Gr2], we get that the ideals in both cases are isomorphic to the direct sum $\oplus^2(C_0(\mathbb{R}_+ \times \mathbb{T}) \otimes \mathbb{K})$.

Next suppose that all orbits in $\mathbb{C} \setminus \{0\}$ are homeomorphic to $\mathbb{R}_+$. Then the crossed product $C_0(\mathbb{C}\setminus\{0\}) \rtimes H_{2n+1}$ has the decomposition $C(\mathbb{T})\otimes(C_0(\mathbb{R}_+)\rtimes H_{2n+1})$. In this case we may assume that all z_{i_k}, w_{i_l} $(1 \le k \le u,\ 1 \le l \le v)$ are real. Moreover, Green's imprimitivity theorem [Gr2] implies that

$$\begin{aligned} C_0(\mathbb{R}_+) \rtimes H_{2n+1} &\cong C_0(H_{2n+1}/(H_{2n+1})_1) \rtimes H_{2n+1} \\ &\cong C^*((H_{2n+1})_1) \otimes \mathbb{K}. \end{aligned}$$

And the stabilizer of $1 \in \mathbb{R}_+$ under $\mathbb{R}^{2n}$ is equal to the hyperplane

$$\{(b,a) \in \mathbb{R}^{2n} : \sum_{k=1}^{u} z_{i_k} b_{i_k} + \sum_{j=1}^{v} w_{i_j} a_{i_j} = 0\},$$

so that $(H_{2n+1})_1$ is isomorphic to one of the following:

$$\text{(C2)} \qquad \begin{cases} (\mathbb{R} \times \mathbb{R}^{n-1}) \rtimes_\alpha \mathbb{R}^n, & \mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^{n-1} \\ (\mathbb{R} \times \mathbb{R}^{n-1}) \rtimes_\gamma \mathbb{R}^n, & \mathbb{R}^{n+1} \rtimes_\rho \mathbb{R}^{n-1} \end{cases}$$

where if $u \neq 0$,

$$\gamma_a(c, (b_i)_{i\neq i_1}^n) = (c - a_{i_1} z_{i_1}^{-1} \sum_{k=2}^{u} z_{i_k} b_{i_k} + \sum_{k \neq i_1}^{n} a_k b_k, (b_i)_{i\neq i_1}^n),$$

and if $u = 0$,

$$\rho_{(a_i)_{i\neq i_1}^n}(c, b) = (c - b_{i_1} w_{i_1}^{-1} \sum_{j=2}^{v} w_{i_j} a_{i_j} + \sum_{k\neq i_1}^{n} a_k b_k, b).$$

For the upper two cases of $(H_{2n+1})_1$, we see that $C^*((H_{2n+1})_1)$ splits into $C_0(\mathbb{R}) \otimes C^*(H_{2(n-1)+1})$, so that

$$0 \to \oplus^2(C_0(\mathbb{R}^2) \otimes \mathbb{K}) \to C^*((H_{2n+1})_1) \to C_0(\mathbb{R}^{2(n-1)+1}) \to 0.$$

For the lower two cases of $(H_{2n+1})_1$, each dual action is given by

$$\begin{aligned} \hat{\gamma}_a(l, (m_i)_{i\neq i_1}^n) &= (l, (m_i + a_i l)_{i\neq i_1}^n + (-a_{i_1} z_{i_k} z_{i_1}^{-1} l)_{k=2}^u) \\ \hat{\rho}_{(a_i)_{i\neq i_1}^n}(l, m) &= (l, (m_i + a_i l)_{i\neq i_1}^n, (m_{i_1} - w_{i_1}^{-1} l \sum_{j=2}^{v} w_{i_j} a_{i_j})). \end{aligned}$$

Then each $C^*((H_{2n+1})_1)$ has the following structure respectively:

$$0 \to \left\{ \begin{array}{l} C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \\ C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \otimes \mathbb{K} \end{array} \right. \to C^*((H_{2n+1})_1) \to C_0(\mathbb{R}^{2n-1}) \to 0.$$

Finally suppose that the action of H_{2n+1} on $\mathbb{C} \setminus \{0\}$ is transitive. Then Green's imprimitivity theorem implies that

$$\begin{aligned} C_0(\mathbb{C} \setminus \{0\}) \rtimes H_{2n+1} &\cong C_0(H_{2n+1}/(H_{2n+1})_1) \rtimes H_{2n+1} \\ &\cong C^*((H_{2n+1})_1) \otimes \mathbb{K}. \end{aligned}$$

