On $ \mathfrak{F}$ -reducers in Finite Solvable Groups

On $\mathfrak{F}$—reducers in Finite Solvable Groups

Haruo NAKAZATO*

1. INTRODUCTION. In this note all groups are finite and solvable. The letter G stands always for such a group. A. Mann constructed in [3], for any subgroup H of G, a subgroup $Q(H)$ which gives a different characterization of the reducer $R(H)$ in G of H, defined by B. Fischer, and he defined in [4] another subgroup $M(H)$ by using a certain concept of equivalence introduced by R. Carter. In [5] he provided an alternative characterization of the Carter subgroups of G as nilpotent subgroups H of G satisfying $H=M(H)$.

In [1] C. J. Graddon introduced the concept of the $\mathfrak{F}$—reducer $R(H;\mathfrak{F})$ in G of a subgroup H of G by defineding $\mathfrak{F}$—basis of G, which gives an alternative characterization of $Q(H;\mathfrak{F})$ which is a generalization of the work of A. Mann [3], and showed some of the basic properties of this subgroup, where $\mathfrak{F}$ is the local (saturated) formation defined by a set of nonempty subgroup closed formations $\{\mathfrak{F}(p)\}$. He showed in [1] that the $\mathfrak{F}$—projector of G are characterized as the $\mathfrak{F}$—subgroup H of G satisfying $H=R(H;\mathfrak{F})$.

In this note we give, for a certain subgroup H of G, an alternative characterization of the $\mathfrak{F}$—reducer $R(H;\mathfrak{F})$ of H in G as the subgroup $M(H;\mathfrak{F})$ which is similar to the subgroup $M(H)$ introduced by A. Mann, and show some properties of $\mathfrak{F}$—reducer $R(H;\mathfrak{F})$ of H in G. In section 2 we give a brief resume of the definitions and properties which we require later in this note, and in section 3 we show some properties of $\mathfrak{F}$—subnormal subgroups of G.

2. PRELIMINARIES. We shall wherever possible, adhere to the notation used in [1]. Throught this note, $\mathfrak{F}$ will denote the integrated formation defined locally by the nonempty subgroup closed formations $\{\mathfrak{F}(p)\}$. Let $\{S^p\}$ be a set of Sylow p—complements of G, one for each prime p dividing $|G|$, and let $\mathfrak{S}$ be a Sylow system of G generated by the S^p. Then the $\mathfrak{F}$—basis of G associated with $\mathfrak{S}$ is the collection $\mathfrak{F}(\mathfrak{S})=\{S^p\cap G_{\mathfrak{F}(p)}\}$ of subgroups of G, where for each prime p, $G_{\mathfrak{F}(p)}$ denotes the $\mathfrak{F}(p)$—residual of G, i.e., the smallest normal subgroup of G with the factor in $\mathfrak{F}(p)$. Let H be a subgroup of G, then, as in [1], $\mathfrak{F}(\mathfrak{S})$ reduces into H if for each prime p, $S^p\cap H_{\mathfrak{F}(p)}=S^p\cap G_{\mathfrak{F}(p)}\cap H_{\mathfrak{F}(p)}$ is a Sylow p—complement of $H_{\mathfrak{F}(p)}$, i.e., if $\{S^p\cap H_{\mathfrak{F}(p)}\}$ is an $\mathfrak{F}$—basis of H.

Thus $\mathfrak{F}(\mathfrak{S})$ reduces into H if and only if there exists a Sylow system $\mathfrak{S}_H=\{H^p\}$ of H such that $S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$ for each prime p. In [1] C. J. Graddon showed

*Dept. of Mathematics Univ. of the Ryukyus

that there always exists at least one $\mathfrak{F}$-basis of G which reduces into H and defined, for given such an $\mathfrak{F}$-basis $\mathfrak{F}(\mathfrak{S})$, the $\mathfrak{F}$-reducer of H in G to be the subgroup

$$R(H;\mathfrak{F})=\langle g\in G\,;\,\mathfrak{F}(\mathfrak{S})^g \text{ reduces into } H\rangle$$

DEFINITION. A maximal subgroup M of G, of index powers of a prime p in G, is said to be $\mathfrak{F}$-normal in G if $M/\mathrm{Core}(M)\in\mathfrak{F}(p)$. M is said to be $\mathfrak{F}$-abnormal otherwise. A subgroup H of G is $\mathfrak{F}$-abnormal in G if every link in each maximal chain joining H to G is $\mathfrak{F}$-abnormal. H is said to be $\mathfrak{F}$-subnormal in G if every link in some maximal chain joining H to G is $\mathfrak{F}$-normal.