By the same way as above, we can deduce that $(H_{2n+1})_1$ is isomorphic to one of the following:

$$\text{(C3)} \qquad \left\{ \begin{array}{ll} (\mathbb{R} \times \mathbb{Z} \times \mathbb{R}^{n-2}) \rtimes \mathbb{R}^n, & (\mathbb{R} \times \mathbb{R}^{n-1}) \rtimes (\mathbb{Z} \times \mathbb{R}^{n-1}), \\ \mathbb{R}^{n+1} \rtimes (\mathbb{Z} \times \mathbb{R}^{n-2}), & (\mathbb{R} \times \mathbb{Z} \times \mathbb{R}^{n-1}) \rtimes \mathbb{R}^{n-1} \end{array} \right.$$

where each action is induced by the equations of the hyperplanes as given in the case of $\mathbb{C} \rtimes \mathbb{R}^n$. When the action is a restriction of α, the stabilizer $(H_{2n+1})_1$ has $\mathbb{R}$ as a direct factor. Thus by the similar argument as the case under the first assumption, the structure of $C^*((H_{2n+1})_1)$ is given by

$$0 \to \oplus^2 (C_0(\mathbb{R}^2 \times \mathbb{T}) \otimes \mathbb{K}) \to C^*((H_{2n+1})_1) \to C_0(\mathbb{T} \times \mathbb{R}^{2n-2}) \to 0.$$

Otherwise, for the upper and lower cases of $(H_{2n+1})_1$, we have that respectively

$$\begin{aligned} 0 \to &\left\{ \begin{array}{l} \oplus^2 (C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K}) \\ \oplus^2 (C_0(\mathbb{R}^2 \times \mathbb{T}) \otimes \mathbb{K}) \end{array} \right. \\ &\to C^*((H_{2n+1})_1) \to C_0(\mathbb{T} \times \mathbb{R}^{2n-2}) \to 0. \end{aligned}$$

Summing up, we get that

Theorem 3.7. *Let G be a nontrivial Lie semi-direct product of $\mathbb{C}$ by H_{2n+1}. Then $C^*(G)$ has the following exact sequence:*

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes H_{2n+1} \to C^*(G) \to C^*(H_{2n+1}) \to 0.$$

Moreover, the above ideal is isomorphic to one of the three cases

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((H_{2n+1})_{1\in\mathbb{T}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((H_{2n+1})_{1\in\mathbb{R}_+}) \otimes \mathbb{K} \\ C^*((H_{2n+1})_{1\in\mathbb{C}\setminus\{0\}}) \otimes \mathbb{K}, \end{cases}$$

where each $(H_{2n+1})_1$ *is given by* (C1), (C2) *and* (C3). *And for each case,* $C^*((H_{2n+1})_1)$ *has the following structure respectively:*

$$0 \to \oplus^2(C_0(\mathbb{R}\times\mathbb{T})\otimes\mathbb{K}) \to C^*((H_{2n+1})_{1\in\mathbb{T}}) \to C_0(\mathbb{T}\times\mathbb{R}^{2n-1}) \to 0,$$

$$0 \to \begin{cases} \oplus^2(C_0(\mathbb{R}^2)\otimes\mathbb{K}) \\ \oplus^2(C_0(\mathbb{R})\otimes\mathbb{K}) \end{cases} \to C^*((H_{2n+1})_{1\in\mathbb{R}_+}) \to C_0(\mathbb{R}^{2n-1}) \to 0,$$

$$0 \to \begin{cases} \oplus^2(C_0(\mathbb{R}^2\times\mathbb{T})\otimes\mathbb{K}) \\ \oplus^2(C_0(\mathbb{R}\times\mathbb{T})\otimes\mathbb{K}) \end{cases}$$
$$\to C^*((H_{2n+1})_{1\in\mathbb{C}\setminus\{0\}}) \to C_0(\mathbb{T}\times\mathbb{R}^{2n-2}) \to 0.$$

Remark. The above theorem says that the building blocks of $C^*(G)$ are given by

$$C_0(\mathbb{R}^{2n}), \quad C_0(\mathbb{T}^k\times\mathbb{R}^s)\otimes\mathbb{K}, \quad (k = 0 \text{ or } 1, \text{ and } s \geq 1).$$

Hence, $C^*(G)$ has no nontrivial projections.

In a more general situation, we obtain that

Proposition 3.8. *Let* G *be a Lie semi-direct product of* $\mathbb{C}$ *by a connected Lie group* N. *Then* $C^*(G)$ *has the following exact sequence:*

$$0 \to C_0(\mathbb{C}\setminus\{0\}) \rtimes N \to C^*(G) \to C^*(N) \to 0.$$

Moreover, the above ideal is isomorphic to one of the following:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((N)_{1\in\mathbb{T}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((N)_{1\in\mathbb{R}_+}) \otimes \mathbb{K} \\ C^*((N)_{1\in\mathbb{C}\setminus\{0\}}) \otimes \mathbb{K}. \end{cases}$$

Proof. The proof follows from the same argument as before Theorem 3.7.

□

Remark. In the above proposition we assume that N is a simply connected nilpotent Lie group. Then by Proposition 3.8 and the similar reason with the proof of Theorem 2.7, we have that if $N/N_1 \cong \mathbb{T}$ or $\mathbb{R}$ (the first and second cases above), then

$$C^*(G) \text{ has } \begin{cases} \text{nontrivial projections if } N = \mathbb{R} \text{ and } (N)_1 = 0, \\ \text{no nontrivial projections if otherwise.} \end{cases}$$

For the third case above, we conjecture that $C^*(G)$ has no nontrivial projections.

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