In [2] it is described that, for a subgroup H of G

(2.1) H is $\mathfrak{F}$-subnormal in G if and only if every $\mathfrak{F}$-basis of G reduces into H.

The following two results are showed by C. J. Graddon in [1] :

(2.2) H is an $\mathfrak{F}$-abnormal subgroup of G if and only if $H=R(H;\mathfrak{F})$.

(2.3) If H is a subgroup of G, then $R(H;\mathfrak{F})$ is self $\mathfrak{F}$-reducing in G.

Let Ω be the collection of $\mathfrak{F}$-bases of G and let $\mathfrak{M}$ be the set of elements of Ω which reduces into the subgroup H of G. Let $\mathfrak{M}_0$ be the block generated by $\mathfrak{M}$ in Ω. Then $Q(H;\mathfrak{F})$ is defined to be the set stabilizer in G of $\mathfrak{M}_0$, i.e., the set of all elements g in G such that $(\mathfrak{M}_0)^g=\mathfrak{M}_0$.

C. J. Graddon showed in [2] that

(2.4) Every $\mathfrak{F}$-bases of G which reduces into the subgroup H of G also reduces into $R(H;\mathfrak{F})$.

(2.5) $\mathfrak{M}_0$ is the set of $\mathfrak{F}$-bases of G which reduces into $R(H;\mathfrak{F})$.

and in [1] that

(2.6) For each subgroup H of G, $R(H;\mathfrak{F})=Q(H;\mathfrak{F})$.

Let H be a subgroup of G. Then an H-composition series of G is a series

$$1=G_n<G_{n-1}<\cdots<G_1<G_0=G$$

in which each subgroup G_i is a maximal H-invariant normal subgroup of G_{i-1}. We say that the factor G_i/G_{i+1} is $\mathfrak{F}$-central if $A_H(G_i/G_{i+1})$, the automorphism group induced by H on G_i/G_{i+1}, belongs to the formation $\mathfrak{F}(p)$. where G_i/G_{i+1} is an elementary abelian p-group. If this is not the case we say this factor is $\mathfrak{F}$-eccentric.

The following result is the structure theorem of $R(H;\mathfrak{F})$, which is obtained by C. J. Graddon in [2].

(2.7) Let H be a subgroup of G. Then (i) $R(H;\mathfrak{F})$ covers each $\mathfrak{F}$-central H-composition factor of G, and (ii) if K is a subgroup of G which contains H and covers every $\mathfrak{F}$-central H-composition factor of G, then K contains $R(H;\mathfrak{F})$.

DEFINITION. Suppose that $H\leq K\leq G$. Then K is an $\mathfrak{F}$-subnormalizer of H in G if

(i) H is $\mathfrak{F}$-subnormal in K, and

(ii) Whenever H is $\mathfrak{F}$-subnormal in a subgroup L of G, then L is contained in K.

The following facts follows from theorem 4.6 of [2].

(2.8) If H is a subgroup of G and the set of $\mathfrak{F}$-bases of G which reduce into H forms a block then $R(H;\mathfrak{F})$ is an $\mathfrak{F}$-subnormalizer of H in G.

3. $\mathfrak{F}$-SUBNORMAL. We show some proporties of $\mathfrak{F}$-subnormal subgroups of G.

PROPOSITION 1. *Suppose that* $H \leq K \leq G$. *If* H *is* $\mathfrak{F}$-*subnormal in* G, *then* H *is* $\mathfrak{F}$-*subnormal in* K.

PROOF. Let $\mathfrak{F}(\mathfrak{S}_K)$ be an $\mathfrak{F}$-basis of K associated with a Sylow system $\mathfrak{S}_K=\{K^p\}$ of K. Then there exists a Sylow system $\mathfrak{S}=\{S^p\}$ of G which is an extension of $\mathfrak{S}_K$, i.e, $K^p=S^p\cap K$ for each prime p. Now $\mathfrak{F}(\mathfrak{S})$ is an $\mathfrak{F}$-basis of G. Since H is $\mathfrak{F}$-subnormal in G, $\mathfrak{F}(\mathfrak{S})$ reduces into H by (2.1). Then there exists a Sylow system $\mathfrak{S}_H=\{H^p\}$ of H such that $S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$ for each prime p. Therefore we have that

$$K^p \cap H_{\mathfrak{F}(p)}=(S^p\cap K)\cap H_{\mathfrak{F}(p)}=S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$$

for each prime p. and thus $\mathfrak{F}(\mathfrak{S}_K)$ reduces into H. This implies that H is $\mathfrak{F}$-subnormal in K.

PROPOSITION 2. *Let* H *be a snbgroup of* G *and suppose that* $\mathfrak{F}(\mathfrak{S})$ *is an* $\mathfrak{F}$-*basis of* G *which reduce into* H. *If* K *is an* $\mathfrak{F}$-*subnormal subgroup of* H, *then* $\mathfrak{F}(\mathfrak{S})$ *reduces into* K.

PROOF. Since $\mathfrak{F}(\mathfrak{S})=\{S^p\cap G_{\mathfrak{F}(p)}\}$ reduce into H, there exists a Sylow system $\mathfrak{S}_H=\{H^p\}$ of H such that $S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$ for each prime p. Therefore if K is $\mathfrak{F}$-subnormal in H, $\mathfrak{F}(\mathfrak{S}_H)$ reduces into K by (2.1), ie., there exists a Sylow system $\mathfrak{S}_K=\{K^p\}$ of K such that $H^p\cap K_{\mathfrak{F}(p)}=K^p\cap K_{\mathfrak{F}(p)}$ for each prime p. Since $\mathfrak{F}(p)$ is subgroup closed, we know that $K\leq H$ implies $K_{\mathfrak{F}(p)}\leq H_{\mathfrak{F}(p)}$. Now we have that, for each prime p,

$$\begin{aligned} S^p\cap K_{\mathfrak{F}(p)}&=(S^p\cap H_{\mathfrak{F}(p)})\cap \mathrm{K}_{\mathfrak{F}(p)}=(H^p\cap H_{\mathfrak{F}(p)})\cap K_{\mathfrak{F}(p)}\\ &=H^p\cap K_{\mathfrak{F}(p)}=K^p\cap K_{\mathfrak{F}(p)}. \end{aligned}$$

Thus $\mathfrak{F}(\mathfrak{S})$ reduces into K.

PROPOSITION 3. *Let* H *be a subgroup and* N *a normal subgroup of* G. *Suppose that* H *is* $\mathfrak{F}$-*subnormal in* G. *Then* HN/N *is* $\mathfrak{F}$-*subnormal in* G/N *and* HN *is* $\mathfrak{F}$-*subnormal in* G.

PROOF. Now each $\mathfrak{F}$-basis of G/N is $\mathfrak{F}(\mathfrak{S}N/N)$ for some Sylow system $\mathfrak{S}$ of G, where $\mathfrak{S}N/N=\{S^pN/N\}$ is a Sylow system of G/N for the Sylow system $\mathfrak{S}=\{S^p\}$ of G. Suppose that H is $\mathfrak{F}$-subnormal in Then every $\mathfrak{F}$-bases $\mathfrak{F}(\mathfrak{S})$ of G reduce into H. Therefore we have that $S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$ for each Sylow p-complement H^p of H, and so by (2.5) and (2.6) of [1],

$$\begin{aligned}(S^pN/N)\cap(HN/N)_{\mathfrak{F}(p)}&=(S^pN/N)\cap(H_{\mathfrak{F}(p)}N/N)=(S^pN\cap H_{\mathfrak{F}(p)}N)/N\\&=(S^p\cap H_{\mathfrak{F}(p)})N/N=(H^p\cap H_{\mathfrak{F}(p)})N/N\\&=(H^pN/N)\cap(H_{\mathfrak{F}(p)}N/N)=(HN/N)^p\cap(HN/N)_{\mathfrak{F}(p)}.\end{aligned}$$

Therefore every $\mathfrak{F}$-basis $\mathfrak{F}(\mathfrak{S}N/N)$ of G/N reduces into HN/N. Hence HN/N is $\mathfrak{F}$-subnormal in G/N by (2.1).

Let

$$HN/N=G_r/N<\cdots<G_0/N=G/N$$

be a maximal chain joining HN/N to G/N such that G_i/N is an $\mathfrak{F}$-normal maximal subgroup of G_{i-1}/N. Now G_i is a maximal subgroup of G_{i-1} if and only if G_i/N is a maximal subgroup of G_{i-1}/N. On the other hand, since $\mathrm{Core}_{G_{i-1}/N}(G_i/N)=\mathrm{Core}_{G_{i-1}}(G_i)/N$, it follows that G_i is $\mathfrak{F}$-normal in G_{i-1} if and only if G_i/N is $\mathfrak{F}$-normal in G_{i-1}/N. Therefore we have a maximal chain joining HN to G such that every normal link is $\mathfrak{F}$-normal :

$$HN=G_r<\cdots<G_0=G.$$

Thus HN is $\mathfrak{F}$-subnormal in G.

4. $\mathfrak{F}$-REDUCER. Suppose that $\mathfrak{F}$ is an integrated formation defined locally by the nonempty subgroup closed formations $\{\mathfrak{F}(p)\}$.

LEMMA 4. *Let $H\leq K\leq G$ and $\mathfrak{S}_K=\{K^p\}$ be a Sylow system of K. Suppose that $\mathfrak{S}=\{S^p\}$ is a Sylow system of G which is a extension of $\mathfrak{S}_K$, i.e., $S^p\cap K=K^p$ for each prime p. Then the $\mathfrak{F}$-basis $\mathfrak{F}(\mathfrak{S}_K)$ of K reduces into H if and only if the $\mathfrak{F}$-basis $\mathfrak{F}(\mathfrak{S})$ of G reduces into H.*

PROOF. Suppose that $\mathfrak{F}(\mathfrak{S}_K)=\{K^p\cap K_{\mathfrak{F}(p)}\}$ reduces into H. Then there exists a Sylow system $\mathfrak{S}_H=\{H^p\}$ of H such that $K^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$ for each prime p. Thus we have that

$$S^p\cap H_{\mathfrak{F}(p)}=S^p\cap K\cap H_{\mathfrak{F}(p)}=K^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$$

for each prime p. Therefore $\mathfrak{F}(\mathfrak{S})$ reduces into H.

Conversely, suppose that $\mathfrak{F}(\mathfrak{S})=\{S^p\cap G_{\mathfrak{F}(p)}\}$ reduces into H. Then there exists a Sylow system $\mathfrak{S}_H=\{H^p\}$ of H such that $S^p\cap H H^p\cap H_{\mathfrak{F}(p)}{}_{\mathfrak{F}(p)}=$ for each prime p. Thus we have that

$$K^p\cap H_{\mathfrak{F}(p)}=S^p\cap K\cap H_{\mathfrak{F}(p)}=S^p\cap H_{\mathfrak{F}(p)}=H^p\cap H_{\mathfrak{F}(p)}$$

for each prime p. Therefore $\mathfrak{F}(\mathfrak{S}_K)$ reduces into H.

DEFINITION. Two subgroups H, K of G are termed $\mathfrak{F}$-equivalent, denoted $H\sim K$, if the set of $\mathfrak{F}$-bases of G reducing into H is the same as the set of $\mathfrak{F}$-bases of G reducing into K.

Remark. If we take $\mathfrak{F}(p)=$the class of unit groups, for all primes p, then $\mathfrak{F}=\mathfrak{N}$, where $\mathfrak{N}$ is the class of finite nilpotent groups, and the above definition is just the definition due to R. Carter, of equivalency of two subgroups of G. (see, Definition in [4]).

PROPOSITION 5. *Let H and K be two subgroups of G. If $H \sim K$ in G, then H is $\mathfrak{F}$-subnormal in $<H, K>$.*

PROOF. If every $\mathfrak{F}$-basis of G reduces into H, then H is $\mathfrak{F}$-subnormal in G by (2.1). Therefore H is $\mathfrak{F}$-subnormal in $<H, K>$ by proposition 1. Thus, if $\mathfrak{M}$ is the set of every $\mathfrak{F}$-basis of G reducing into H, we can assume that $\mathfrak{M}$ does not contain all $\mathfrak{F}$-bases of G. Let L be the stabilizer of $\mathfrak{M}$ in G, i.e.,

$$L = \{g \epsilon G \mid \mathfrak{M} = \mathfrak{M}^g\}.$$

Now let $\mathfrak{F}(\mathfrak{S})$ be an $\mathfrak{F}$-basis of G reducing into H, i.e., $\mathfrak{F}(\mathfrak{S}) \epsilon \mathfrak{M}$. For $h \epsilon H$, since $(S^p)^h \cap H_{\mathfrak{F}(p)} = (S^p \cap H_{\mathfrak{F}(p)})^h = (H^p \cap H_{\mathfrak{F}(p)})^h = (H^p)^h \cap H_{\mathfrak{F}(p)}$ for each prime p, where $S^p \epsilon \mathfrak{S}$ and H^p is Sylow p-complement of H, $\mathfrak{F}(\mathfrak{S})^h = \mathfrak{F}(\mathfrak{S}^h)$ reduces into $H = H^h$. Therefore $\mathfrak{M} = \mathfrak{M}^h$ and hence H is a subgroup of L. Since $H \sim K$ in G. $\mathfrak{M}$ is the set of all $\mathfrak{F}$-bases of G reducing into K and hence, by the same reason as above, K is a subgroup of L, Let $\mathfrak{F}(\mathfrak{S}')$ be any $\mathfrak{F}$-basis of G. Then, since any two $\mathfrak{F}$-bases of G are conjugate in G, there is g in G such that $\mathfrak{F}(\mathfrak{S}') = \mathfrak{F}(\mathfrak{S})^g$. Now suppose that $L = G$. Then $\mathfrak{F}(\mathfrak{S}')$ reduces into H which contradicts the hypotheses of $\mathfrak{M}$, since L is the stabilizer of $\mathfrak{M}$. Thus $L \neq G$.

If $\mathfrak{F}(\mathfrak{S}_L)$ is an $\mathfrak{F}$-basis of L reducing into H, then, for a Sylow system $\mathfrak{S}$ of G which is the extension of $\mathfrak{S}_L$, $\mathfrak{F}(\mathfrak{S})$ is an $\mathfrak{F}$-basis of G reducing into H by lemma 4. Since $H \sim K$ in G, $\mathfrak{F}(\mathfrak{S})$ reduces into K. Hence $\mathfrak{F}(\mathfrak{S}_L)$ reduces into K by lemma 4. Similary, if $\mathfrak{F}(\mathfrak{S}_L)$ is an $\mathfrak{F}$-basis of L reducing into K, then $\mathfrak{F}(\mathfrak{S}_L)$ reduces into H. Therefore $H \sim K$ in L. We will prove the proposition by using induction on the group order. Since $|L| < |G|$, we see that, by working on L, H is $\mathfrak{F}$-subnormal in $<H, K>$.

LEMMA 6. *Let H be a subgroup of G. Let $\mathfrak{M}$ be the set of all $\mathfrak{F}$-bases of G reducing into H and L be the stabilizer of $\mathfrak{M}$. Then H is an $\mathfrak{F}$-subnormal subgroup of L.*

PROOF. In the proof of above proposition, we showed that H is a subgroup of L. Let $\mathfrak{F}(\mathfrak{S}_L)$ be any $\mathfrak{F}$-basis of L. Then there exists a Sylow system $\mathfrak{S}$ of G which is an extension of the Sylow system $\mathfrak{S}_L$ of L. Now $\mathfrak{S}_L$ reduces into some conjugate of H in L, say H^l. Hence $\mathfrak{S}$ reduces into H^l by lemma 4. Therefore $\mathfrak{S}^{l^{-1}}$ reduces into H and $\mathfrak{F}(\mathfrak{S})^{l^{-1}} = \mathfrak{F}(\mathfrak{S}^{l^{-1}})$ reduces into H. Since L is the stabilizer of $\mathfrak{M}$, $\mathfrak{F}(\mathfrak{S})$ reduces into H and hence $\mathfrak{F}(\mathfrak{S}_L)$ reduces into H by lemma 4. Thus H is $\mathfrak{F}$-subnormal in L by (2.1).

We need the following result of H. Wielandt. A subgroup H of G is said to be subnormal in G if H is $\mathfrak{F}$-subnormal in G for $\mathfrak{F} = \mathfrak{N}$.

LEMMA 7. [6, THEOREM 6.5] *If H and K are subnormal subgroups of G, then $<H, K>$ is a subnormal subgroup of G.*

PROPOSITION 8. *Let H be a subgroup of G. Let $\mathfrak{M}$ be the set of all $\mathfrak{F}$-bases of G reducing into H and L be the stabilizer in G of $\mathfrak{M}$. Suppose that $\mathfrak{M}$ forms a block. Then if H is subnormal in L, the equivalence class which contains H has a maximal element.*

PROOF. We will show that, if $H \sim K$ in G, then $H \sim <H, K>$ in G. Then it follows that $M(H;\mathfrak{F}) = <K; H \sim K$ in G, K is a subgroup of $G>$ is a maximal element in the equivalence class which contains H.

Let $\mathfrak{F}(\mathfrak{S})$ be any $\mathfrak{F}$-basis of G reducing into $<H, K>$. Since $H \sim K$ in G, H is $\mathfrak{F}$-subnormal in $<H, K>$ by proposition 5 and hence $\mathfrak{F}(\mathfrak{S})$ reduces into H by proposition 2. Therefore we need show that any $\mathfrak{F}$-basis reducing into H reduces into $<H, K>$.

Let $\mathfrak{F}(\mathfrak{S})$ be any $\mathfrak{F}$-basis of G reducing into H, i. e., $\mathfrak{F}(\mathfrak{S}) \in \mathfrak{M}$. Then $\mathfrak{F}(\mathfrak{S})$ reduces into L^g for some g in G. Since H is $\mathfrak{F}$-subnormal in L by lemma 6, H^g is $\mathfrak{F}$-subnormal in L^g Therefore $\mathfrak{F}(\mathfrak{S})$ reduces into H^g and $\mathfrak{F}(\mathfrak{S}) \in \mathfrak{M} \cap \mathfrak{M}^g$. Thus, since $\mathfrak{M}$ is a block, $\mathfrak{M} = \mathfrak{M}^g$ and hence g is in L and so $\mathfrak{F}(\mathfrak{S})$ reduces into L. Since H is a subnormal subgroup of L, K is a subnormal subgroup of L and hence $<H, K>$ is subnormal in L by lemma 7. Therefore $\mathfrak{F}(\mathfrak{S})$ reduces into $<H, K>$ by proposition 2.

LEMMA 9. *Let H be a subgroup of G. Suppose that the set of all $\mathfrak{F}$-bases of G reducing into H forms a block. Then $H \sim R(H;\mathfrak{F})$ in G.*

PROOF. This lemma follows from the definition of $\mathfrak{F}$-equivalent and (2.5).

PROPOSITION 10. *Let H be a subgroup of G, Let $\mathfrak{M}$ be the set of all $\mathfrak{F}$-bases of G reducing into H and L be the stabilizer in G of $\mathfrak{M}$. Suppose that H is subnormal in L and $\mathfrak{M}$ forms a block. Then we have that $M(H;\mathfrak{F}) = R(H;\mathfrak{F})$.*

PROOF. Since $\mathfrak{M}$ is a block, we have $R(H;\mathfrak{F}) = Q(H;\mathfrak{F}) = L$ by (2.6). Now by lemma 9, $H \sim R(H;\mathfrak{F})$ in G, and hence we have $R(H;\mathfrak{F}) \subseteq M(H;\mathfrak{F})$ by the construction of $M(H;\mathfrak{F})$. On the other hand, since $M(H;\mathfrak{F})$ is a subgroup of L, it follows that $M(H;\mathfrak{F}) \subseteq R(H;\mathfrak{F})$. Therefore $M(H;\mathfrak{F}) = R(H;\mathfrak{F})$.

PROPOSITION 11. *Let H be a subgroup of G, Suppose that the set of all $\mathfrak{F}$-bases of G reducing into H forms a block. Then we have that $N_G(R(H;\mathfrak{F})) = R(H;\mathfrak{F})$.*

PROOF. By (2.8), $R(H;\mathfrak{F})$ is an $\mathfrak{F}$-subnormalizer of H in G. Therefore H is $\mathfrak{F}$-subnormal in $R(H;\mathfrak{F})$ and hence H is $\mathfrak{F}$-subnormal in $N_G(R(H;\mathfrak{F}))$ since $R(H;\mathfrak{F})$ is normal in $N_G(R(H;\mathfrak{F}))$. Thus we have $N_G(R(H;\mathfrak{F})) \subseteq R(H;\mathfrak{F})$, so that $N_G(R(H;\mathfrak{F})) = R(H;\mathfrak{F})$.

PROPOSITION 12. *Let H be a subgroup of G. Suppose that the set of all $\mathfrak{F}$-bases of G reducing into H forms a block. Then $R(H;\mathfrak{F})$ is the least $\mathfrak{F}$-abnormal subgroup K of G such that every $\mathfrak{F}$-basis of G reducing into H reduces also into K.*

PROOF. It follows from (2.2), (2.3) and (2.4) that $R(H;\mathfrak{F})$ is the $\mathfrak{F}$-abnormal subgroup of G such that every $\mathfrak{F}$-basis of G reducing into H reduces also into $R(H;\mathfrak{F})$.

Let $\mathfrak{F}(\mathfrak{S})$ be any $\mathfrak{F}$-basis of G which reduces into H and let a be any element of $R(H;\mathfrak{F})$. Then, by lemma 9, $\mathfrak{F}(\mathfrak{S})^a$ is an $\mathfrak{F}$-basis of G which reduces into H. Now suppose K as in the theorem. Then $\mathfrak{F}(\mathfrak{S})^a$ reduces into K and thus a is in $R(K;\mathfrak{F})$. Therefore, $R(H;\mathfrak{F})\subseteq R(K;\mathfrak{F})$. Hence we have $R(H;\mathfrak{F})\leq K$ by (2.2) since K is $\mathfrak{F}$-abnormal and the proof is complete.

PROPOSITION 13. *Let H be a subgroup of G and let K be a subgroup of G which contains $R(H;\mathfrak{F})$. Suppose that the set of all $\mathfrak{F}$-bases of G reducing into H forms a block. Then $R(H;\mathfrak{F})$ is the $\mathfrak{F}$-reducer of H in K.*

PROOF. Let A/B be an $\mathfrak{F}$-central H-composition factor of G. Then, by (2.7), $R(H;\mathfrak{F})$ covers A/B and hence K covers A/B. Now A/B is isomophic to $A\cap K/B\cap K$ as H-groups, so therefore $A\cap K/B\cap K$ is an $\mathfrak{F}$-central H-composition factor of K. Thus $R_K(H;\mathfrak{F})$ covers $A\cap K/B\cap K$ by (2.7), where $R_K(H;\mathfrak{F})$ denote the $\mathfrak{F}$-reducer of H in K. Therefore $R_K(H;\mathfrak{F})$ covers A/B. Hence $R(H;\mathfrak{F})\subseteq R_K(H;\mathfrak{F})$ by (2.7).

Conversely, now let $\mathfrak{F}(\mathfrak{S}_K)$ be any $\mathfrak{F}$-basis of K which reduces into H. Then there exists a Sylow system $\mathfrak{S}$ of G which is extension of $\mathfrak{S}_K$ and $\mathfrak{F}(\mathfrak{S})$ reduces into H by lemma 4. Therefore $\mathfrak{F}(\mathfrak{S})$ reduces into $R(H;\mathfrak{F})$ by (2.4), and so $\mathfrak{F}(\mathfrak{S}_K)$ reduces into $R(H;\mathfrak{F})$ by lemma 4. Thus by Proposition 12, $R_K(H;\mathfrak{F})\subseteq R(H;\mathfrak{F})$ and the proof is complete.

References

1. C. J. Graddon, *$\mathfrak{F}$-reducers in finite soluble groups,* J. Algebra, 18(1971), 574–587.
2. C. J. Graddon, *The relation between $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers in finite soluble groups,* J. London Math. Soc., (2) 4 (1971) 51–61.
3. A. Mann, *On subgroups of finite solvable groups,* Proc. Amer. Math. Soc., 22(1969), 214–216.
4. A. Mann, *On subgroups of finite solvable groups* II, J. Algegra,, 22(1972), 233–240.
5. A. Mann, *A characterization of Carter subgroups,* J. London Math. Soc. (2) 5 (1972), 517–518.
6. D. S. Passman, "Permutation Groups." W. A. Benjamin, New York, 1968.

(Recieved : April 30, 1977)